Assertions, hypotheses, conjectures, expectations: Rough-sets semantics and proof-theory.

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to Dag Prawitz

1 Preface

In this paper we reconsider some notions and results in the project of a "Logic for Pragmatics", initiated by the philosopher Carlo Dalla Pozza and by the phisicist Claudio Garola [18] and later continued by Dalla Pozza and Bellin [7] and others; in particular we revise the discussion of the logical properties of assertive and conjectural reasoning presented in "Towards a logic for pragmatics. Assertions and conjectures" [5] by the author and Corrado Biasi. The revisions meet some philosophical objections raised to that paper, involving a fundamental aspect, the modal translation of our intuitionistic logic of assertions and conjectures into classical **S4**: in a nutshell, the justification of an *assertion* requires *epistemic necessity* of the truth of the propositional content; expressing a *doubt* about a proposition is justified by the *epistemic possibility* of its falsity; for the justification of a conjecture we need *the possibility of epistemic necessity* of its truth, not just epistemic possibility. This conceptual clarification not only gives a more convincing characterization of the logic in [5] as a treatment of *assertions and hypotheses*, but also opens the way to distinct representations of *conjectures* both in the epistemic **S4** semantics and *rough sets* interpretation. The latter provides a possible link with Piero Pagliani's analysis of "intrinsic co-Heyting boundaries" [40], a notion by William Lawvere of great significance.[1]

[1] We wish to thank Prof Andrew Pitts and Dr Valeria de Paiva for their advice on various aspects of this research and Dr Piero Pagliani for his expert support in the Rough Sets semantics. We are much indebted with Tristan Crolard for his intriguing work on bi-intuitionistic and classical logic and with Hugo Herbelin for important suggestions about the distributed calculus for co-intuitionistic logic. Thanks to Carlo Dalla Pozza, Kurt Ranalter, Corrado Biasi and Graham White for their cooperation in the "logic for pragmatics" enterprise and thanks to Ugo

In the proof-theoretic treatment we focus on a small subsystem of the sequent calculus in [5] and look at a term assignment to co-intuitionistic logic, starting from Tristan Crolard's treatment [14], but developing it outside the framework of the $\lambda\mu$ calculus. We believe that these results justify further efforts towards a full proof-theory of bi-intuitionistic logic (including eventually a categorical semantics for it).

1.1 Logic for Pragmatics: Dalla Pozza and Garola's approach

The aim of Dalla Pozza and Garola's "logic for pragmatics" is to capture the logical properties of what are called *illocutionary acts* – asserting, conjecturing, commanding, promising, and so on. Consider assertions. In their framework there is a logic of propositions and a logic of assertions. Propositions can be either true or false, according to classical semantics, assertions are acts that can be *justified* or *unjustified*, *felicitous* or *infelicitous*. They propose a two-layer theory with a distinctive informal interpretation, according to which propositions have *truth conditions*, i.e., a semantics, whereas assertions have *justification conditions*, belonging to pragmatics. As a consequence, we can form logical combinations of propositions, which are given a classical semantics as usual, but we can also form logical combinations of assertions, and interpret these combinations along the familiar lines of Heyting's interpretation of intuitionistic connectives. This is Dalla Pozza and Garola's *pragmatic interpretation of intuitionistic logic*: if α denotes a proposition, the *elementary expression* $\vdash\alpha$ stands for an assertion and $\vdash\alpha$ is justified just in case we have *conclusive evidence* that α is true; in the case of a mathematical statement α, "conclusive evidence" is a proof of α. Moreover, an assertive expression of conditional type $A \supset B$ is justified by providing a method that transforms a justification of an assertive type A into a justification of an assertive type B.

It should be noticed that intuitionistic logic is represented in Dalla Pozza and Garola's framework as a theory of pragmatic validity only if the justification of elementary expressions $\vdash \alpha$ does not depend on the logical structure of the *radical expression* α as a classical proposition – e.g., we shall not allow α to be $p \vee \neg p$. Thus in every investigation of intuitionistic theories within the framework of Dalla Pozza and Garola [18] it is assumed that *elementary expressions have atomic radicals, i.e.,* $\alpha = p$. This convention is essential also for the present investigation of our co-intuitionistic and bi-intuitionistic logic.

The novelty of Dalla Pozza and Garola's work is that Heyting's semantics is applied to *illocutionary types* of acts, not to propositions; if the justification of an assertion of atomic type $\vdash\alpha$ is related to the semantics of the propositional content α, a complex type has only a *pragmatic* justification value,

Solitro for useful discussions. I am grateful to Dag Prawitz, my first marvellous supervisor in Stockholm 1978 and to Luiz Carlos Pereira, a fellow student and a supportive colleague.

not a *semantic* one. To recover propositions and semantic values one considers *semantic projections* given by the Gödel, McKinsey, Tarksi and Kripke's translation:

$$(\vdash\alpha)^M = \Box\alpha \qquad\qquad (A \supset B)^M = \Box(A^M \to B^M)$$

This modal formalism can be given the usual interpretation through an *epistemic* view of Kripke **S4** semantics. Thus in a Kripke model $(W, R, \Vdash)$ for **S4** every $w \in W$ is seen as a stage of human knowledge and the accessibility relation expresses ways in which our knowledge may evolve; at each stage atomic propositions are locally true or false according to $\Vdash$; reflexivity of R means that what we know must be true also locally and transitivity of R expresses the fact that human knowledge cannot be forgotten or falsified, and so on.[2]

The basic approach of Dalla Pozza and Garola seems to stand as a helpful conceptual clarification, following Quine saying that *a change of logic reflects a change of the subject matter of the logic*. The remarkable technical developments of the proof-theory of classical logic in the last decades suggest the possibility of a pragmatic interpretation of classical methods of inference; despite some hints in [5], section 5, and the result in section 2.5 below, this remains an essentially unfinished business.

Justification and felicity conditions.

Going back to the basic texts of modern pragmatics, such as Austin[2] and Levinson[31], every speech act has a *propositional content*, an *illocutionary force* (or *pragmatic* mood) and *perlocutionary effects*. Now it seems that the *felicity* or *infelicity* conditions of a speech act essentially depend on the actual

[2] The interpretation of intuitionistic logic as a logic of *assertions* appears already in Dummett's work. Martin-Löf regards his intuitionistic theory of types as expressing *judgements* about the truth of propositions; in his system well-formed complex types are *propositions* and the terms inhabiting them are witnesses of their truth, intuitionistically understood. This view is disputed by Dalla Pozza: for him only *atomic types* assert the truth of propositions, but complex types neither *are* propositions nor *assert* propositions. To recover a proposition corresponding to the complex type

$$(\S) \qquad\qquad \vdash\alpha \supset (\vdash\beta \supset \vdash\alpha)$$

we need a *semantic projection*, i.e., $\Box(\Box\alpha \to \Box(\Box\beta \to \Box\alpha))$; but justification of the assertion type

$$(\S\S) \qquad\qquad \vdash \Box(\Box\alpha \to \Box(\Box\beta \to \Box\alpha))$$

is a semantic argument for a sentence of classical **S4** while (§) is justified by something like a program $\lambda x.\lambda y.x$, where $x : \vdash\alpha$ and $y : \vdash\beta$ are variable ranging over proofs of the truth of α and of β.

circumstances of its performance and on its intended or unintended perlocutionary effects. Thus a formalization of the felicity or infelicity conditions of a statement would be based on a *formal theory of actions* including a representation of the *agent* and the *addressees* of a speech act and also its *preconditions* and *postconditions* (for a first formulation of such a theory, see [60]).

On the contrary, the contribution of the *illocutionary mood* to the pragmatics of speech acts can be characterized by *abstracting away* from the actual agents and addressees and from their specific context, effects and goals. Thus an *impersonal* illocutionary operator of an intensional logic may suffice to express illocutionary force, if the justification of the illocutionary mood of such type of acts makes reference to a relatively stable and uniform context (e.g., scientific knowledge in a given time, obligations within an established legal system, unambiguous linguistic acts within a linguistic community, and so on).

In this framework, several works have explored the "logic for pragmatics" of obligations (Dalla Pozza [19]) and then the logic of assertions, obligations with causal reasoning (Bellin, Dalla Pozza and Ranalter [7, 6, 48, 49]). In general, the development of such logics requires an identification of the appropriate modal operators or non-classical connectives used in the modal projection and their Kripke semantics; then one proceeds to a more abstract treatment of the proof theory, as in Ranalter's work.

2 PART I. Conceptual analysis: assertions, hypotheses and conjectures

In extending Dalla Pozza and Garola's framework to a logic of hypothetical and conjectural moods, we encounter a variety of moods with different linguistic and logical properties.[3] It is familiar the distinction in Latin between three kinds of *if* clauses, the first one using the indicative to express the condition as a matter of fact, the second the present subjunctive to express possibility of the condition and, finally, the third one using the past subjunctive for counterfactuals. Also consider *theory of argumentation*. Here six *proof-standards* have been identified from an analysis of legal practice: *scintilla of evidence*, *preponderance of evidence*, *clear and convincing evidence*, *beyond reasonable doubt* and *dialectical validity*, in a linear order of strength [24].[4]

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[3] The conceptual development traced results from cooperation with other researchers, in particular with Corrado Biasi, whose doctoral dissertation at Queen Mary, University of London is still unfinished.

[4] In the formal treatment of Carneades model of argumentation, *proof-standards* occur in the definition of what it means for an argument with conclusion c from premises P and exceptions E to be *applicable* in a *Carneades argument evaluation structure* $\mathcal{S} = \langle \textit{arguments, assumptions, weights, standard} \rangle$. The definition relies on a non-logical real-valued function *weights* ranging over arguments. The notion of applicability is recursive, as it depends on the notion of a proposition p

It is essential to remember that "legal reasoning is not primarily deductive, but rather a modelling process of shaping an understanding of the facts, based on evidence, and an interpretation of the legal sources, to construct a theory for some legal conclusion" (Jon Bing [12] cited in [24]). More precisely, in order to decide whether to accept or reject each element of a given set of "claims", one constructs a consistent "theory of the generalizations of the domain and the facts of the particular case", together with "a *proof* justifying the decision of each issue, showing how the decision is supported by the theory" [24].

Thus in Argumentation Theory one starts with an inconsistent knowledge base and a set of claims to build a consistent theory. But later, when deriving the claims from the theory, it might be desirable to use a logic which retains essential pragmatic information such as the standards of evidence of the premises, omitted in classical logic. The relations between "theory searching" and deductive reasoning are a subject of research in Argumentation Theory; here we use the notion of "standards of proof" in an informal way and point at the possibility of incorporating a *theory of positive evidence* in our framework as a suggestion for future work.

2.1 First attempt: assertive and hypothetical types

In Bellin and Biasi [5] we have given a logic of *hypothetical types* parallel to Dalla Pozza and Garola's logic of assertions. We start with *elementary illocutionary acts of hypothesis*, denoted by $\mathcal{H}\alpha$: here α is a proposition which is presented as possibly true; such an act is justified if there are grounds for believing that α may be true in some circumstances. Next we consider connectives building up complex hypothetical types from elementary ones. For instance, through the connective of *subtraction* we build the hypothetical expression *possibly* C *but not* D (written $C \smallsetminus D$); such an expression is justified if it is justified to believe the truth of the hypothetical expression C while also believing that the hypothesis D may never be true; the disjunction $C \curlyvee D$ of the hypothetical expressions C and D is also a hypothetical expression, and so on.

The *modal projection* of hypothetical expressions is also in classical **S4**:

$$(\mathcal{H}\alpha)^M = \Diamond\alpha \qquad (C \smallsetminus D)^M = \Diamond(C^M \wedge \neg D^M)$$

Namely, the modal translations of assertions A^M and hypotheses C^M are both interpreted in models $(W, R, \Vdash)$ where R is transitive and reflexive. This choice is crucial for the approach of [5]: other modal candidates are possible as discussed in [5] and in more detail below.

being *acceptable* in an argument evaluation structure $\mathcal{S}$. Here a proposition p is *acceptable* with a *scintilla of evidence* if there is at least one applicable argument for p and p is *acceptable* as *dialectically valid* if there is an applicable argument for p and no applicable argument against p. All other proof standards require comparing the weights of arguments for and against p. See [24, 13].

In natural language illocutionary acts of hypothesis may be embedded into a context consisting of illocutionary act of assertion, for instance

> *Arturo is the best pianist of his generation and will not refuse to play in this town, although the audience may be slightly noisy;*

an assertive conjunction of two assertions and a hypothetical statement; conversely, assertions may be embedded in a hypothetical context:

> *We may not hear Arturo playing, because he has very high standards and if the audience is slightly noisy then he may refuse to play.*

containing a hypothetical implication with an assertive antecedent and hypothetical consequent. Taking this idea seriously, one obtains a rather unmanageable family of *mixed connectives* [5]; in this paper we shall consider only the role of *mixed negations* turning assertive expressions into hypothetical ones and conversely.

Three methodological principles.

Our logical treatment of *assertions and hypotheses* is based on the notion of a *duality* between these two illocutionary moods: informally it is a familiar idea, since a proof of a proposition may be obtained as a refutation of the conjecture that its dual is true. In a formal treatment, there are many aspects to this duality, which is certainly satisfied by the modal translation in **S4**. In [5], Section 1.1, three methodological principles are stated for a logic expressing the duality between assertive and hypothetical types:

1. The grounds that justify *asserting* a proposition α certainly suffice also for *conjecturing* it, whatever these grounds may be;
2. in any situation, the ground that justify the assertion of α are also necessary and sufficient to regard the conjecture that $\neg\alpha$ as unjustified;
3. the justification of non-elementary assertive or hypothetical types, built up from elementary types using pragmatic connectives, depends on the justification of the component types, possibly using intensional operations.

The third principle requires a sort of compositionality of justification: this is certainly satisfied by the intended informal interpretation of the connectives.

As it stands, the second principle is inadequate. On one hand, it is indisputable that the grounds allowing one to regard the assertion of α as justified must override any ground in favour of the conjecture of $\neg\alpha$; on the other hand, it is wrong and contrary to common sense to say that if the *conjecture* of $\neg\alpha$ is unjustified then the *assertion* of α is justified: the grounds we may have to dismiss the conjecture that $\neg\alpha$ may be the case may not be strong enough to justify the assertion that α is true. There are at least two issues here.

Firstly, we must distinguish between the illocutionary force of a mere *hypothesis* and that of a *conjecture*, a distinction we shall develop later in this paper. Let us split the second principle in two parts, replacing "hypothesis" for "conjecture":

2.i If the assertion of α is justified, then the hypothesis that $\neg\alpha$ is true cannot be justified.
2.ii If the hypothesis that $\neg\alpha$ is true is unjustified, then the assertion of α is justified.

Except for the case of counterfactuals, which are not our concern here, (2.*i*) is still correct; as for (2.*ii*), it becomes plausible if we assume that a hypothesis $\mathcal{H}\neg\alpha$ may be justified by a mere *cognitive possibility* of a situation, no matter how unlikely it may be, in which $\neg\alpha$ is true. The epistemic interpretation of the modal interpretation in **S4** validates this reading of (2.*ii*).

This raises a second issue: in our framework there is no theory of *positive evidence*; nevertheless we must be able to distinguish illocutionary forces whose justification depends on different strengths of evidence. Thus the logic of hypothetical reasoning in [5] reduces to a *refutation calculus*; although pure refutation does correspond to common-sense reasoning – indeed it seems to be very close to the medieval practice of disputation [1] [5] – it may not suffice for applications, e.g., to a theory of laws and to legal reasoning.

Finally, the first principle is true for any reading of $\mathcal{H}\alpha$, e.g., as *hypothesis* or *conjecture*. Also it is true in argumentation theory: the assertion $\vdash\alpha$ must be justified by "standards of proof" at least as strong as those justifying the hypothesis $\mathcal{H}\alpha$. Notice that this principle shows a basic asymmetry between assertions and hypotheses.

A logic of assertions and hypotheses: the language $\mathcal{L}^{AH}$

The core fragment of the logic of assertions and hypotheses in [5] is a propositional language built from a countable set of atomic formulas p, p_1, p_2, ... and symbols of illocutionary force yielding elementary formulas $\vdash p$ (*certainly* p) and $\mathcal{H}p$ (*perhaps* p). It consists of two dual parts:

- an assertive part $\mathcal{L}^A$ built from elementary assertions $\vdash p$, a sentential constant for *validity* ($\curlyvee$), using *assertive conjunction* ($\cap$) and *assertive implication* ($\supset$) and
- a hypothetical part $\mathcal{L}^H$ built from elementary hypotheses $\mathcal{H}p$ and a constant for *absurdity* ($\curlywedge$), using *hypothetical disjunction* ($\curlyvee$), and *hypothetical subtraction* ($\smallsetminus$).

[5] We are grateful to an anonymous referee to [5] for making the point clear and for indicating the reference. The same referee, acknowledging that our "refutation calculus" is dual to intuitionistic logic, questioned whether a calculus based on a theory of positive evidence could be co-intuitionistic: we come back to this issue below.

Thus $\mathcal{L}^A$ and $\mathcal{L}^H$ are negation-free fragments of the language of intuitionistic and co-intuitionistic logic. Let **abs** be an absurd statement in $\mathcal{L}^A$ and **val** is a valid statement in $\mathcal{L}^H$. Then $\sim X =_{def} X \supset \mathbf{abs}$ expresses assertively the existence of a method to turn a justification of X into a justification of an absurdity. Similarly $\frown Y =_{def} \mathbf{val} \smallsetminus Y$ expresses the *doubt* that Y may be true, namely, the hypothesis that a valid statement **val** may be compatible with the negation of Y. Thus we have *four negations*:

1. if X is an assertive expression, then $\sim X$ is the usual *intuitionistic negation*;
2. if Y is a hypothetical expression, $\frown Y$ is *co-intuitionistic supplement*;
3. if X is a *hypothetical*, then the mixed expression $X \supset \mathbf{abs}$ is an assertive type;
4. if Y is assertive, then $\mathbf{val} \smallsetminus Y$ is a hypothetical type.[6]

Our logic is therefore *bi-intuitionistic*, in the sense that it has intuitionistic and co-intuitionistic connectives, but it is *polarized*, as elementary formulas are either intuitionistic ($\vdash p$) or co-intuitionistic ($\mathcal{H} p$), *but not both*, and connectives, with the possble exception of negations, preserve the polarity. Thus we have the following grammar of the language of *polarized bi-intuitionistic* logic for the pragmatics of assertions and hypotheses $\mathcal{L}^{AH}$:

$$A, B \quad := \quad \vdash p \;\mid\; \curlyvee \;\mid\; A \supset B \;\mid\; A \cap B \;\mid\; \sim C$$

$$C, D \quad := \quad \mathcal{H} p \;\mid\; \curlywedge \;\mid\; A \smallsetminus B \;\mid\; A \curlyvee B \;\mid\; \frown A$$

2.2 Second attempt: more general modal translations

In order to approximate alternative treatments of a logic of *assertions*, *hypotheses* and *conjectures*, we consider more general modal translations in *bimodal* **S4**.

Translations in Bimodal S4

Definition 1. *(i) Let p range over a denumerable set of propositional variables* $\mathbf{Var} = \{p_1, p_1, \ldots\}$. *The bimodal language* $\mathcal{L}_{\Box,\boxminus}$ *is defined by the following grammar.*

$$\alpha := p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \to \alpha \mid \Box\alpha \mid \;\boxminus\; \alpha$$

[6] As in [5], to these pragmatic negations one should add classical negation in the radical part $\neg\alpha$; but no logical property of the radical part can be used in the treatment of intuitionistic pragmatics. To avoid confusions with the “polarized classical logic” in [5], section 5, in the treatment of dualities we shall assume that the atoms occurring in the radical part are either positive p_i^+ or negative p_i^-, i.e., that there is an involution without fixed point on atoms exchanging p_i^+ and p_i^-.

Define $\Diamond\alpha =_{df} \neg\Box\neg\alpha$ *and* $\diamondminus\alpha =_{df} \neg \boxminus \neg\alpha$.

(ii) Let $\mathcal{F} = (W, R, S)$ *be a multimodal frame, where* W *is a set,* R *and* S *are preorders on* W. *Given a valuation function* $V : \mathbf{Var} \to \wp(W)$, *the forcing relations are defined as follows:*

- $w \Vdash \Box\alpha$ *iff* $\forall w'.wRw' \Rightarrow w' \Vdash \alpha$,
- $w \Vdash \boxminus \alpha$ *iff* $\forall w'.wSw' \Rightarrow w' \Vdash \alpha$.

(iii) We say that a formula A *in the language* $\mathcal{L}_{\Box,\boxminus}$ *is* valid in bimodal **S4** *if* A *is valid in all bimodal frames* $\mathcal{F} = (W, R, S)$ *where* R *and* S *are preorders.*

Lemma 1. *Let* $\mathcal{F} = (W, R, S)$ *be a multimodal frame, where* R *and* S *are preorders.*

(i) *The following are valid in* $\mathcal{F}$

$$\Box\boxminus\Box\alpha \to \Box\alpha \qquad \textit{and} \qquad \boxminus\Box\boxminus\alpha \to \boxminus\alpha$$

$(ii)(a)$ *The following are equivalent:*

1.a: $S \subseteq R$;

2.a: the following scheme is valid in $\mathcal{F}$:

(Ax.a) $\Box\alpha \to \Box\boxminus\Box\alpha$;

3.a: the following rule is valid in $\mathcal{F}$:

(R.a)
$$\frac{\diamondminus\beta \Rightarrow \diamondminus\neg\Box\alpha}{\Box\alpha \Rightarrow \Box\neg\diamondminus\beta}$$

$(ii)(b)$ *The following are equivalent*

1.b: $R \subseteq S$;

2.b: the following scheme is valid in $\mathcal{F}$:

(Ax.b) $\boxminus\alpha \to \boxminus\Box\boxminus\alpha$

3.b: the following rule is valid in $\mathcal{F}$:

(R.b)
$$\frac{\Box\neg\diamondminus\beta \Rightarrow \Box\alpha}{\diamondminus\neg\Box\alpha \Rightarrow \diamondminus\beta}$$

Proof of $(ii)(a)$. (1.a $\Rightarrow$ 2.a) is obvious. (2.a $\Rightarrow$ 1.a): If S is not a subset of R, then given wSv and not wRv define a model on $\mathcal{F}$ where $w' \Vdash p$ for all w' such that wRw' but $v \nVdash p$; thus $\Box p \to \Box\boxminus\Box p$ is false at w. (2.a $\Rightarrow$ 3.a): If $\diamondminus\beta \Rightarrow \diamondminus\neg\Box\alpha$ is valid in $\mathcal{F}$ then so is $\Box\neg\diamondminus\neg\Box\alpha \Rightarrow \Box\neg\diamondminus\beta$ and the conclusion of $(R.a)$ is valid because of $(Ax.a)$. From the conclusion of $(R.a)$ the premise follows using part (i). (3.a $\Rightarrow$ 2.a): $(Ax.a)$ is obtained by applying $(R.a)$ downwards to $\diamondminus\neg\Box\alpha \Rightarrow \diamondminus\neg\Box\alpha$. The other parts are similar.

2.3 Bimodal interpretations of $\mathcal{L}^{AH}$

Definition 2. *The interpretation* $(\ )^M$ *of* $\mathcal{L}^{AH}$ *into* $\mathcal{L}_{\Box,\boxminus}$ *is defined inductively thus:*

$$\begin{array}{rlllrll}
(\curlywedge)^M & =_{df} & \bot & \quad & (\curlyvee)^M & =_{df} & \top \\
(\vdash p)^M & =_{df} & \Box p & & (\mathcal{H} p)^M & =_{df} & \Diamond p \\
(A \supset B)^M & =_{df} & \Box(A^M \rightarrow B^M) & & (C \smallsetminus D)^M & =_{df} & \Diamond(C^M \wedge \neg D^M) \\
(A_1 \cap A_2)^M & =_{df} & A_1^M \wedge A_2^M & & (C_1 \curlyvee C_2)^M & =_{df} & C_1^M \vee C_2^M \\
(\sim C)^M & =_{df} & \Box\neg C^M & & (\frown A)^M & =_{df} & \Diamond\neg A^M
\end{array}$$

It is easy to prove that $A^M \iff \Box A^M$ *and* $C^M \iff \Diamond C^M$ *in the semantics of (bimodal)* **S4**.

(i) The propositional theory **PBL** *(polarized bi-intuitionistic logic) is the set of all formulas* δ *in the language* $\mathcal{L}^{AH}$ *such that* δ^M *is valid in every preordered bimodal frame (i.e, in any frame* (W, R, S) *where* R *and* S *are arbitrary preorders).*

(ii) The propositional theory **APBL** *(asymmetric polarized bi-intuitionistic logic) is the set of all formulas* δ *in the language* $\mathcal{L}^{AH}$ *such that* δ^M *is valid in every preordered bi-modal frame* (W, R, S) *where* $S \subseteq R$.

(iii) The propositional theory **AHL** *(bi-intuitionistic logic of assertions and hypotheses) is the set of all formulas* δ *in the language* $\mathcal{L}^{AH}$ *such that* δ^M *is valid in every preordered bi-modal frame* (W, R, S) *where* $R = S$*. In other words, in the modal translation let* $\Diamond X =_{df} \neg\Box\neg X$*; then* δ *is in* **AHL** *if and only if* δ^M *is valid in* **S4**.

Remark 1. (i) **PBL** is the most *abstract* theory of *bi-intuitionistic* logic where all formulas are *polarized* as assertive or hypothetical. **PBL** is not a suitable candidate for our logic of assertions and hypotheses, since the pair $(\vdash p)^M$, $(\frown \vdash p)^M$ is consistent in bi-modal **S4**, contrary to the accepted principle $(2.i)$. We won't speculate about possibility of interpreting $\frown \vdash p$ as a counterfactual.

(ii) On the contrary, the *asymmetric* logic **APBL** satisfies $(2.i)$, but not $(2.ii)$[7]. Thus **APBL** may be the right context for studying assertive and hypothetical reasoning where hypothetical statements have different degrees of positive evidence and thus are not representable in a pure refutation calculus.

(iii) Finally, our *canonical* system is the *bi-intuitionistic logic of assertions and hypotheses* **AHL** – poorly named *intuitionistic logic for pragmatics* **ILP** in [5] – satisfying both conditions $(2.i)$ and $(2.ii)$. It is motivated by the epistemic interpretation of (uni-modal) **S4**, where hypotheses are seen as mere epistemic possibilities and assertions as epistemic necessities.

[7] Condition $S \subseteq R$ guarantees that if $w \Vdash \Box p$ then $w \nVdash \Diamond \neg p$. To see that $w \Vdash \Box p$ is not a valid consequence of $w \nVdash \Diamond \neg p$, consider a model $\mathcal{M} = (W, R, S, \Vdash)$ with $W = \{w, w'\}$, R and S reflexive and transitive and such that wRw' but not wSw', and $w \Vdash p$ but $w' \Vdash \neg p$. Notice that $\mathcal{H}\neg p$ is not an expression of the language $\mathcal{L}^{AH}$, but the same remark applies for $(\vdash p)^M = \Box p$ and $(\frown \vdash p)^M = \Diamond \neg\Box p$.

Dualities

Definition 3. *Let* $(\ )^{\perp}$: **Atoms** $\rightarrow$ **Atoms** *be an involution without fixed points on the atomic formulas* p_i. Intuitively, we may think of $p_i^{\perp}$ as $\neg p_i$, but intuitionistic dualites are defined best without any reference to the classical part. *We extend* $(\ )^{\perp}$ *to maps* $F : \mathcal{L}^A \rightarrow \mathcal{L}^H$ *and* $G : \mathcal{L}^H \rightarrow \mathcal{L}^A$ *letting*

$$
\begin{array}{lrl}
(a) & F(\vdash p) = \mathcal{H} p^{\perp} & G(\mathcal{H} p) = \vdash p^{\perp} \\
(b) & F(A \cap B) = F(A) \curlyvee F(B) & G(C \curlyvee D) = G(C) \cap G(D) \\
(c) & F(A \supset B) = F(B) \smallsetminus F(A) & G(C \smallsetminus D) = G(D) \supset G(C)
\end{array}
$$

Lemma 2. *In* **AHL** *let* $F(A) = \frown A$ *and* $G(C) = \sim C$. *Then*

1. *if we interpret* $(p^{\perp})^M$ *as* $\neg p$, *then the modal translations of conditions (a)-(c) are valid equivalences in* **S4**;
2. $GF(A) \equiv A$ *and* $FG(C) \equiv C$;
3. $\dfrac{A \Rightarrow G(C)}{C \Rightarrow F(A)}$ *and* $\dfrac{G(C) \Rightarrow A}{F(A) \Rightarrow C}$

Proof. By definition of the modal translation we have

$$
\begin{array}{ll}
(1)(a): & (\frown \vdash p)^M = \Diamond\neg\Box p \equiv \Diamond\neg p = (\mathcal{H} p^{\perp})^M \\
& (\sim \mathcal{H} p)^M = \Box\neg\Diamond p \equiv \Box\neg p = (\vdash p^{\perp})^M
\end{array}
$$

$$
\begin{array}{ll}
(1)(b): & \Big(\frown (A \cap B)\Big)^M = \Diamond\neg(A^M \wedge B^M) \equiv (\Diamond\neg A^M) \vee (\Diamond\neg B^M) = \Big((\frown A) \curlyvee (\frown B)\Big)^M \\
& \Big(\sim (C \curlyvee D)\Big)^M = \Box\neg(C^M \vee D^M) \equiv (\Box\neg C^M) \wedge (\Box\neg D^M) = \Big((\sim C) \cap (\sim D)\Big)^M
\end{array}
$$

$$
\begin{array}{ll}
(1)(c): & \Big(\frown (A \supset B)\Big)^M = \Diamond\neg\Box(A^M \rightarrow B^M) \equiv \Diamond(\Diamond\neg B^M \wedge \neg\Diamond\neg A^M) = \Big((\frown B) \smallsetminus (\frown A)\Big)^M \\
& \Big(\sim (C \smallsetminus D)\Big)^M = \Box\neg\Diamond(C^M \wedge \neg D^M) \equiv \Box(\Box\neg D^M \rightarrow \Box\neg C^M) = \Big((\sim D) \supset (\sim C)\Big)^M
\end{array}
$$

(2): The conditions

$$
\sim\frown A \equiv A \qquad \text{and} \qquad C \equiv \frown\sim C \tag{1}
$$

follow from Lemma 1.(i) and (ii). The conditions in (3) follow from rules (R.a) and (R.b) in Lemma 1.(ii).

Remark 2. (i) Lemma 2 fails for **PBL** and **APBL**.

(ii) As the only mixed formulas in $\mathcal{L}^{AH}$ are negations, Lemma 2 gives us a (meta-theoretic) "method for eliminating mixed formulas in **AHL**" *modulo the atomic involution* $(\ )^{\perp}$, (interpreted in the modal translation as classical negation $\neg p$). E.g., the mixed expression $A \cap \sim(\mathcal{H} p \curlyvee \mathcal{H} q)$ is equivalent in the **S4** semantics to the purely assertive expression $A \cap (\vdash p^{\perp} \cap \vdash q^{\perp})$.

(iii) Sometimes we shall write $A^{\perp}$ and $C^{\perp}$ for $F(A)$ and $G(C)$, respectively.

Proposition 1. (restricted substitution) *Let σ be a map*

$$\vdash p_i \mapsto A_i \qquad \mathcal{H} p_j \mapsto C_j$$

sending a vector $\overline{\eta_a}$ of assertive elementary formulas *to a vector $\overline{A}$ of assertive formulas and a vector $\overline{\eta_h}$ of* hypothetical elementary formulas *to a vector $\overline{C}$ of hypothetical formulas. Then $X(\overline{\eta_a}, \overline{\eta_h})$ is a theorem of* **AHL** [**PBL**, **APBL**] *if and only if $X(\sigma(\overline{\eta_a}), \sigma(\overline{\eta_h}))$ is a theorem of* **AHL** [**PBL**, **APBL**].

On the other hand, the theories **AHL**, **PBL** and **APBL** are not closed under substitution of hypothetical formulas for assertive elementary formulas (and symmetrically). An example is the following:

$\sim\sim\sim \vdash p \Rightarrow \sim \vdash p$ is valid, but $\sim\sim\sim \mathcal{H} p \Rightarrow \sim \mathcal{H} p$ is not.
Indeed $\Box\Diamond\Box\Diamond\neg p \Rightarrow \Box\Diamond\neg p$ is valid, but $\Box\Diamond\Box\neg p \Rightarrow \Box\neg p$ is invalid in **S4**.

2.4 Sequent Calculi for PBL, APBL, AHL

The logics **PBL**, **APBL**, **AHL** can be formalized in **G3**-style sequent calculi [59], where the rules of Weakening and Contraction are implicit, as in [5]. One then proves that the rules of Weakening and Contraction are admissible preserving the depth of the derivation.

Definition 4. *All the sequents S are of the form*

$$\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon \tag{2}$$

where

- *Θ is a sequence of assertive formulas $A_1, \ldots, A_m$;*
- *Υ is a sequence of hypothetical formulas $C_1, \ldots, C_n$;*
- *ϵ is hypothetical and ϵ' is assertive and exactly one of ϵ, ϵ' occurs.*

The bi-intuitionistic logic of assertions and conjectures **AHL** *is formalized in the sequent calculus given by the axioms and rules in the table (1)* [8]*. Let us call this fragment standard* **AH-G3**.

The polarized bi-intuitionistic logic **PBL** *and the asymmetric polarized bi-intuitionistic logic* **APBL** *are formalized by restricting the rules of canonical* **AH-G3** *as indicated below: the restrictions only modify the rules* $\supset$-right, $\sim$-right, $\smallsetminus$-left *and* $\frown$-left. *Let us call the resulting sequent calculi abstract* **PB-G3** *and asymmetric* **APB-G3**, *respectively.*

[8] This calculus is essentially the system *Intuitionistic Logic for Pragmatics* **ILP** presented and studied in [5], Section 3, restricted to the language $\mathcal{L}^{AH}$ - namely, a sequent calculus with axioms and rules for *assertive validity, implication and conjunction*, *hypothetical absurdity, subtraction and disjunction* and two *mixed negations*.

Sequent Calculus AH-G3: axioms and rules

identity rules

logical axiom:

$$A, \Theta \; ; \; \Rightarrow \; A \; ; \; \Upsilon$$

logical axiom:

$$\Theta \; ; \; C \; \Rightarrow \; ; \; \Upsilon, C$$

cut$_1$:

$$\frac{\Theta \; ; \; \Rightarrow \; A \; ; \; \Upsilon \qquad A, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon'}{\Theta, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \Upsilon'}$$

cut$_2$:

$$\frac{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, C \qquad \Theta' \; ; \; C \; \Rightarrow \; \Upsilon'}{\Theta, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \Upsilon'}$$

ASSERTIVE LOGICAL RULES

validity axiom:

$$\Theta \; ; \; \Rightarrow \; \curlyvee \; ; \; \Upsilon$$

right $\supset$:

$$\frac{\Theta, A_1 \; ; \; \Rightarrow \; A_2 \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; A_1 \supset A_2 \; ; \; \Upsilon}$$

left $\supset$:

$$\frac{A_1 \supset A_2, \Theta ; \; \Rightarrow \; A_1 \; ; \; \Upsilon \qquad A_2, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{A_1 \supset A_2, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

right $\cap$:

$$\frac{\Theta \; ; \; \Rightarrow \; A_1 \; ; \; \Upsilon \qquad \Theta \; ; \; \Rightarrow \; A_2 \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; A_1 \cap A_2 \; ; \; \Upsilon}$$

left $\cap$:

$$\frac{A_0, A_1, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{A_0 \cap A_1, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

CONJECTURAL RULES

absurdity axiom:

$$; \; \curlywedge \; \Rightarrow \; ; \; \Upsilon$$

right $\smallsetminus$:

$$\frac{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, C_1 \qquad \Theta \; ; \; C_2 \; \Rightarrow \; ; \; \Upsilon, C_1 \smallsetminus C_2}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, C_1 \smallsetminus C_2}$$

left $\smallsetminus$:

$$\frac{\Theta ; \; C_1 \; \Rightarrow \; ; \; \Upsilon, C_2}{\Theta \; ; \; C_1 \smallsetminus C_2 \; \Rightarrow \; ; \; \Upsilon}$$

right $\curlyvee$:

$$\frac{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; C, \upsilon_0, C_1}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, C_0 \curlyvee C_1}$$

left $\curlyvee$:

$$\frac{\Theta \; ; \; C_1 \; \Rightarrow \; ; \; \Upsilon \qquad \Theta \; ; \; C_2 \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; C_1 \curlyvee C_2 \; \Rightarrow \; ; \; \Upsilon}$$

MIXED-TYPE NEGATIONS:

right $\sim$:

$$\frac{\Theta \; ; \; C \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \sim C \; ; \; \Upsilon}$$

left $\sim$:

$$\frac{\sim C, \Theta ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, C}{\sim C, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

right $\frown$:

$$\frac{\Theta, A \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \frown A}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \frown A}$$

left $\frown$:

$$\frac{\Theta ; \; \Rightarrow \; A \; ; \; \Upsilon}{\Theta \; ; \; \frown A \; \Rightarrow \; ; \; \Upsilon}$$

Table 1. The sequent calculus **AH-G3**

AH-G3:

$$\supset\text{-R}\ \frac{\Theta, A_1\ ;\ \Rightarrow\ A_2\ ;\ \Upsilon^*}{\Theta\ ;\ \Rightarrow\ A_1 \supset A_2\ ;\ \Upsilon} \qquad \smallsetminus\text{-L}\ \frac{\Theta^{**}\ ;\ C_1\ \Rightarrow\ ; C_2, \Upsilon}{\Theta\ ;\ C_1 \smallsetminus C_2 \Rightarrow\ ;\ \Upsilon}$$

$$\sim\text{-R}\ \frac{\Theta\ ;\ C\ \Rightarrow\ ;\ \Upsilon^*}{\Theta\ ;\ \Rightarrow\ \sim C\ ;\ \Upsilon} \qquad \frown\text{-L}\ \frac{\Theta^{**}\ ;\ \Rightarrow\ A\ ;\ \Upsilon}{\Theta\ ;\ \frown A\ \Rightarrow\ ;\ \Upsilon}$$

Υ^* not allowed in **PB-G3**, **APB-G3**	Θ^{**} not allowed in **PB-G3**

To see why in the *asymmetric* **APB-G3** and in the *canonical* **AH-G3** systems the formulas in Θ are allowed in the antecedent of the sequent-premise of $\smallsetminus$-*left* and of $\frown$-*left*, notice that by the valid scheme $(Ax.a)$ of Lemma 1.$(ii)(a)$

$$A \Rightarrow \sim\frown A \quad \text{is valid in the semantics of } \mathbf{APB} \text{ and of } \mathbf{AHL} \tag{3}$$

Thus the unrestricted rule $\frown$-*left* of **AP-G3** and **AH-G3** becomes derivable in **PB-G3** using *cut* with the scheme (1) taken as axiom:

$$\text{cut}\ \cfrac{\overset{(1)}{B\ ;\ \Rightarrow\ \sim\frown B\ ;} \qquad \sim\text{-L}\ \cfrac{\frown\text{-L}\ \cfrac{\frown\text{-R}\ \cfrac{B\ ;\ \Rightarrow\ A\ ;\ \Upsilon}{;\ \Rightarrow\ A\ ;\ \Upsilon, \frown B}}{;\ \frown A\ \Rightarrow\ ;\ \Upsilon, \frown B}\ (\P)}{\sim\frown B\ ;\ \frown A\ \Rightarrow\ ;\ \Upsilon}}{B\ ;\ \frown A\ \Rightarrow\ ;\ \Upsilon}$$

Similarly, using the fact that

$$\frown\sim C \Rightarrow C \quad \text{is valid in the semantics of } \mathbf{AHL} \tag{4}$$

we show that in **AH-G3** Υ is allowed in the succedent of the sequent premise of $\supset$-*right* and of $\sim$-*right*.

Using the methods of [5] one may prove the following result.

Theorem 1. *The sequent calculi* **PB-G3** [**APB-G3**, **AH-G3**] *without the rules of cut are sound and complete with respect to the interpretation of* **PBL** [**APBL**, **AHL**, *respectively*] *in* bimodal **S4**.

2.5 First conclusions: assertions and conjectures

Although our approach to the logic for pragmatics does not provide a theory of positive evidence, the epistemic reading of the modal interpretation in **S4** does suggest a way to characterize different degrees of evidence, through the essential distinction between *hypotheses* and *conjectures*. While "epistemic possibility", namely, the mere knowledge of a situation in which α happens to be true, does provides enough evidence to justify the hypothesis of α, conjecturing the truth of α requires knowing conditions in which α would be "epistemically necessary". We write $\mathcal{C}\alpha$ to express the *conjecture* that α is true.

Moreover, consider circumstances in which it is unjustified to conjecture the truth of α. This is certainly the case when no matter how our present knowledge evolves, it always reaches a state in which α fails to be true: we may call this epistemic condition *safe expectation* that $\neg\alpha$ eventually becomes true. We write $\mathcal{E}\alpha$ to express the *safe expectation* of α.

Setting $(\mathcal{C}\alpha)^M = \Diamond\Box\alpha$ and $(\mathcal{E}\alpha)^M = \Box\Diamond\alpha$, we have a modal interpretation in **S4** that fits nicely in the above informal interpretation. In Table 2 we find the map of all distinct modalities in **S4**; arrows indicate valid implications between non-equivalent modalities.

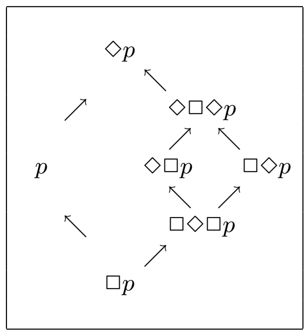

Table 2. The modalities of **S4**

It is clear that the illocutionary operators of *conjecturing* $\mathcal{C}$ and *expecting* $\mathcal{E}$ are not definable *intuitionistically* through the other illocutionary operators and pragmatic connectives of the language $\mathcal{L}^{AH}$. Indeed we have $(\mathcal{C}p)^M = \Diamond\Box p \equiv (\frown \mathcal{H}\neg p)^M$ and $(\mathcal{E}p)^M = \Box\Diamond p \equiv (\sim \vdash \neg p)^M$, as indicated in [5], but these expressions make essential use of *classical negation* "$\neg$" in the radical part. Table 3 presents all pragmatic expressions corresponding to modalities of **S4** and the valid implications between them.

We shall not develop a full theory of assertions, hypotheses, conjectures and expectation with four corresponding types of pragmatic connectives. We are interested in theories obtained by extending the polarized language $\mathcal{L}^{AH}$ of *assertions and hypotheses* with new elementary expressions $\mathcal{C}p$ for *conjectures* and dually, expressions $\mathcal{E}p$ for expectations. Let us write $\mathcal{L}^{AHC}$ [$\mathcal{L}^{AHCE}$] for the extension of $\mathcal{L}^{AH}$ with elementary expressions $\mathcal{C}p$ for conjectures [and $\mathcal{E}p$ for expectations].

Let **AHCE** be the set of all expressions in $\mathcal{L}^{AHCE}$ that are valid in the **S4** modal translation. We conjecture in order to axiomatize **AHCE** in a cut-free sequent calculus it suffices to extend **AH-G3** with the following rules:

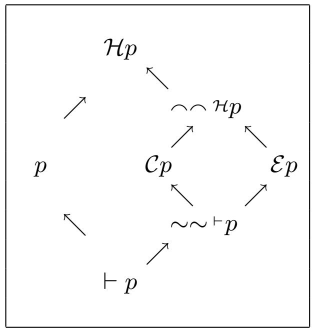

Table 3. Assertions, conjectures, expectations and hypotheses

$$\mathcal{C}\text{-R}\ \frac{\Theta\ ;\ \Rightarrow\ {}^{\vdash}p\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ ;\ \mathcal{C}p, \Upsilon} \qquad \mathcal{C}\text{-L}\ \frac{\Theta, {}^{\vdash}p\ ;\ \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \mathcal{C}p\ \Rightarrow\ ;\ \Upsilon}$$

$$\mathcal{E}\text{-R}\ \frac{\Theta\ ;\ \Rightarrow\ ;\ {}^{\mathcal{H}}p,\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \mathcal{E}p\ ;\ \Upsilon} \qquad \mathcal{E}\text{-L}\ \frac{\Theta\ ;\ {}^{\mathcal{H}}p\ \Rightarrow\ ;\ \Upsilon}{\Theta, \mathcal{E}p\ ;\ \Rightarrow\ ;\ \Upsilon}$$

Duality between safe expectations and conjectures

Clearly the **S4** translations of conjectures and of assertions are not dual from the viewpoint of modal logic, but the modal translations of *conjectures* and *safe expectations* certainly are; if in definition 3 and in Lemma 2 we replace the illocutionary operators "$\mathcal{C}$" (conjectures) and "$\mathcal{E}$" (safe expectations) for the operators "${}^{\mathcal{H}}$" (hypotheses) and "${}^{\vdash}$" (assertions), respectively, then clearly the conditions of duality are expressible through negations within a logic **AHCEL** of *assertions, hypotheses, conjectures and safe expectations*. For instance, the base case becomes:

(a)	setting	$F(\mathcal{E}p) = \mathcal{C}p^{\perp}$	and	$G(\mathcal{C}p) = \mathcal{E}p^{\perp}$
	we have	$\frown \mathcal{E}p \equiv \mathcal{C}p^{\perp}$	and	$\sim \mathcal{C}p \equiv \mathcal{E}p^{\perp}$
	since	$\Diamond\neg\Box\Diamond p \equiv \Diamond\Box\neg p$	and	$\Box\neg\Diamond\Box p \equiv \Box\Diamond\neg p.$

The logic of safe expectations is classical

Let $\mathcal{L}^E$ be the language defined by the grammar

$$E, F \quad := \quad \mathcal{E}p \mid\ \curlyvee\ \mid E \supset F \mid E \cap F$$

and let bf EL be the set of all formulas δ in the language $\mathcal{L}^E$ such that the modal translation δ^M is valid in **S4**.

Proposition 2. *The theory* **EL** *(*logic of safe expectations*) is closed under the double negation rule, i.e.,* $\sim\sim E \Rightarrow E$ *is a valid axiom of* **EL**.

The proof shows by induction on the logical complexity that the double negation rule for molecular formulas can be reduced to applications of the double negation rule for elementary formulas (essentially, as in [46]). The base case is then given by the following equivalence:

$$(\sim\sim \mathcal{E}p)^M \quad = \quad \Box\Diamond\Box\Diamond p \quad \equiv \quad \Box\Diamond p \quad = \quad (\mathcal{E}p)^M.$$

On the other hand, if we extend $\mathcal{L}^E$ with intuitionistic disjunction ($\cup$), then $E\cup \sim E$ is not a theorem of the logic of safe expectations extended in this way. Indeed

$$(\mathcal{E}p\cup \sim \mathcal{E}p)^M = \Box\Diamond p \vee \Box\Diamond\Box\neg p$$

is not valid in **S4**.

Historical Note.

In Appendix B of [46] Prawitz considers an extension of the language of intuitionistic logic with an involutory negation $\neg$ and then extends intuitionistic natural deduction $\mathbf{NJ}^{\supset\cap}$ with rules $\neg\supset$-I, $\neg\supset$-E, $\neg\cap$-I and $\neg\cap$-E; these new rules are presented as an axiomatization of Nelson's logic of *constructible falsity* [38][9]. Thomason [58] provides a Kripke semantics for Nelson's logic of constructible falsity, where $w \Vdash \neg p$ if and only if $w' \Vdash \neg p$ for all w' with wRw'; this implies that the evaluation function must be *partial*. Miglioli, Moscato, Ornaghi and Usberti [37] introduce an operator $\mathbf{T}$ which represents classical truth within the context of Nelson's logic of constructive negation: in particular we have A is classically valid if and only if $\sim\sim A$ is intuitionistically valid (by Gödel's translation) if and only if $\mathbf{T}A$ is valid in the constructive extended system. In [37] a Kripke semantics for the constructive logic with $\mathbf{T}$ operator is presented, where Thomason's semantics is restricted to frames satisfying the additional condition that from each world w a terminal world w' is reached where all atoms and negations of atoms are evaluated. Then the forcing conditions for $\mathbf{T}p$ by Miglioli *et al.* are expressible as $w \Vdash \mathbf{T}p$ if and only if $w \Vdash \Box\Diamond p$ and $w \Vdash \neg\mathbf{T}p$ if and only if in all w' with wRw' we have that p is either not evaluated or false in w'. Comparing the operator $\mathbf{T}$ to our operator $\mathcal{E}$ of *safe expectation*, when applied to atomic formulas, we can say that their properties are similar, but in the context of a *polarized* bi-intuitionistic system they can be expressed in a simpler way. We cannot discuss the intriguing work by Miglioli and his co-workers in more detail here; a recent discussion of their approach is in Pagliani's book [39].

[9] The negation "$\neg$" corresponds to the orthogonality $(\ )^{\perp}$, as in remark 2.

3 PART II. Rough Set Semantics

Proofs and Refutations

The idea that a characterisation of constructive logic must include a definition not only of what proofs of a formula A are but also of *refutations* of A goes back at least to D.Nelson [38] and comes up in various contexts related to game semantics and in particular the construction of Chu spaces. Thus we may say that *a proof* of $A \supset B$ is a method transforming a proof of A into a proof of B and that a *refutation* of $A \supset B$ is a pair consisting of a proof of A together with a refutation of B; in some contexts instead of proofs and refutations we may speak of *evidence for* and *against* A. To study bi-intuitionistic logic and its dualities one may say that a proof of $C \setminus D$ is a pair consisting of a proof of C and of a refutation of D and that a refutation of $C \setminus D$ is a method transforming a proof of C into a proof of D. But we will not go very far if the spaces of proofs and of refutations of A coincide with the spaces of refutations and of proofs of $A^{\perp}$, respectively. This is certainly not the case if we consider the semantics of *assertions, hypotheses and conjectures* rather than that of assertions and hypotheses, as discussed informally in section 2.1. Moreover it turns out that *Rough Set semantics* applied to our canonical polarized system **AHCB** does provide new insight and also a bridge to geometric models [57].

3.1 Rough Sets

As pointed out in [5], any topological space provides a mathematical model of bi-intuitionistic logic, thus also of our canonical system **AHL**, if we interpret the assertive expressions by *open sets* and the hypothetical ones by *closed sets*. A more interesting suggestion comes from the interpretation in terms of Rough Sets, following Piero Pagliani's work (in particular, see [40, 41] and Lech Polkowski's book [45], Chapter 12).

Definition 5. *Given an indiscernibility space* (U, E)*, where* U *is a finite set and* $E \subseteq U \times U$ *an equivalence relation, identifying objects that may be indiscernible from some point of view, let* $\mathbf{AS}(U)$ *be the atomic Boolean algebras having the set of equivalence classes* U/E *as atoms; then* $(U, \mathbf{AS}(U))$ *is a topological space, called the Approximation Space of* (U, E)*, which induces an interior operator and a closure operator* $\mathcal{I}, \mathcal{C} : \wp(U) \to \mathbf{AS}(U)$*. If two subsets* $G', G'' \subseteq U$ *have the same interior and the same closure, then they are roughly equal, i.e., indistinguishable either by the coarsest classification given by* $\mathcal{C}$*, or by the finest classification* $\mathcal{I}$*; thus each subset* G *is a representative of a class of subsets identified by the pair* $(\mathcal{I}(G), \mathcal{C}(G))$*; only a clopen* G *for which* $\mathcal{I}(G) = \mathcal{C}(G)$ *is fully characterised in* (U, E)*.*

For our purpose it is more convenient the *disjoint representation* $(\mathcal{I}(G), -\mathcal{C}(G))$ using the *complement of the closure* of G, the set of object different from G

even for the coarser classification, instead of $\mathcal{C}(G)$. Thus we may regard the two clopen sets ($\mathcal{I}(G)$ and $-\mathcal{C}(G)$) as representing the space of proofs and of refutations of an intuitionisitc formula.

Following Pagliani, we can define the following data and operations on pairs

1 $1 = (U, \emptyset)$, $\quad 0 = (\emptyset, U)$;
2 $(X^+, X^-) \wedge (Y^+, Y^-) = (X^+ \cap Y^+, X^- \cup Y^-)$ (*conjunction*);
3 $(X^+, X^-) \vee (Y^+, Y^-) = (X^+ \cup Y^+, X^- \cap Y^-)$ (*disjunction*);
4 $(X^+, X^-) \rightarrow (Y^+, Y^-) = (-X^+ \cup Y^+, X^+ \cap Y^-)$ (*Nelson's implication*)
5 $\vdash (X^+, X^-) = (-X^+, X^+)$ (*weak negation* or *supplement*);
6 $(X^+, X^-)^\perp = (X^-, X^+)$ (*orthogonality*);
7 $(X^+, X^-) \Rightarrow (Y^+, Y^-) = ((-X^+ \cup Y^+) \cap (-Y^- \cup X^-), -X^- \cap Y^-)$ (*Heyting's implication*);
8 $\dashv (X^+, X^-) = (X^+, X^-) \Rightarrow (\emptyset, U) = (X^-, -X^-)$ (*intuitionistic negation*);
9 $(X^+, X^-) \smallsetminus (Y^+, Y^-) = (X^+ \cap -Y^+, (-X^+ \cup Y^+) \cap (-Y^- \cup X^-)$ (*co-intuitionistic subtraction*).

(see Pagliani[41], Polkowski [45], p. 363 – with an equivalent definition of Heyting implication).[10]

Of course one will not obtain a complete semantics for intuitionistic logic starting from a finite base of clopen sets. Thus we need to look at general topological spaces. Since the language of our logic of assertions, hypotheses and conjectures **AHCL** is *polarized*, in order to turn Pagliani's operations into a topological model of **AHCL** we need to make sure that the interpretation of an assertive expression is an *open set* and a hypothetical expression is assigned a *closed set*; this is not always the case for Pagliani's operations, in particular implications and negations, which have to be modified as follows.

[10] Notational decisions are nightmarish if we try to match the uses in the literature of Rough Sets, in Rauszer's bi-intuitionistic logic and our own.
In our polarized bi-intuitionistic logic [5] we used $\sim A$ for intuitionistic negation and $\frown C$ for co-intuitionistic supplement, leaving $\neg\alpha$ for classical negation, as required in Dalla Pozza and Garola's framework and following the meaning originally given to the symbol "$\neg$" by Frege.
C.Rauszer uses $\dashv A$ for intuitionistic negation and $\vdash A$ for co-intuitionistic supplement; but in later literature on bi-intuitionistic logic $\sim A$ is used for co-intuitionistic supplement.
In the literature of Rough Sets, weak-negation is sometimes written $\neg C$; intuitionistic negation is written in various ways (Pagliani uses $\div A$, in Polkowski's book there is $\dagger A$), while the symbol $\sim A$ is used exactly in the sense of *orthogonality* $A^\perp$.
However it is unnecessary to make notations uniform across three areas, where similar connectives have different meanings: e.g., in Rough Sets negations are defined in a more general algebraic setting than Heyting algebras.
Hence it seems reasonable for us to retain the notation of [5] for our polarized logic, while using "$\dashv$", "$\vdash$" and "$(\ )^\perp$" for intuitionistic negation, co-intuitionistic supplement and orthogonality in Rough Sets.

Definition 6. *Let (U, O) be a topological space, where O is the collection of open sets on U, and $\mathcal{I}(X)$ and $\mathcal{C}(X)$ are the interior and the closure of X*[11]*, respectively. We write (A_o^+, A_c^-) and (C_c^+, C_o^-) for pairs of disjoint sets of the types (open, closed) and (closed, open), respectively. We define the rough set interpretation $(\ )^R$ of the language of assertions, hypotheses and conjectures $\mathcal{L}^{AC}$ (in the disjoint representation) as follows.*

Fix an assignment R : $(\vdash p_i)^R = (A_{i_o}^{\ +}, A_{i_c}^{\ -})$ and $(\mathcal{H} p_i)^R = (C_{i_c}^{\ +}, C_{i_o}^{\ -})$ to the elementary expressions of $\mathcal{L}^{AH}$. Then

1 $\curlyvee^R = (U, \emptyset)$ and $\curlywedge^M = (\emptyset, U)$ (clopen, clopen);
2 $(A \cap B)^R = (A_o^+, A_c^-) \wedge (B_o^+, B_c^-) = (A_o^+ \cap B_o^+, A_c^- \cup B_c^-)$;
3 $(C \curlyvee D)^R = (C_c^+, C_o^-) \vee (D_c^+, D_O^-) = (C_c^+ \cup D_c^+, C_o^- \cap D_o^-)$;
4 $(A_o^+, A_c^-) \to (B_o^+, B_c^-) = \Big(\mathcal{I}(-A_o^+ \cup B_o^+), \mathcal{C}(A_o^+ \cap B_c^-)\Big)$ [12];
5 $(\frown C)^R = \ulcorner (C_c^+, C_o^-) = \Big(\mathcal{C}(-C_c^+), \mathcal{I}(C_c^+)\Big)$ and
$(\frown A)^R = \ulcorner (A_o^+, A_c^-) = (-A_o^+, A_o^+)$;
6 $(A_o^+, A_c^-)^\perp = (A_c^-, A_o^+)$ and $(C_c^+, C_o^-)^\perp = (C_o^-, C_c^+)$[13];
7 $(A \supset B)^R = (A_o^+, A_c^-) \Rightarrow (B_o^+, B_c^-) =$
$= \Big(\mathcal{I}(-A_o^+ \cup B_o^+) \cap \mathcal{I}(-B_c^- \cup A_c^-), \mathcal{C}(-A_c^- \cap B_c^-)\Big)$;
8 $(\sim A)^R = \urcorner (A_o^+, A_c^-) = \Big(\mathcal{I}(A_c^-), \mathcal{C}(-A_c^-)\Big)$ and
$(\sim C)^R = \urcorner (C_c^+, C_o^-) = (C_o^-, -C_o^-)$;
9 $(C \smallsetminus D)^R = (C_c^+, C_o^-) \smallsetminus (D_c^+, D_c^-) =$
$= \Big(\mathcal{C}(C_c^+ \cap -D_c^+), \mathcal{I}(-C_c^+ \cup D_c^-) \cap \mathcal{I}(-D_o^- \cup C_o^-)\Big)$.

Let $\mathcal{L}^{AHC}$ a language of assertions, hypotheses and conjectures built from a set of propositional atoms p_0, p_1, ... and let $(\ )^\perp$ be an involution without fixed points on the atoms. A rough set interpretation *$\mathcal{M} = (U, O, R)$ of the language $\mathcal{L}^{AHC}$ (with an involution $(\ )^\perp$ on the atoms) is a topological space (U, O) together with an assignment R to the elementary expressions of disjoint pairs of the following forms:*

$$(\vdash p)^R = (A_o^+, A_c^-);$$
$$(\mathcal{H} p)^R = (C_c^+, C_o^-);$$
$$(\mathcal{C} p_i)^R = \big(\mathcal{C}(X^+), \mathcal{I}(X^-)\big), \qquad \text{where } (\vdash p_i)^R = (X^+, X^-).$$

Lemma 3. *Let $\mathcal{M} = (U, O, R)$ be an interpretation of $\mathcal{L}^{AHC}$, with an involution $(\ )^\perp$ on the atoms. Then $\mathcal{M}$ is a model of* **AHCL** *if and only if the assignment R to elementary expressions of $\mathcal{L}^{AHC}$ satisfies the following duality conditions:*

---

[11] The notation $\mathcal{C}X$ is overloaded, for the illocutionary operator of conjecture in the syntax of the language of pragmatics and for the closure operator in a topological space. No confusion is possible, given the difference of context.

[12] There is no connective to represent Nelson's implication as distinct from intuitionistic implication in $\mathcal{L}^{AH}$.

[13] There is no specific connective for orthogonality in $\mathcal{L}^{AH}$.

$$
\begin{array}{ll}
(\vdash p^{\perp})^R = (C_o^-, C_c^+) = (C_c^+, C_o^-)^{\perp} & \textit{where } (\mathcal{H}p)^R = (C_c^+, C_o^-) \\
(\mathcal{H}p^{\perp})^R = (A_c^-, A_o^+) = (A_o^+, A_c^-)^{\perp} & \textit{where } (\vdash p)^R = (A_o^+, A_c^-)
\end{array}
$$

and moreover for every (A_o^+, A_c^-) *and* (C_c^+, C_o^-) *in* R *we have*

$$A_c^- = -A_o^+ \qquad \textit{and} \qquad C_o^- = -C_c^+. \tag{5}$$

Proof. Concerning conditions $\sim\frown A \equiv A$ and $\frown\sim C \equiv C$ of Lemma 2, notice that

$$\dashv\vdash\ (A_o^+, A_c^-) = \dashv (-A_o^+, A_o^+) = (A_o^+, -A_o^+) = (A_o^+, A_c^-)$$

if and only if $A_c^- = -A_o^+$ and similarly

$$\vdash\ \dashv (C_c^+, C_o^-) = \vdash (C_o^-, -C_o^-) = (-C_o^-, C_o^-) = (C_c^+, C_o^-)$$

where the last equality holds if and only if $C_c^+ = -C_o^-$. Moreover the conditions $(b) - (c)$ in the definition of duality between $\mathcal{L}^A$ and $\mathcal{L}^H$ (definition 3) are clearly satisfied by the standard Rough Set definition. As for condition (a), given the involution $(\ )^{\perp}$ on the atoms, we have

$$(\frown \vdash p)^R = \vdash (A_o^+, A_c^-) = (-A_o^+, A_o^+) = (A_c^-, A_o^+) = (\mathcal{H}p^{\perp})^R$$

where the third equality holds by condition (5) and the fourth by the condition of duality in a model. Similarly

$$(\sim \mathcal{H}p)^R = \dashv (C_c^+, C_o^-) = (C_o^-, -C_o^-) = (C_o^-, C_c^+) = (\vdash p^{\perp})^R$$

as required.

Remark 3. In a model $\mathcal{M} = (U, O, R)$ for **AHCL** *intuitionistic negation* and *Nelson's negation* coincide:

$$
\begin{aligned}
(A_o^+, A_c^-) \supset (B_o^+, B_c^-) &= (\mathcal{I}(-A_o^+ \cup B_o^+) \cap \mathcal{I}(-B_c^- \cup A_c^-), \mathcal{C}(-A_c^- \cap B_c^-)) \\
&= (\mathcal{I}(A_c^- \cup B_o^+), \mathcal{C}(A_o^+ \cap B_c^-))
\end{aligned}
$$

Thus to exploit Rough Set semantics in full, we may want to consider notions of duality where condition (5) does not hold.

3.2 Algebra of Regions

A main reason of interest in bi-intuitionistic logic are its topos-theoretic models studied by F. W. Lawvere [32] G. Reyes and H. Zolfaghari [52], recently reconsidered by J. Stell and M. Worboys [57] in their "algebra of regions". It is clearly impossible here to compare Reyes and Zolfaghari's modal logic to our polarized bi-intuitionistic systems, but we must say something about Stell and Worboys' geometric examples.

The first one is Reyes and Zolfaghari's motivating example [52]: it provides a model of bi-intuitionistic logic based on the subgraphs of arbitrary *undirected* graphs. It ought to be possible to define graphic models of **AHL** and **PBL**, but we shall not attempt this here. On the other hand, "two stages sets" in the second example are just a geometric representation of the basic notion of "rough equality": in an approximation space each subset G of the universe is identified only by the pair $(\mathcal{I}(G), \mathcal{C}(G))$ – or with $(\mathcal{I}(G), -\mathcal{C}(G))$ in the disjoint representation – where the interior and closure operator result from two stages of process of classification.

Now it is evident that condition (5) on models of **AHL** restricts the interpretation to sets G that are fully characterised in (U, E), i.e., such that $\mathcal{I}(G) = \mathcal{C}(G)$. We illustrate more interesting semantics applications with an example. Consider the Kripke model K for **S4** obtained from the reflexive and transitive closure of the graph in Figure 1.

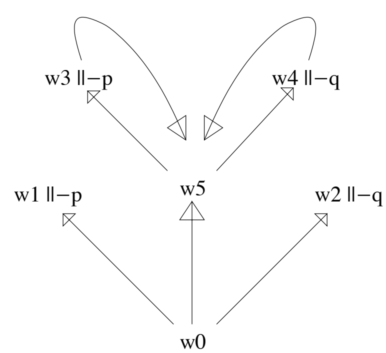

Fig. 1. "Kripke model".

Writing α_K for the set of possible worlds satisfying α, we have $(\vdash p)_K = \{w_1\}$, $(\mathcal{C}p)_K = \{w_0, w_1\}$ and $(\mathcal{H} p)_K = \{w_0, w_1, w_3, w_4, w_5\} = K \setminus \{w_2\}$. We are satisfied with the Rough Set interpretation of *assertions* in the disjoint representation as $(\vdash p)^R = (\{w_1\}, K \setminus \{w_1\})$: after all, the grounds for an assertion ought to be a "stable" state of knowledge; by duality the representation of *hypotheses* as $(\mathcal{H}p)^R = (K \setminus \{w_2\}, \{w_2\})$ is appropriate. On the other hand, the state of knowledge justifying *conjectures* is "unstable"; thus there seems to be a meaningful "two-stage set" representation of *conjectures* of the form $(\mathcal{C}p)^R = (\{w_1\}, K \setminus \{w_0, w_1\})$, of type (*open, open*). We notice that such an interpretation is possible for the logic **AHCL** of assertions, conjectures and hypotheses, as it does not interfere with the basic symmetry between assertions and hypotheses.

4 PART III. Proof theory

We shall start with the definition of a sequent-style *single-assumption multiple-conclusions natural deduction* system for the subtractive-disjunctive fragment

$\mathbf{NJ}^{\smallsetminus\curlyvee}$ of co-intuitionistic logic. We have sequents of the form

$$A \vdash C_1, \ldots, C_m$$

where A indicates the only open assumption in a derivation with the multiset C_1, ..., C_m of open conclusions. The rules of inference are in Table 4.

assumption

$$A \vdash A$$

$\curlyvee_0$-I

$$\frac{E \vdash \Gamma, C_0}{E \vdash \Gamma, C_0 \curlyvee C_1}$$

$\curlyvee_1$-I

$$\frac{E \vdash \Gamma, C_0}{E \vdash \Gamma, C_0 \curlyvee C_1}$$

$\curlyvee$-E

$$\frac{E \vdash \Gamma, C_0 \curlyvee C_1 \qquad C_0 \vdash \Gamma_0 \qquad C_1 \vdash \Gamma_1}{E \vdash \Gamma, \Gamma_0, \Gamma_1}$$

$\smallsetminus$-I

$$\frac{E \vdash \Gamma, C \qquad D \vdash \Delta}{E \vdash \Gamma, C \smallsetminus D, \Delta}$$

$\smallsetminus$-E

$$\frac{E \vdash \Gamma, C \smallsetminus D \qquad C \vdash (D)^k, \Delta}{E \vdash \Gamma, \Delta}$$

$(D)^k$ is a multiset with k occurrences of D.

Table 4. Natural Deduction $\mathbf{NJ}^{\smallsetminus\curlyvee}$

Definition 7. *We say that $C_1, \ldots, C_m$ is derivable from A if there is a natural deduction derivation of the sequent $A \vdash \Gamma$ where all formulas in the multiset Γ are among $C_1, \ldots, C_m$.*

Remark 4. (i) Looking at the deduction rules in Table 4, notice that $\smallsetminus$-*introductions*, $\curlyvee$-*eliminations* and $\smallsetminus$-*eliminations* discharge the open assumption(s) of the sequent-premise(s) to the right, but a $\smallsetminus$-*elimination* discharges also a *multiset of open conclusions*. As a consequence, $\smallsetminus$-*eliminations* are the only inferences that cannot be permuted freely with other inferences. From another point of view, here we have a limit to the "parallelization of the syntax", a *box* in the sense of Girard. To remove such a box, a device is needed to discharge open conclusions preserving as much as possible the geometry of proofs. In this section we recover *Prawitz trees* as an appropriate representation of proofs in $\mathbf{NJ}^{\smallsetminus\curlyvee}$.

(ii) As in Prawitz's natural deduction *weakening* is not explicitly represented in proof-trees and *contraction* appears only in the discharging of conclusions in a $\smallsetminus$-E inference.

Definition 8. (i) (active and passive formula-occurrences) *In assumptions and in rules of inference the indicated formula-occurrences in the succedent of a sequent are* active *and all occurrences in the multisets Γ, Γ_i, Δ are* passive. *Also the discharged assumptions in a $\smallsetminus$-I, $\curlyvee$-E and $\smallsetminus$-E are* active*, all other* *assumptions are* passive. *An active formula in the sequent-conclusion of an inference is also called the* conclusion *of the inference.*

(ii) (segments) *If Υ occurs in the premise and in the conclusion of an inference then an occurrence $D_i \in \Upsilon$ in the premise is the* immediate ancestor *of the occurrence D_i in the conclusion. Then as in Prawitz* [46] *we define a* segment *as a sequence $D_1, \ldots, D_m$ of occurrences of the same formula where D_1 and D_m are active, and D_i is the immediate ancestor of D_{i+1}, for $i < m$.*

(iii) *Thus we may speak of a* segment *as the conclusion or the premise of some inference.*

(iv) *A* maximal segment *is the conclusion of an introduction rule which is premise of an elimination. A derivation is* normal *if it does not have maximal segments.*

4.1 Structure of normal proofs

The structure of normal deductions in co-intuitionistic logic $\mathbf{NJ}^{\smallsetminus,\curlyvee}$ mirrors that of normal deductions in intuitionistic logic $\mathbf{NJ}^{\supset,\cap}$.

Definition 9. (i) *A* Prawitz path *in a normal deduction is a sequence C_1, …, C_i, …, C_n of segments such that*

- *C_1 is an assumption, either open or discharged by a $\smallsetminus$-introduction;*
- *for j with $1 \leq j < i$, $C_j = C \smallsetminus D$ is a premise of a $\curlyvee$- or $\smallsetminus$-elimination and $C_{j+1} = C$ is an assumption discharged by the inference;*
- *for j with $i \leq j < n$, C_j is a premise of a $\curlyvee$- or $\smallsetminus$-introduction with conclusion C_{j+1};*
- *C_n is a conclusion of the derivation, either open or discharged by a $\smallsetminus$-E.*

(ii) *The collection of all Prawitz paths in a derivation is a graph, called the* tree of Prawitz paths *τ. If we collapse segments to their formulas, the resulting tree yields a graphical representation of proofs which we shall call* Prawitz tree *for $\mathbf{NJ}^{\smallsetminus\curlyvee}$. Such trees are similar to those in Prawitz-style Natural Deduction derivation for $\mathbf{NJ}^{\supset\cap}$, but in $\mathbf{NJ}^{\smallsetminus\curlyvee}$ the logical flow goes from the root to the leaves, rather than from the leaves to the root as in $\mathbf{NJ}^{\supset\cap}$.*

(iii) *The definition of the* depth *of a path π in a tree τ is familiar: the depth of π is 0 if its first formula C_1 is open; the depth of π is $n+1$ if C_1 is discharged by a $\smallsetminus$-introduction with conclusion in a path of depth n.*

From this analysis we derive as usual the *subformula property* for normal deductions:

Proposition 3. *Every formula occurring in a normal deduction of $A \vdash C_1, \ldots, C_m$ is a subformula either of A or of C_i for some i.*

Example. We constuct a derivation d in $\mathbf{NJ}^{\smallsetminus\curlyvee}$ of

$$C \smallsetminus A \vdash ((C \smallsetminus (B \curlyvee D)) \smallsetminus A, ((B \smallsetminus A) \curlyvee (D \smallsetminus A))^2$$

It may be helpful to think of the dual derivation in $\mathbf{NJ}^{\supset\cap}$ of

$$(A \supset B) \cap (A \supset D), A \supset ((B \cap D) \supset C) \vdash A \supset C.$$

We write $\mathbf{F}$ for $(B \smallsetminus A) \curlyvee (D \smallsetminus A)$ and $\mathbf{G}$ for $C \smallsetminus (B \curlyvee D)$.

$$\dfrac{\dfrac{\dfrac{C \vdash C \qquad B \curlyvee D \vdash B \curlyvee D}{C \vdash C \smallsetminus (B \curlyvee D), B \curlyvee D}\ \smallsetminus\text{-I} \qquad B \vdash B \qquad D \vdash D}{C \vdash C \smallsetminus (B \curlyvee D), B, D}\ \curlyvee\text{-E} \qquad A \vdash A}{C \vdash \mathbf{G} \smallsetminus A, A, B, D}\ \smallsetminus\text{-I}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots \qquad A \vdash A}{C \vdash \mathbf{G} \smallsetminus A, (A)^2, B \smallsetminus A, D}\ \smallsetminus\text{-I} \qquad A \vdash A}{C \vdash \mathbf{G} \smallsetminus A, (A)^3, B \smallsetminus A, D \smallsetminus A}\ \smallsetminus\text{-I} \qquad A \vdash A}{C \vdash \mathbf{G} \smallsetminus A, (A)^3, (B \smallsetminus A) \curlyvee (D \smallsetminus A), D \smallsetminus A}\ \curlyvee_0\text{-I}}{C \smallsetminus A \vdash C \smallsetminus A \qquad C \vdash (A)^3, \mathbf{G} \smallsetminus A, (\mathbf{F})^2}\ \curlyvee_1\text{-I}}{C \smallsetminus A \vdash \mathbf{G} \smallsetminus A, (\mathbf{F})^2}\ \smallsetminus\text{-E}$$

In Fig. 2 we find the tree-structure of "Prawitz' paths" of the derivation d.

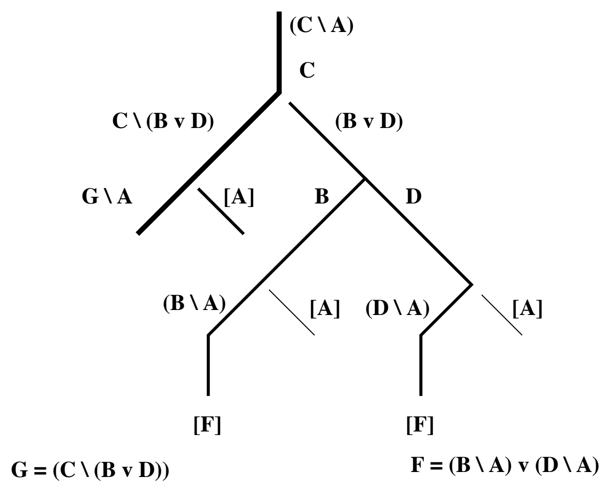

Fig. 2. A Prawitz tree.

4.2 Sequents with tail formula

A very perspicuous representations of derivations in co-intuitionistic logic is through *sequent calculus with tail formula* q-$\mathbf{LJ}^{\smallsetminus\curlyvee}$, the exact dual of the well-known sequent calculus with head formula t-$\mathbf{LJ}^{\supset\cap}$[14]. Here sequents have the form

$$E \Rightarrow \Upsilon \,; C$$

with one formula in the antecedent, a multiset in the *ordinary area* and at most one formula in the *linear area* (*stoup*) of the succedent. The principal formulas of the *right*-rules are in the stoup and *left*-rules require empty stoup in the sequent-premies[15]. The rules of q-$\mathbf{LJ}^{\smallsetminus\curlyvee}$ are given in Table 5.
The following fact is the dual of a well-known correspondence between Natural Deduction derivations in $\mathbf{NJ}^{\supset\cap}$ and Sequent Calculus derivations in t-$\mathbf{LJ}^{\supset\cap}$. For sequent calculi with head formulas or tail formulas see, for instance, [20].

Proposition 4. *There is a bijection between trees of Prawitz paths of normal derivations in* $\mathbf{NJ}^{\smallsetminus\curlyvee}$ *and cut-free derivations in q-*$\mathbf{LJ}^{\smallsetminus\curlyvee}$ *(modulo the order of structural inferences).*

Proof. Given a Prawitz tree τ, by induction on τ we construct a q-$\mathbf{LJ}^{\smallsetminus\curlyvee}$ derivation with the property that the formula in the stoup (*tail formula*), if any, is the conclusion of a path of depth 0 (*main path*) of τ. If τ begins with an elimination rule, the result is immediate by the inductive hypothesis applied to the immediate subtree(s) from the top, since we may assume that the corresponding cut-free derivations have conclusions with empty stoup. If τ begins with an introduction rule, then there is only one main path and we remove the last inference of it: if the concusion was a formula $C \smallsetminus D$, the inductive hypothesis yields two q-$\mathbf{LJ}^{\smallsetminus\curlyvee}$ derivations; in one the endsequent must have C in the stoup, since C belongs to the main path; in the other the endsequent has D in the antecedent and we may assume that it has no formula in the stoup, by applying dereliction if necessary. Therefore we may apply $\smallsetminus$-R to obtain the desired derivation. The other cases are obvious.
The fact that two derivations d' and d'' corresponding to the same tree τ can only differ for the order of structural inferences is due to the fact that in q-$\mathbf{LJ}^{\smallsetminus\curlyvee}$ logical inferences cannot be permuted with each other. Indeed, the principal formulas of all inferences occur either in the antecedent or in the stoup, and the rule of dereliction is irreversible.

[14] The "q" in q-$\mathbf{LJ}^{\smallsetminus\curlyvee}$ stands for *queue*, tail, as the "t" in t-$\mathbf{LJ}^{\supset\cap}$ stands for *tête*, head.

[15] It ought to be clear that the use of *focalization* in the sequent calculus q-$\mathbf{LJ}^{\smallsetminus\curlyvee}$ and in the dual t-$\mathbf{LJ}^{\supset\cap}$ (see Table 7), Appendix III), is unrelated to the use of the "stoup" in our sequent calculi **AH-G3**, **PB-G3** and **APB-G3** for bi-intuitionistic logic, where it is used simply to highlight the restrictions of intuitionistic systems.

Sequent Calculus q-LJ$^{\smallsetminus\curlyvee}$

identity rules

logical axiom:

$$C \;\Rightarrow\;\; ;\; C$$

tail-cut:

$$\dfrac{E \;\Rightarrow\; \Upsilon_1 \;;\; C \qquad C \;\Rightarrow\; \Upsilon_2 \;;\; D}{E \;\Rightarrow\; \Upsilon_1, \Upsilon_2 \;;\; D}$$

central-cut:

$$\dfrac{E \;\Rightarrow\; \Upsilon_1, C \;; \qquad C \;\Rightarrow\; \Upsilon_2 ;}{E \;\Rightarrow\; \Upsilon_1, \Upsilon_2 \;;}$$

structural rules

contraction:

$$\dfrac{E \;\Rightarrow\; \Upsilon, C, C \;;\; D}{E \;\Rightarrow\; \Upsilon, C \;;\; D}$$

weakening:

$$\dfrac{E \;\Rightarrow\; \Upsilon \;;\; D}{E \;\Rightarrow\; \Upsilon, C \;;\; D}$$

dereliction:

$$\dfrac{E \;\Rightarrow\; \Upsilon \;;\; D}{E \;\Rightarrow\; \Upsilon, D \;;}$$

logical rules

$\smallsetminus$ *right:*

$$\dfrac{E \;\Rightarrow\; \Upsilon_1 \;; C \qquad D \;\Rightarrow\; \Upsilon_2 \;;}{E \;\Rightarrow\; \Upsilon_1 \;;\; C \smallsetminus D}$$

$\smallsetminus$ *left*:

$$\dfrac{C \;\Rightarrow\; \Upsilon, D \;;}{C \smallsetminus D \;\Rightarrow \Upsilon \;;}$$

right $\curlyvee_i$*:*

$$\dfrac{E \;\Rightarrow\; \Upsilon \;;\; C_i}{E \Rightarrow\; \Upsilon \;;\; C_0 \curlyvee C_1}$$

left $\curlyvee$*:*

$$\dfrac{C_0 \;\Rightarrow\; \Upsilon; \qquad C_1 \;\Rightarrow\; \Upsilon;}{C_0 \curlyvee C_1 \;\Rightarrow\; \Upsilon;}$$

Table 5. The sequent calculus q-**LJ**$^{\smallsetminus\curlyvee}$

Example. (cont.) The following sequent derivation d_q corresponds to the natural deduction derivation d:

$$\dfrac{\dfrac{\dfrac{C \Rightarrow;\; C \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow;\; B \quad \dfrac{A \Rightarrow;\; A}{A \Rightarrow A\;;}\,der}{B \Rightarrow A\;;\; B \smallsetminus A}\,\smallsetminus\text{-R}}{B \Rightarrow A\;;\; \mathbf{F}}\,\curlyvee_0\text{-R}}{B \Rightarrow A, \mathbf{F}\;;}\,der \qquad \dfrac{\dfrac{\dfrac{D \Rightarrow;\; D \quad \dfrac{A \Rightarrow;\; A}{A \Rightarrow A\;;}\,der}{D \Rightarrow A\;;\; D \smallsetminus A}\,\smallsetminus\text{-R}}{D \Rightarrow A\;;\; \mathbf{F}}\,\curlyvee_1\text{-R}}{D \Rightarrow A, \mathbf{F}\;;}\,der}{B \curlyvee D \Rightarrow A, A, \mathbf{F}, \mathbf{F}\;;}}{C \Rightarrow A, A, \mathbf{F}, \mathbf{F}\;;\; C \smallsetminus (B \curlyvee D)}\,\smallsetminus\text{-R} \qquad \dfrac{A \Rightarrow\;;\; A}{A \Rightarrow A\;;}\,der}{\dfrac{\dfrac{C \Rightarrow A, A, A, \mathbf{F}, \mathbf{F}\;;\; \mathbf{G} \smallsetminus A}{C \Rightarrow A, \mathbf{F}, \mathbf{G} \smallsetminus A\;;}\,contr,\; der}{C \smallsetminus A \Rightarrow \mathbf{F}, \mathbf{G} \smallsetminus A\;;}\,\smallsetminus\text{-L}}$$

5 PART IV. Term assignment for co-intuitionistic logic

In a tantalising pair of papers [42, 44] Michel Parigot introduced *Free Deduction*, a formalism consisting of *elimination rules only*, with the property that both Natural Deduction and the Sequent Calculus could be represented in it simply by restricting the order of deduction, e.g., by permutations of inferences. Free Deduction was conceived to study the computational properties of classical logic, but it can be adapted to intuitionistic and co-intuitionistic logic through the analogue of Gentzen's restrictions on sequents.

For instance, although they do not appear in this form in [42], the rules for *multiplicative implication and subtraction* can be formulated as follows:

multiplicative implication

$$\frac{\Gamma, \mathbf{A} \to \mathbf{B} \ \vdash \ \Delta \qquad \Pi, \mathbf{A} \vdash \mathbf{B}, (\Sigma^{\P})}{\Gamma, \Pi \vdash \Delta, \Sigma} \to \textit{elim left}$$

$$\frac{\Gamma \vdash \Delta, \mathbf{A} \to \mathbf{B} \qquad \Pi \ \vdash \Sigma, \mathbf{A} \qquad \Pi', \mathbf{B} \ \vdash \Sigma'}{\Gamma, \Pi, \Pi' \vdash \Delta, \Sigma, \Sigma'} \to \textit{elim right}$$

multiplicative subtraction

$$\frac{\Gamma, \mathbf{A} \smallsetminus \mathbf{B} \vdash \Delta \qquad \Pi \vdash \Sigma, \mathbf{A} \qquad \Pi', \mathbf{B} \ \vdash \ \Sigma'}{\Gamma, \Pi, \Pi' \vdash \Delta, \Sigma, \Sigma'} \smallsetminus \textit{elim left}$$

$$\frac{\Gamma \vdash \Delta, \mathbf{A} \smallsetminus \mathbf{B} \qquad (\Pi^{\P}), \mathbf{A} \vdash \mathbf{B}, \Sigma}{\Gamma, \Pi \vdash \Delta, \Sigma} \smallsetminus \textit{elim right}$$

The intuitionistic restriction $(\Sigma^{\P})$, namely, that Σ is empty, applies to the secondary premise of the $\to$-*left elimination* rule, and the dual restriction holds for $\smallsetminus$-*right elimination*. The sequent calculus rules are obtained by *killing the main premise* (i.e., keeping it only as an axiom). Here are the rules for subtraction:

subtraction rules, as in the sequent calculus

$$\frac{\mathbf{A} \smallsetminus \mathbf{B} \vdash A \smallsetminus B \qquad \Pi \vdash \Sigma, \mathbf{A} \qquad \Pi', \mathbf{B} \vdash \Sigma'}{\Pi, \Pi' \vdash A \smallsetminus B, \Sigma, \Sigma'} \smallsetminus\text{-R}$$

$$\frac{A \smallsetminus B \ \vdash \mathbf{A} \smallsetminus \mathbf{B} \qquad [\Pi^{\P}], \mathbf{A} \vdash \Sigma, \mathbf{B}}{A \smallsetminus B, \Pi \vdash \Sigma} \smallsetminus\text{-L}$$

Natural Deduction, on the other hand, is given by *keeping all inputs on the left*. Namely: for left elimination rules, *we kill the main premise*; for right elimination rules, *we kill the secondary premises which have only a left active formula*. Thus no premise is killed in *subtraction elimination right*.

subtraction rules, as in natural deduction

$$\frac{\mathbf{A} \smallsetminus \mathbf{B} \vdash A \smallsetminus B \qquad \Pi \ \vdash \ \mathbf{A}, \Sigma \qquad \Pi', \mathbf{B} \vdash \Sigma'}{\Pi, \Pi' \vdash A \smallsetminus B, \Sigma'} \smallsetminus \textit{intro}$$

$$\frac{\Gamma \vdash \mathbf{A} \smallsetminus \mathbf{B}, \Delta \qquad [\Pi^{\P}], \mathbf{A} \vdash \mathbf{B}, \Sigma}{\Gamma, \Pi \vdash \Delta, \Sigma} \smallsetminus \textit{elim}$$

Since Free Deduction yields a *multiple conclusion* natural deduction system in a very straightforward way, one would expect that a term assignment to Free Deduction might be *distributed* to all formula in the succedent of sequents. On the contrary in 1992 Michel Parigot introduced the λ-μ calculus as "an algorithmic interpretation of classical Natural Deduction", which is based on a notion of "central control". In the last part of this paper we propose a distributed term assignment to co-intuitionistic logic.

5.1 Term assignment to the subtraction rules in the λ-μ calculus

Recently the proof theory of bi-intuitionistic (*subtractive*) logic has been studied by T. Crolard [15, 16]: in [16] a Natural Deduction system is presented with a *calculus of coroutines* as term assignment. [16] Crolard works in the framework of Parigot's $\lambda\mu$-calculus: sequents may be written in the form[17] $\Gamma \vdash t : A \mid \Delta$, with contexts $\Gamma = x_1 : C_1, \ldots, x_m : C_m$ and $\Delta = \alpha_1 : D_1, \ldots, \alpha_n : D_n$, where the x_i are *variables* and the α_j are *μ-variables*. In addition to the rules of the simply typed lambda calculus, there are *naming rules*

$$\frac{\Gamma \vdash t : A \mid \alpha : A, \Delta}{\Gamma \vdash [\alpha]t : \bot \mid \alpha : A, \Delta}[\alpha] \qquad\qquad \frac{\Gamma \vdash t : \bot \mid \alpha : A, \Delta}{\Gamma \vdash \mu\alpha.t : A \mid \Delta}\mu$$

It is well-know that the $\lambda\mu$-calculus provides a computational interpretation of classical logic and a typing system for functional programs with continuations (see, e.g., [17, 54]).

Crolard extends the $\lambda\mu$ calculus with introduction and elimination rules for subtraction:[18]

$$\frac{\Gamma \vdash t : A \mid \Delta}{\Gamma \vdash \texttt{make-coroutine}(t, \beta) : A \smallsetminus B \mid \beta : B, \Delta} \smallsetminus I$$

$$\frac{\Gamma \vdash t : A \smallsetminus B \mid \Delta \qquad \Gamma,\ x : A \vdash u : B \mid \Delta}{\Gamma \vdash \texttt{resume}\ t\ \texttt{with}\ x \mapsto u : C \mid \Delta} \smallsetminus E$$

The reduction of a redex of the form $\texttt{resume}(\texttt{make-coroutine}(t, \beta))\ \texttt{with}\ x \mapsto u : C$ yields $\mu\gamma.[\beta]u[t/x]$, where the μ-variables are typed as $\beta : B$ and $\gamma : C$. Namely,

[16] This part is joint work with Corrado Biasi and incorporates important contributions from his still unfinished doctoral dissertation at Queen Mary, University of London.

[17] Parigot and Crolard actually write sequents in the form $t : \Gamma \vdash \Delta; A$, where the term t is given the type of the formula A in the stoup, if such a formula exists. If the stoup is empty, the notation allows one to think of t as being assigned to the entire sequent or to a formula $\bot$ implicitly present in the stoup.

[18] In Crolard [16] the introduction rule corresponds to the more general form $\smallsetminus$-I given above, and more general *continuation contexts* occur in place of β; the above formulation suffices for our purpose here.

$$\dfrac{\dfrac{\Gamma \vdash t : A \mid \Delta}{\Gamma \vdash \texttt{make-coroutine}(t, \beta) : A \setminus B \mid \beta : B, \Delta}\ \setminus\text{-I} \qquad \Gamma, x : A \vdash u : B \mid \Delta}{\Gamma \vdash \texttt{resume}\ (\texttt{make-coroutine}(t, \beta))\ \texttt{with}\ x \mapsto u\ \ : C \mid \beta : B, \Delta}\ \setminus\text{-E}$$

reduces to

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash t : A \mid \Delta \qquad \Gamma, x : A \vdash u : B \mid \Delta}{\Gamma \vdash u[t/x] : B \mid \Delta}\ \textit{substitution}}{\Gamma \vdash [\beta]u[t/x] : \bot \mid \beta : B, \gamma : C, \Delta'}\ [\beta]}{\Gamma \vdash \mu\gamma.[\beta]u[t/x] : C \mid \beta : B, \Delta'}\ \mu$$

Working with the full power of classical logic, if a constructive system of bi-intuitionistic logic is required, then the implication right and subtraction left rules must be restricted by considering *relevant dependencies*.[19] Crolard is able to show that the term assignment for such a restricted logic is a calculus of *safe coroutines*, namely, terms in which no coroutine can access the local environment of another coroutine.

5.2 A distributed term assignment for the subtractive fragment

When we consider a term assignment for the Natural Deduction system $\mathbf{NJ}^{\setminus\curlyvee}$ of dual intuitionistic logic only, we are led to ask what Crolard's calculus becomes when separated from its $\lambda\mu$ context. Indeed the naming rules of the $\lambda\mu$ calculus allow us to represent the action of an operating system jumping from one thread of computation to another: when a name β for a coroutine has been created by `make-coroutine`, it can be later accessed by the system and the coroutine executed.

On the contrary in our proposal different terms are simultaneously assigned to the multiple conclusions of a sequent in a sequent-style Natural Deduction, (or in the Sequent Calculus with tail formula). There is no mechanism to simulate the passage of control from one "thread" to another. A process is stopped by the operator assigned to subtraction elimination (called here `postpone` rather than Crolard's `resume`) and becomes active only in the normalization process. Thus in presence of different processes running in parallel, one wonders whether our system can still be regarded as a calculus of *coroutines*: it is perhaps closer to an abstract representation of a *multiprocessing system.*

Before giving formal definitions, let us survey the most distinctive features of our calculus for the terms assignment to the *subtractive* fragment only. Most characteristic is the treatment of *variables*: there is no operator for *explicitly binding* variables or *delimiting the scope* of an implicitly binding operation. We may say that a *computational context* is characterized by exactly one free

[19] For instance, in the derivation of the right premise of a subtraction elimination ($\setminus E$), there should be no relevant dependency between the formula B and the assumptions in Γ, but only between B and A.

variable and that a free variable a becomes bound when its computational context $\mathcal{S}_a$ is plunged into the computational context $\mathcal{S}_b$ associated with another variable b. In this case, the variable a is replaced everywhere by $\mathtt{a}(t)$ for some term t containing b; here the function $\mathtt{a}$ is vaguely reminiscent of a *Herbrand function*. In the normalization process the term $\mathtt{a}(t)$ may later be replaced by another term u throughout the new computational context; thus we assume that a mechanism is in place for *broadcasting* substitutions throughout an environment.

We have the following operators:

- the term $\mathtt{mkc}(t, \mathtt{y})$, which is assigned to the conclusion of a $\smallsetminus$-*introduction*, connects two disjoint computational contexts, say, $\mathcal{S}_x$ and $\mathcal{S}_y$. Every term in $\mathcal{S}_x$ contains exactly one free variable x, and we assume that the term t represents a *thread* starting from x[20]. The computational context $\mathcal{S}_y$ contains the free variable y and all threads starting from y. When the term $\mathtt{mkc}(t, \mathtt{y})$ is introduced, the substitution $y := \mathtt{y}(t)$ must be performed throughout the environment $\mathcal{S}_y$. Thus the term $\mathtt{mkc}(t, \mathtt{y})$ represents a *jump* extending the thread t to all threads in $\mathcal{S}_y\{y := \mathtt{y}(t)\}$; the substitution of $\mathtt{y}(t)$ for y throughout $\mathcal{S}_y$ has the effect that the extended computational context contains only the free variable x. Here we retain Crolard's name `make-coroutine` for historic reasons; a more precise but more redundant description would be the following:

  $$\mathtt{mkc}(t, \mathtt{y}) \qquad \text{stands for} \qquad \mathtt{extend\ thread}\ t\ \mathtt{from}\ \mathtt{y}(t).$$

- The term $\mathtt{postp}(\mathtt{z} \mapsto \ell, t)$, which is assigned to the conclusion of a $\smallsetminus$-*elimination*, takes a computational context $\mathcal{S}_z$ containing the only free variable z, and plunges it into another context $\mathcal{S}_x$ where the only free variable is x; this is done by selecting the list ℓ of threads starting from z and the term t with free variable x, replacing z with $\mathtt{z}(t)$ throughout $\mathcal{S}_z$ and *freezing* $\ell\{z := \mathtt{z}(t)\}$ until through normalization the term t is transformed to a term of the form `extend thread`. A fuller description is therefore the following:

  $$\mathtt{postp}(\mathtt{z} \mapsto \ell, w)\ \text{stands for}\ \mathtt{postpone\ subthreads}\ \ell\{z := \mathtt{z}(w)\}\ \mathtt{until}\ w.$$

Let $\mathbf{M}$ be $\mathtt{mkc}(t, \mathtt{y})$ and let $\mathbf{P}(v)$ be $\mathtt{postp}(\mathtt{z} \mapsto \ell, v)$. Then

$$\mathbf{P}(\mathbf{M}) \ = \ \mathtt{postp}(\mathtt{z} \mapsto \ell, \mathtt{mkc}(t, \mathtt{y}))$$

is a *redex*. The main idea of a reduction is to replace the jump from t to $\mathtt{y}(t)$ with each one of the subthreads in ℓ. But such an operation has important side effects. A redex $\mathbf{P}(\mathbf{M})$ occurs in a *computational context* $\mathcal{S}_x$ of the form

[20] Here we use the term "thread" in the sense of Prawitz [46], p.25; namely, a *thread* is a branch in the proof-tree from the a leaf to the root. The equivalent notion here is that of a branch in *Prawitz' tree* τ from the root to the leaf. No claim is made here about the computer science usage of the term "thread".

$$\mathcal{S}_x : \qquad \texttt{postp}(\texttt{z} \mapsto \ell, \texttt{mkc}(t, \texttt{y})), \quad \overline{\kappa}, \quad \overline{\zeta}_{\texttt{y}}, \quad \overline{\xi}_{\texttt{z}}$$

where $\overline{\zeta}_{\texttt{y}}$ is a sequence of terms containing $\texttt{y}(t)$, $\overline{\xi}_{\texttt{z}}$ a sequence of terms containing $\texttt{z}(\texttt{mkc}(t, \texttt{y}))$ and $\overline{\kappa}$ a sequence containing neither $\texttt{y}(t)$ nor $\texttt{z}(\texttt{mkc}(t, \texttt{y}))$. Thus the *side effects* consist in the replacement of $\texttt{z}(\texttt{mkc}(t, \texttt{y}))$ with t in $\overline{\xi}_{\texttt{z}}$ and in each subthread s_k of ℓ; let $\ell' = s'_1, \ldots, s'_n$ be the resulting sequence. Finally, we replace $\texttt{y}(t)$ in $\overline{\zeta}_{\texttt{y}}$ with each one of the subthreads s'_k, thus *expanding* the sequence $\overline{\zeta}_{\texttt{y}}$ in a sense to be made precise below. To indicate such a rewriting process we shall use the notation

$$\mathcal{S}' = \mathcal{S}_x - \mathbf{P}(\mathbf{M}) \quad \{z := t\} \quad \{y := \ell\{z := t\}\}$$

where $z = \texttt{z}(\texttt{mkc}(t, \texttt{y}))$ and $y = \texttt{y}(t)$.
In an enterprise where notation is in danger of growing out of control, readability is essential. The notations $\texttt{mkc}(t, \texttt{y})$ and $\texttt{postp}(\texttt{z} \mapsto \ell, w)$ are already effective abbreviations, as from them we can recover the terms $\texttt{y}(t)$ and $\texttt{z}(w)$ present in the context. Further simplification is given by Corrado Biasi's elegant notations:

$$t \to \texttt{y} \quad \text{for} \quad \texttt{mkc}(t, \texttt{y}) \quad \text{and} \quad \xleftarrow{\texttt{z} \mapsto \ell} t \quad \text{for} \quad \texttt{postp}(\texttt{z} \mapsto \ell, t).$$

If we consider the typed version of the above rewriting we have[21]:

$$\dfrac{\dfrac{x : E \vdash \pi_0 : \bullet \mid \overline{\kappa} : \Delta, t : C \qquad y : D \vdash \pi_1 : \bullet \mid \overline{\zeta} : \Upsilon}{x : E \vdash \pi_0, \pi'_1 : \bullet \mid \overline{\kappa} : \Delta, \overline{\zeta}' : \Upsilon, (t \to \texttt{y}) : C \smallsetminus D}\smallsetminus\text{-I} \qquad z : C \vdash \pi_2 : \bullet \mid \overline{\xi} : \Xi, \ell : D}{x : E \vdash \pi_0, \pi'_1, \pi'_2, \ \xleftarrow{\texttt{z} \mapsto \ell} (t \to \texttt{y}) \ : \bullet \mid \overline{\kappa} : \Delta, \ \overline{\zeta}' : \Upsilon, \ \overline{\xi}' : \Xi}\smallsetminus\text{-E}$$

$$\text{where } \pi'_1 = \pi_1\{y := \texttt{y}(t)\}, \ \overline{\zeta}' = \overline{\zeta}\{y := \texttt{y}(t)\},$$
$$\pi'_2 = \pi_2\{z := \texttt{z}((t \to \texttt{y}))\}, \ \overline{\xi}' = \overline{\xi}\{z := \texttt{z}((t \to \texttt{y}))\}$$

reduces to

$$\dfrac{\dfrac{x : E \vdash \pi_0 : \bullet \mid \overline{\kappa} : \Delta, t : C \qquad z : C \vdash \pi_2 : \bullet \mid \overline{\xi} : \Xi, \ell : D}{x : E \vdash \pi_0, \pi''_2 : \bullet \mid \overline{\kappa} : \Delta, \overline{\xi}'' : \Xi, \ell\{z := t\} : D}\,sub \qquad y : D \vdash \pi_1 : \bullet \mid \overline{\zeta} : \Upsilon}{x : E \vdash \pi_0, \pi''_1, \pi''_2 \ \bullet \ \mid \overline{\kappa} : \Delta, \overline{\zeta}'' : \Upsilon, \overline{\xi}'' : \Xi}\,sub$$

$$\text{where } \pi''_2 = \pi_2\{z := t\}, \ \overline{\xi}'' = \overline{\xi}\{z := t\}$$
$$\pi''_1 = \pi_1\{y := \ell\{z := t\}\}, \ \overline{\zeta}'' = \overline{\zeta}\{y := \ell\{z := t\}\}$$

[21] The expression $\bullet$ is not a formula, but a non-logical expression, which cannot be part of other formulas; its meaning could be though of as an absurdity.

6 A distributed term assignment for co-intuitionistic logic $\mathbf{NJ}^{\prec\curlyvee}$

We present the grammar and the basic definitions of our distributed calculus for the fragment of co-intuitionistic logic with subtraction and disjunction.

Definition 10. *We are given a countable set of* free variables *(denoted by x, y, z ...), and a countable set of* unary functions *(denoted by* $\mathtt{x}, \mathtt{y}, \mathtt{z}, \ldots$*).*

(i) *Terms and lists of terms are defined by the following grammar:*

$$\begin{aligned} t &:= x \mid \mathtt{x}(t) \mid \mathtt{inl}(t) \mid \mathtt{inr}(t) \mid \mathtt{casel}(t) \mid \mathtt{caser}(t) \mid \mathtt{mkc}(t, \mathtt{x}) \\ \ell &:= () \mid t \cdot \ell \end{aligned}$$

(ii) *Let t_1, t_2, ... an enumeration in a given order of all the* terms *freely generated by the above grammar starting with a special symbol $*$ and no variables (a selected variable a would also do the job). Thus we have a fixed* bijection $t_i \mapsto x_i$ *between terms and free variables.*

(iii) *Moreover, if t is a term and ℓ is a list such that for each term $u \in \ell$, y occurs in u, then* $\mathtt{postp}(\mathtt{y} \mapsto \ell\{y := \mathtt{y}(t)\}, t)$ *is a p-term.*

We use the abbreviations $(t \to \mathtt{y})$ *for* $\mathtt{mkc}(t, \mathtt{y})$ *and* $\xleftarrow{\mathtt{z}\mapsto\ell} w$ *for* $\mathtt{postp}(\mathtt{z} \mapsto \ell, w)$.

Thus a p-term cannot be a subterm of other terms. In the official definition above lists appear only as arguments of postp[22] It is notationally convenient to extend the above definition so that our operators apply to *lists* in addition to terms:

Definition 11. *Let* $\mathtt{op}(\ )$ *be one of* $\mathtt{x}(\ )$, $\mathtt{inl}(\ )$, $\mathtt{inr}(\ )$, $\mathtt{casel}(\ )$, $\mathtt{caser}(\ )$, $\mathtt{mkc}((\ ), \mathtt{x})$, $\mathtt{postp}(\mathtt{x} \mapsto \ell, (\ ))$.
Then the term expansion $\mathtt{op}(\ell)$ *is the list of terms defined inductively thus:*

$$\mathtt{op}(()) = () \qquad\qquad \mathtt{op}\ (t \cdot \ell) = \mathtt{op}\ (t) \cdot \mathtt{op}\ (\ell)$$

Remark 5. By term expansion, a term consisting of an operator applied to a list of terms is turned into a list of terms; thus terms may always be transformed into an expanded form where operators are applied only to terms, *except for expressions ℓ* occurring in terms of the form $\mathtt{postp}(\mathtt{y} \mapsto \ell, u)$.

Definition 12. (i) *The* free variables $FV(\ell)$ *in a list of terms ℓ are defined as follows:*

[22] In our definition we use lists of terms where multisets are intended. A multiset can be represented as a list $\ell = (t_1, \ldots, t_n)$ with the action of the group of permutations $\sigma : n \to n$ given by $\ell_\sigma = (t_{\sigma(1)}, \ldots, t_{\sigma(n)})$.

$$\begin{aligned} FV(()) &= \emptyset \\ FV(t\cdot\ell) &= FV(t)\cup FV(\ell) \\ FV(x) &= \{x\} \\ FV(\mathtt{x}(t) &= FV(t) \\ FV(\mathtt{inl}(t)) = FV(\mathtt{inr}(t)) &= FV(t) \\ FV(\mathtt{casel}(t)) = FV(\mathtt{caser}(t)) &= FV(t) \\ FV(\mathtt{mkc}\ (t,\mathtt{x}) &= FV(t) \\ FV(\mathtt{postp}(\mathtt{x}\mapsto\ell,t) &= FV(\ell)\cup FV(t). \end{aligned}$$

(ii) *A* computational context $\mathcal{S}_x$ *is a set of terms and p-terms containing the free variable x and no other free variable.*

Definition 13. *Substitution of a term t for a free variable x in a term u is defined as follows:*

$$\begin{array}{rlrl} x\{x:=t\} &= t, & y\{x:=t\} &= y \text{ if } x\neq y; \\ \mathtt{y}(u)\{x:=t\} &= \mathtt{y}(u\{x:=t\}); & & \\ \mathtt{inl}(r)\{x:=t\} &= \mathtt{inl}(r\{x:=t\}), & \mathtt{inr}(r)\{x:=t\} &= \mathtt{inr}(r\{x:=t\}); \\ \mathtt{casel}(r)\{x:=t\} &= \mathtt{casel}(r\{x:=t\}), & \mathtt{caser}(r)\{x:=t\} &= \mathtt{caser}(r\{x:=t\}); \\ \mathtt{mkc}(r,\mathtt{y})\{x:=t\} &= \mathtt{mkc}(r\{x:=t\},\mathtt{y}), & & \\ \mathtt{postp}(\mathtt{y}\mapsto(\ell),s)\{x:=t\} &= \mathtt{postp}(\mathtt{y}\mapsto(\ell\{x:=t\}),s\{x:=t\}). & & \end{array}$$

We define substitution of a list of terms ℓ for a variable x in a list of terms κ:

$$\begin{array}{ll} ()\{x:=\ell\}=() & (t\cdot\kappa)\{x:=\ell\}=t\{x:=\ell\}\cdot\kappa\{x:=\ell\} \\ t\{x:=()\}=() & t\{x:=u\cdot\ell\}=t\{x:=u\}\cdot t\{x:=\ell\} \end{array}$$

If $\overline{\zeta}$ is a vector of lists $\ell_1,\ldots,\ell_m$, then $\overline{\zeta}\{x:=\ell\}=\ell_1\{x:=\ell\},\ldots,\ell_m\{x:=\ell\}$.

Definition 14. *β-reduction of a* redex $\mathcal{R}ed$ *in a* computational context $\mathcal{S}_x$ *is defined as follows.*
(i) *If $\mathcal{R}ed$ is a term u of the following form, then the reduction is local and consists of the rewriting $u \rightsquigarrow_\beta u'$ in $\mathcal{S}_x$ as follows:*

$$\begin{array}{ll} \mathtt{casel}\ (\mathtt{inl}(t)) \rightsquigarrow_\beta\ t; & \mathtt{caser}\ (\mathtt{inr}(t)) \rightsquigarrow_\beta\ t. \\ \mathtt{casel}\ (\mathtt{inr}(t)) \rightsquigarrow_\beta (); & \mathtt{caser}\ (\mathtt{inl}(t)) \rightsquigarrow_\beta (); \end{array}$$

(ii) *If $\mathcal{R}ed$ has the form* $\xleftarrow{\mathtt{z}\mapsto\ell}(t\to\mathtt{y})$ *, i.e.,* $\mathtt{postp}(\mathtt{z}\mapsto\ell,\mathtt{mkc}(t,\mathtt{y}))$*, then $\mathcal{S}_x$ has the form*

$$\mathcal{S}_x \;=\; \mathcal{R}ed,\quad \overline{\kappa},\quad \overline{\zeta}_{\mathtt{y}},\quad \overline{\xi}_{\mathtt{z}}$$

where $\mathtt{y}(t)$ occurs in $\overline{\zeta}_{\mathtt{y}}$ and $\mathtt{z}((t\to\mathtt{y}))$ occurs in $\overline{\xi}_{\mathtt{z}}$ and neither $\mathtt{y}(t)$ nor $\mathtt{z}((t\to\mathtt{y}))$ occurs in $\overline{\kappa}$. Writing $y=\mathtt{y}(t)$ and $z=\mathtt{z}((t\to\mathtt{y}))$, a reduction *of $\mathcal{R}ed$ transforms the computational context as follows:*

$$\mathcal{S}_x \quad\rightsquigarrow\quad \overline{\kappa},\quad \overline{\zeta}\{y:=\ell\{z:=t\}\},\quad \overline{\xi}\{z:=t\}.$$

Thus for $\overline{\zeta}=u_1,\ldots,u_k$, for $\overline{\xi}=r_1,\ldots,r_m$ and for $\ell=s_1,\ldots,s_n$ we have:

$$\begin{array}{rcl}\overline{\xi}\{z := t\} & = & r_1\{z := t\}, \ldots, r_m\{z := t\};\\ \overline{\zeta}\{y := \ell\{z := t\}\} & = & u_1\{y := s_1\{z := t\}\}, \ldots, u_1\{y := s_n\{z := t\}\}, \ldots\\ & \ldots & u_k\{y := s_1\{z := t\}\}, \ldots, u_k\{y := s_n\{z := t\}\};\\ & = & \overline{\zeta}\{y := s_1\{z := t\}\}, \ldots, \overline{\zeta}\{y := s_n\{z := t\}\}.\end{array}$$

Given the correspondence between Prawitz style Natural Deduction derivations in $\mathbf{NJ}^{\supset\cap}$ and sequent derivations in t-$\mathbf{LJ}^{\supset\cap}$, and the dual correspondence between Prawitz trees for $\mathbf{co}-\mathbf{NJ}^{\smallsetminus\curlyvee}$ and sequent derivations in q-$\mathbf{LJ}^{\smallsetminus\curlyvee}$, we find it convenient to define the term assignment directly to sequent calculus in q-$\mathbf{LJ}^{\smallsetminus\curlyvee}$, given in Appendix III, Table 6.

Definition 15. (**term assignment**) *The assignment of terms of the distributed calculus to sequent calculus derivation in q-$\mathbf{LJ}^{\smallsetminus\curlyvee}$ is given in Appendix III, Table 6. In Table 7 we give the familiar assignment of λ-terms to sequent calculus with head formulas t-$\mathbf{LJ}^{\supset\cap}$.*

Remark on free variables and α conversion. Since in our calculus the binding of a free variable x is expressed through its substitution with a term $\mathtt{x}(t)$, the so-called "capture of free variables" takes a different form. Suppose a free variable y has been replaced by $\mathtt{y}(t)$ in the construction of a term $M = \mathtt{mkc}(t, \mathtt{y})$ or $P(t) = \mathtt{postp}(\mathtt{y}, \ell\{y := \mathtt{y}(t)\}, t)$: all other occurrences of y in the previous context have been replaced with $\mathtt{y}(t)$ in the current context, represented, say, by a vector $\overline{\ell}$, and we may say that M or $P(t)$ is a *binder* of $\mathtt{y}(t)$ in $\overline{\ell}$.

In the process of normalization such a "bound" term $\mathtt{y}(t)$ may be replaced by another term u. It would be natural to think of such a replacement as a two-steps process, first recovering the free variable y and then applying a substitution $\{y := u\}$ to the current computational context. However, it may also happen that in the process of normalization different occurrences of the term $\mathtt{y}(t)$ evolve to $\mathtt{y}(t')$ and to $\mathtt{y}(t'')$ so that distinct variables y' and y'' are needed for distinct substitutions. For this reason we have established a bijection between freely generated terms and free variables.

This may not solve all problems: indeed in the *untyped* formulation of our calculus it might happen that the same free variable y has been replaced with $\mathtt{y}(t)$ in the construction of *two distinct terms* of $\overline{\ell}$: our syntax may not have tools to disambiguate the "scope" of the bindings and some further restrictions may be needed to block such pathologies. However, if the calculus is used for assigning term to derivations in $\mathbf{NJ}^{\smallsetminus\curlyvee}$, then to avoid "capture of free variables" it is enough to set the following condition.

Convention. We assume that

- Derivations have the *pure parameter property*, i.e., that in a derivation free variables assigned to distinct open assumptions are distinct;

Since to distinct free variables x, y there correspond distinct unary functions $\mathtt{x}$, $\mathtt{y}$, then it is clear that in the term assignment to derivations with the *pure* *parameter property* the above indicated ambiguity cannot occur. Moreover, a derivation resulting by normalization from a derivation with the pure parameter property can be transformed again into a derivation with the pure parameter property. Indeed, the set of terms assigned to a $\mathbf{NJ}^{\setminus \curlyvee}$ derivation encode a tree-structure, and it is easy to see that if different occurrences of the term $\mathtt{y}(t)$ evolve to $\mathtt{y}(t')$ and to $\mathtt{y}(t'')$ *in a tree*, then the terms t' and t'' are distinct as they encode distinct threads. Thus once again apply the bijection between terms and free variables can be used to produce a derivation with the pure parameter property.

Examples. (i) Assigning terms to the derivation d_q in section 4.2 we obtain the following assignment to the endsequent:

$$z : C \setminus A \Rightarrow \xleftarrow{\mathtt{c} \mapsto (a', a'', a''')}, z \;: \bullet \mid (t', t'') : \mathbf{F}, ((\mathtt{c}(z) \to \mathtt{e}) \to \mathtt{a}''') : \mathbf{G} \setminus A$$

where we have
$a' = \mathtt{a}_1(\mathtt{casel}(\mathtt{e}(\mathtt{c}(z))))$, $a'' = \mathtt{a}_2(\mathtt{caser}(\mathtt{e}(\mathtt{c}(z))))$, $a''' = \mathtt{a}_3((\mathtt{c}(z) \to \mathtt{e})) : A$;
$t' = \mathtt{inl}((\mathtt{casel}(\mathtt{e}(\mathtt{c}(z))) \to \mathtt{a}_1))$, $t'' = \mathtt{inl}((\mathtt{caser}(\mathtt{e}(\mathtt{c}(z))) \to \mathtt{a}_2)) : \mathbf{F}$,
$\mathbf{F} = (B \setminus A) \curlyvee (D \setminus A)$, $\mathbf{G} = (C \setminus (B \curlyvee D))$.

(ii) Applying cut-elimination to the derivation

$$\dfrac{\dfrac{\dfrac{a : A \Rightarrow\; ;\; a : A}{a : A \Rightarrow :\; \mathtt{inl}(a) : A \curlyvee B}}{\vdots} \qquad \dfrac{a' : A \Rightarrow\; ;\; a' : A \qquad \dfrac{b : B \Rightarrow\; ;\; b : B \qquad c : C \Rightarrow\; c : C\; ;}{b : B \Rightarrow\; \mathtt{c}(b) : C\; ;\; (b \to \mathtt{c}) : B \setminus C}}{e : A \curlyvee B \Rightarrow\; \mathtt{casel}(e) : A, \mathtt{c}(\mathtt{caser}(e)) : C, (\mathtt{caser}(e) \to \mathtt{c}) : B \setminus C\; ;}}{a : A \Rightarrow\; t_1 : A, t_2 : C, t_3 : B \setminus C\; ;}$$

we obtain the following rewritings: $t_1 = \mathtt{casel}(\mathtt{inl}(a)) \leadsto a$;

$$t_2 = \mathtt{c}(\mathtt{caser}(\mathtt{inl}(a)))) \leadsto (), \qquad t_3 = (\mathtt{caser}(\mathtt{inl}(a)) \to \mathtt{c}) \leadsto ()$$

and the term assignment

$$a : A \Rightarrow\; a : A, () : C, () : B \setminus C\; ;$$

6.1 Duality between the distributed calculus and the simply typed λ calculus

Consider the term assignment in Appendix III, Tables 6 and 7. In this setting the following facts are clear:

- given a sequent S in q-$\mathbf{LJ}^{\setminus \curlyvee}$, there is a dual sequent $S^{\perp}$ in h-$\mathbf{LJ}^{\supset \cap}$, and conversely;
- given a derivation d of S in q-$\mathbf{LJ}^{\setminus \curlyvee}$, there is a dual derivation $d^{\perp}$ of $S^{\perp}$ in h-$\mathbf{LJ}^{\supset \cap}$, and conversely.

Therefore any cut-elimination procedure in h-$\mathbf{LJ}^{\supset\cap}$ induces a cut-elimination procedure for q-$\mathbf{LJ}^{\smallsetminus\curlyvee}$; clearly the steps of such reduction procedure for q-$\mathbf{LJ}^{\smallsetminus\curlyvee}$ must be seen as "macro" instructions for several steps of rewriting, which may nevertheless be seen as a unit. Thus we have the following fact:

Theorem 2. *There is a correspondence between reduction sequences starting from a derivation d of S in q-$\mathbf{LJ}^{\smallsetminus\curlyvee}$ and reduction sequences from a derivation $d^{\perp}$ of $S^{\perp}$, and conversely.*

In this setting this result seems obvious and its proof straightforward. Going through the details of the construction, as done in [8], does give an insight into the structure of terminating computations in our distributed calculus. Assigning terms to derivations in q-$\mathbf{LJ}^{\smallsetminus\curlyvee}$ as in in Appendix III, Tables 6 makes the structure of the calculus more clear and provides a bridge to the representation of computations in the graphical notation of *Prawitz trees* as in Appendix II.

7 Conclusions

In this paper we have given an account of research in the logic for pragmatics of *assertions* and *conjectures*, following the paper Bellin and Biasi [5] and also of work in the proof-theory of co-intuitionistic logic aiming at defining natural deduction system and a distributed term-assignment for it.

A conceptual clarification of the distinction between *hypotheses* and *conjectures* with respect to their interpretation in epistemic **S4**, where hypotheses are justified by mere *epistemic possibility* of the truth of their propositional content and conjectures require *possible necessity*, has shown connections with other areas of logic and semantics. On one hand, within our framework we can make distinctions which may be relevant to work on *standards of evidence* in the theory of argumentation [24, 13]. On the other hand, the semantics of *rough sets* and the notion of an *approximation space* provide another semantics to a theory of assertions, hypotheses, conjectures and expectations, in addition to Kripke models through the translation in epistemic **S4** and in *bi-modal* **S4**, as in [5]. Rough sets point at promising connections with research by P. Pagliani [40, 41].

Abstract relations between functional programming and concurrent programming have been studied extensively, e.g., through translations of the λ calculus into R. Milner's π-calculus. Abstract forms of the continuation-passing style, e.g., as in Thielecke's work, have been typed in classical logic, suggesting an interpretation of these relations as a logical duality between classical and intuitionistic logic. In this way, the $\lambda\mu$ calculus is naturally invoked here. In [8] and this paper we propose the duality between intuitionistic and co-intuitionistic logic as the most basic type theoretic setting for studying the relations between distributed and functional programming calculi. Our calculus distributed displays exactly the programming features that are required in order to implement such a logical duality. In this way this paper and other still unpublished work by Corrado Biasi give a type-theoretic framework for studying the relations between *safe* and *unsafe coroutines* in the sense of Crolard: typically, safe coroutines are those which can be represented *as constructs of a distributed calculus* without making essential use of the $\lambda\mu$ calculus *and* can be typed in co-intuitionistic logic. Thus the term assignment to proofs in co-intuitionistic logic can be seen as a contribution to a challenging problem, namely, providing a logical foundation to distributed calculi by means of a typing system, in the Curry-Howard approach. Clearly solving such a problem has a clear interest in computer science, if only to ensure properties of such systems such as termination and confluence.

All the paths followed in this research are open and point at possible directions of work, as already suggested. Other projects could explore the proof-net representation of co-intuitionistic logic and the construction of a term model for co-Cartesian Closed Categories. The proof theory of classical logic is the framework of Crolard's investigations [15, 16] and the concern of Bellin, Hyland, Robinson and Urban [9]: it is expected that eventually research in bi-intuitionistic logic may improve our understanding of classical logic. But this is now a good point to take a rest.

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8 APPENDIX I. Polarized Rauszer's logic

The main stream of bi-intuitionistic logic follows the tradition of Cecylia Rauszer, who created the theory of *bi-Heyting algebras* [50, 51], and defined its Kripke semantics, later studied with categorical methods by Lawvere [32], Makkai, Reyes and Zolfaghari [33, 52]; more recently, proof theoretic treatments of *subtractive* or *bi-intuitionistic logic* have been given by Rajeev Gore [25], Tristan Crolard [14, 15] and others.

In Rauszer's possible-world semantics the forcing conditions for *implication* refer to *up-sets* of possible worlds with respect to the accessibility relation, while the forcing conditions for subtraction refer to *down-sets*. Namely, $(A \supset B)^M = \Box(A^M \to B^M)$ is true in a world w if for all w' such that wRw' $A^M \to B^M$ is true in w'; on the other hand, $(C \smallsetminus D)^M = \diamondminus(C^M \wedge \neg D^M)$ is true in a world w if for some w' such that $w'Rw$ we have $C^M \wedge \neg D^M$ is true in w'; in other words, modal translations are interpreted in models $\mathcal{M} = (W, R, S, \Vdash)$ where R and S are pre-orders such that $S = R^{-1}$. This suggests a *temporal dimension* in the bi-modal translation: the forcing condition for the operator $\Box$ may be seen as referring to "future knowledge" and those for $\diamondminus$ to "past knowledge".

We see at once that Rauzer's bi-intuitionistic logic is as inadequate for a representation of assertions and hypotheses as **PBL**: letting $\vdash p = \Box p$ and $\mathcal{H}\neg p = \diamondminus \neg p$, it is consistent to assert p (with respect to "the future") and also to conjecture $\neg p$ (in the past). Although the issue is beyond the range of the present paper, it may be interesting to catch a glimpse of what *polarized* Rauszer's logic looks like in our framework.

Tense-sensitive polarization

Is there a *pragmatic interpretation* of dual intuitionistic logic which retains such a temporal element and is thus closer to Rauszer's tradition? The question does make sense. Clearly the justification conditions for assertions and conjectures concerning the future and the past are different in several important ways: for instance, direct observations of some future events will be possible, but never of past events. Thus it would be plausible to introduce *tense-sensitive* illocutionary operators, giving *assertive* force to statements about the *future* ($\vdash^{\bullet} \alpha$) and about the *past* (${}^{\bullet}\!\vdash \alpha$) and, similarly, *conjectural* force to statements about the *future* ($\mathcal{H}^{\bullet}\alpha$) and about the *past* (${}^{\bullet}\mathcal{H}\alpha$). Moreover, we would have *strong negation* about the future ($\sim^{\bullet}$) and about the past (${}^{\bullet}\!\sim$) and *weak negation* about the future ($\frown^{\bullet}$) and about the past (${}^{\bullet}\!\frown$). More generally, all pragmatic formulas would become tense-sensitive and could polarized in four ways,

$(A^{\bullet})$ *future-assertive* ,		$({}^{\bullet}A)$ *past-assertive*,
$(C^{\bullet})$ *future-hypothetical*	and	$({}^{\bullet}C)$ *past-hypothetical*:

We define a language $\mathcal{L}_t^{PB}$ according to the grammar

$$\begin{array}{rcl} A^{\bullet} & := & \vdash^{\bullet} p \mid A^{\bullet} \supset B^{\bullet} \mid A^{\bullet} \cap B^{\bullet} \mid \sim^{\bullet} X \\ C^{\bullet} & := & \mathcal{H}^{\bullet} p \mid C^{\bullet} \smallsetminus D^{\bullet} \mid C^{\bullet} \curlyvee D^{\bullet} \mid \frown^{\bullet} X \\ {}^{\bullet}A & := & {}^{\bullet}\!\vdash p \mid {}^{\bullet}A \supset {}^{\bullet}B \mid {}^{\bullet}A \cap {}^{\bullet}B \mid {}^{\bullet}\!\sim X \\ {}^{\bullet}C & := & {}^{\bullet}\!\frown p \mid {}^{\bullet}C \smallsetminus {}^{\bullet}D \mid {}^{\bullet}C \curlyvee {}^{\bullet}D \mid {}^{\bullet}\!\frown X \end{array}$$

This would lead to the development of a *tense-sensitive polarized bi-intuitionistic logic* ($\mathbf{PB}_t$). The "semantic reflection" of $\mathcal{L}_t^{PB}$ is in *temporal* **S4**, where formulas of $\mathcal{L}_{\Box,\boxminus}$ are interpreted in bimodal frames $\mathcal{F} = (W, R, S)$ with R a preorder and $S = R^{-1}$.

The following fact is standard (see, e.g., Ryan and Shobbens [53]):

Proposition 5. *Given a bimodal frame $\mathcal{F}$, the following are equivalent:*

1. $S = R^{-1}$;
2. $\alpha \to \Box \diamondsuit \alpha$ *and* $\diamondsuit \Box \alpha \to \alpha$ *are valid in every Kripke model over* $\mathcal{F}$;
3. *the following rule is valid and semantically invertible in* $\mathcal{F}$

$$\frac{\diamondsuit \alpha \to \beta}{\alpha \to \Box \beta}$$

Therefore the following are equivalent:

1. *the modal interpretation* $(\ )^M$ *of the language* $\mathcal{L}_{\vartheta\upsilon}$ *is in temporal S4;*
2. *for any polarized formula* δ, $\delta \Rightarrow \sim^\bullet {}^\bullet\!\sim \delta$ *and* ${}^\bullet\!\frown\frown^\bullet \delta \Rightarrow \delta$ *are valid axioms of* $\mathbf{PBL_t}$.

One could then define a sequent calculus for $\mathbf{PBL}_t$ with rules of the form

$$\supset^\bullet\text{-R}\ \frac{{}^\bullet\Theta_1, \Theta_2^\bullet, A_1^\bullet \ \Rightarrow A_2^\bullet, \Upsilon_1^\bullet, {}^\bullet\Upsilon_2}{{}^\bullet\Theta_1, \Theta_2^\bullet \ \Rightarrow \ A_1^\bullet \supset A_2^\bullet,\ \Upsilon_1^\bullet, {}^\bullet\Upsilon_2} \quad {}^\bullet\supset\text{-R}\ \frac{\Theta_1^\bullet, {}^\bullet\Theta_2, {}^\bullet A_1 \ \Rightarrow \ {}^\bullet A_2, {}^\bullet\Upsilon_1, \Upsilon_2^\bullet}{\Theta_1^\bullet, {}^\bullet\Theta_2 \ \Rightarrow \ {}^\bullet A_1 \supset {}^\bullet A_2, {}^\bullet\Upsilon_1, \Upsilon_2^\bullet}$$

and similarly for $\smallsetminus^\bullet$-*left* and ${}^\bullet\smallsetminus$-*left* and for negations. Notice that the rule $\supset^\bullet$-*right* is derived from a rule of the form

$$\supset^\bullet\text{-R'}\ \frac{\Theta^\bullet,\ A_1^\bullet \ \Rightarrow \ A_2^\bullet,\ \Upsilon^\bullet}{\Theta^\bullet,\ \Rightarrow \ A_1^\bullet \supset A_2^\bullet,\ \Upsilon^\bullet}$$

using $\sim^\bullet$-*left* and $\frown^\bullet$-*right* and *cut* with axioms of the form

$${}^\bullet A \Rightarrow \sim^\bullet {}^\bullet\!\sim {}^\bullet A \quad \text{and} \quad \frown^\bullet {}^\bullet\!\frown {}^\bullet C \Rightarrow {}^\bullet C$$

9 APPENDIX 2. Example of computation.

In this section we consider an example of computation that is dual to a familiar reduction sequence for Church's numerals.

Two times zero

We consider the dual of a computation of the term representing $\mathbf{2} \times \mathbf{0}$:

$$(\lambda m.\lambda n.\lambda f.m(nf))(\lambda g.\lambda x.g(gx))(\lambda h.\lambda z.z) : (A \supset A) \supset (A \supset A)$$

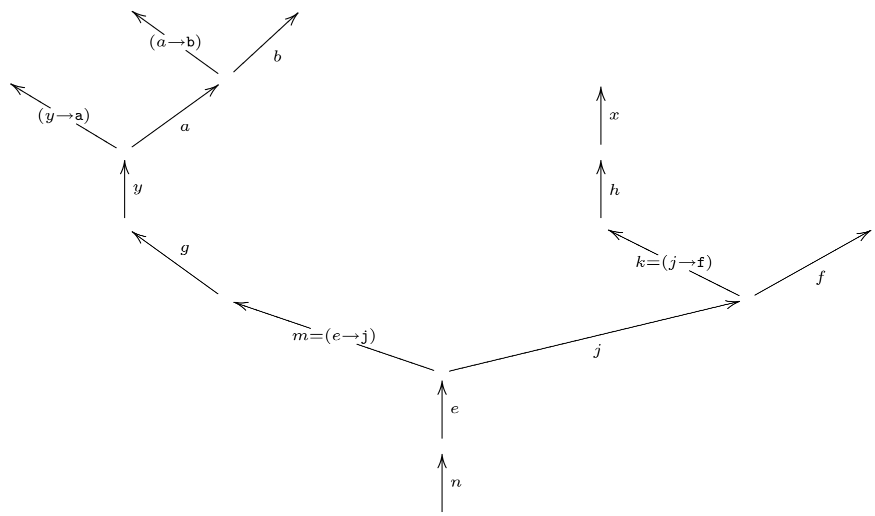
We follow a *call by value* strategy:

$$\begin{aligned}
\lambda f.(\lambda g.\lambda x.g(gx))((\lambda h.\lambda z.z)f) &\rightsquigarrow \lambda f.(\lambda g.\lambda x.g(gx))(\lambda z.z) && (i)\\
&\rightsquigarrow \lambda f.\lambda x.(\lambda z.z)((\lambda z'.z')x))) && (ii)\\
&\rightsquigarrow \lambda f.\lambda x.((\lambda z.z)x)) && (iii)\\
&\rightsquigarrow \lambda f.\lambda x.x && (iv)
\end{aligned}$$

9.1 Labelled Prawitz' trees

As trees in Prawitz style Natural Deduction $\mathbf{NJ}^{\supset}$ can be decorated with λ terms, so we can assign terms of our dual calculus to Prawitz trees of subformulas for $\mathbf{co}-\mathbf{NJ}^{\smallsetminus}$ derivations. For convenience, we still draw trees with the root at the bottom, keeping in mind that here derivations are built from bottom up. We shall use Biasi's notation $(t \to \mathtt{a})$ for $\mathtt{mkc}(t, a)$ and $\xrightarrow{e \mapsto \ell} t$ for $\mathbf{postp}(e \mapsto \ell, t)$.

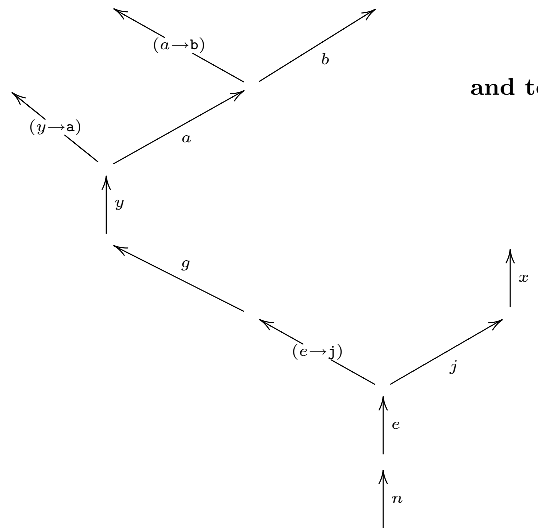
$$\mathcal{S}_0 : \xleftarrow{\mathtt{y}\mapsto(b)} g, \quad \xleftarrow[\mathcal{R}ed_1]{\mathtt{g}\mapsto((y\to\mathtt{a}),(a\to\mathtt{b}))} (e \to \mathtt{j}), \quad \xleftarrow{\mathtt{x}\mapsto(x)} h, \quad \xleftarrow[\mathcal{R}ed_0]{\mathtt{h}\mapsto()} (j \to \mathtt{f}), \xleftarrow{\mathtt{e}\mapsto(f)} n$$



reduces to

$$\mathcal{S}_1 \xleftarrow[\mathcal{R}ed_1]{\mathtt{g}\mapsto((y\to\mathtt{a}),(a\to\mathtt{b}))} (e \to \mathtt{j}),$$

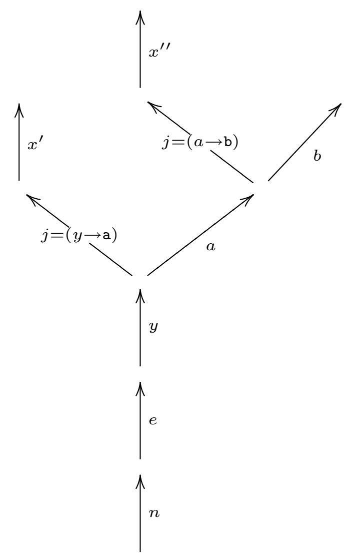
$$\xleftarrow{\mathtt{y}\mapsto(b)} g, \quad \xleftarrow{\mathtt{x}\mapsto(x)} j, \quad \xleftarrow{\mathtt{e}\mapsto()} n$$



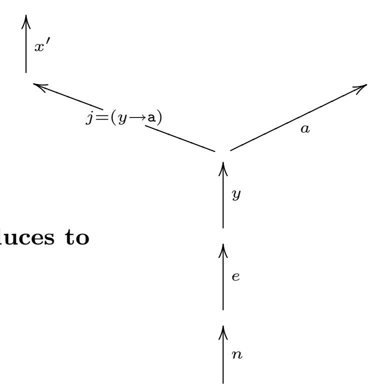
and to

$$\mathcal{S}_2 : \xleftarrow[\mathcal{R}ed_2]{\mathtt{x}'\mapsto(x')} (y \to \mathtt{a}) \qquad \xleftarrow{\mathtt{y}\mapsto(b)} e$$

$$\xleftarrow[\mathcal{R}ed_3]{\mathtt{x}''\mapsto(x'')} (a \to \mathtt{b}) \qquad \xleftarrow{\mathtt{e}\mapsto()} n$$



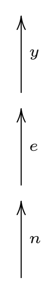
$$\mathcal{S}_3 : \xleftarrow[\mathcal{R}ed_2]{\mathtt{x}'\mapsto(x')} (y \to \mathtt{a}) \quad \xleftarrow{\mathtt{y}\mapsto(a)} e \quad \xleftarrow{\mathtt{e}\mapsto()} n$$



reduces to

$$\mathcal{S}_4 : \xleftarrow{\mathtt{y}\mapsto(y)} e \quad \xleftarrow{\mathtt{e}\mapsto()} n$$

reduces to



We show here the steps of the computation:

$$\mathcal{S}_0 : \xleftarrow{\mathtt{y}\mapsto(b)} g, \quad \xleftarrow[\mathcal{R}ed_1]{\mathtt{g}\mapsto((y\to\mathtt{a}),(a\to\mathtt{b})}(e \to \mathtt{j}), \quad \xleftarrow{\mathtt{x}\mapsto(x)} h, \quad \xleftarrow[\mathcal{R}ed_0]{\mathtt{h}\mapsto()} (j \to \mathtt{f}), \quad \xleftarrow{\mathtt{e}\mapsto(f)} n \quad (o)$$

where $e = \mathbf{e}(n)$, $j = \mathbf{j}(e)$, $f = \mathbf{f}(j)$, $h = \mathbf{h}((j \to \mathtt{f}))$, $x = \mathbf{x}(h)$
$g = \mathbf{g}((e \to \mathtt{j}))$, $y = \mathbf{y}(g)$, $a = \mathbf{a}(y)$, $b = \mathbf{b}(a)$.

Reducing $\mathcal{R}ed_0$: $\mathcal{S}_1 = \mathcal{S}_0 - \mathcal{R}ed_0 \quad \{h := j\}\ \{f := ()\}$

$$\mathcal{S}_1 : \xleftarrow{\mathtt{y}\mapsto(b)} g, \quad \xleftarrow[\mathcal{R}ed_1]{\mathtt{g}\mapsto((y\to\mathtt{a}),(a\to\mathtt{b})}(e \to \mathtt{j}), \quad \xleftarrow{\mathtt{x}\mapsto(x)} j, \quad \xleftarrow{\mathtt{e}\mapsto()} n \quad (i)$$

where $e = \mathbf{e}(n)$, $j = \mathbf{j}(e)$, $x = \mathbf{x}(j)$,
$g = \mathbf{g}((e \to \mathtt{j}))$, $y = \mathbf{y}(g)$, $a = \mathbf{a}(y)$ $b = \mathbf{b}(a)$.

Reducing $\mathcal{R}ed_1$: $\mathcal{S}_2 = \mathcal{S}_1 - \mathcal{R}ed_1 \quad \{g := e\}\ \{j := ((y \to \mathtt{a}), (a \to \mathtt{b}))\{g := e\}\}$.

$$\mathcal{S}_2 : \xleftarrow{\mathtt{y}\mapsto(b)} e, \quad \xleftarrow[\mathcal{R}ed_2]{\mathtt{x}\mapsto(x')} (y \to \mathtt{a}), \quad \xleftarrow[\mathcal{R}ed_3]{\mathtt{x}\mapsto(x'')} (a \to \mathtt{b}), \quad \xleftarrow{\mathtt{e}\mapsto()} n \quad (ii)$$

where $e = \mathbf{e}(n)$, $y = \mathbf{y}(e)$, $a = \mathbf{a}(y)$, $b = \mathbf{b}(a)$,
and, crucially, $x' = \mathbf{x}((y \to \mathtt{a}))$, $x'' = \mathbf{x}((a \to \mathtt{b}))$.

Reducing $\mathcal{R}ed_3$: $\mathcal{S}_3 = \mathcal{S}_2 - \mathcal{R}ed_3 \quad \{x'' := a\}\ \{b := (x'')\{x'' := a\}\}$.

$$\mathcal{S}_3 : \xleftarrow{\mathtt{y}\mapsto(a)} e, \quad \xleftarrow[\mathcal{R}ed_2]{\mathtt{x}\mapsto(x')} (y \to \mathtt{a}), \quad \xleftarrow{\mathtt{e}\mapsto()} n \quad (iii)$$

where $e = \mathbf{e}(n)$, $y = \mathbf{y}(e)$, $a = \mathbf{a}(y)$, $x' = \mathbf{x}((y \to \mathtt{a}))$.

Reducing $\mathcal{R}ed_2$: $\mathcal{S}_4 = \mathcal{S}_3 - \mathcal{R}ed_2 \quad \{x' := y\}\ \{a := (x')\{x' := y\}\}$.

$$\mathcal{S}_4 : \xleftarrow{\mathtt{y}\mapsto(y)} e, \quad \xleftarrow{\mathtt{e}\mapsto()} n\ , \qquad \text{where } e = \mathbf{e}(n), \quad y = \mathbf{y}(e). \quad (iv)$$

10 APPENDIX III. Term assignment to q-LJ$^{\smallsetminus\curlyvee}$

Sequent Calculus q-LJ$^{\smallsetminus\curlyvee}$

identity rules

logical axiom:

$$x : C \;\Rightarrow\; \pi : \bullet \mid \;;\; x : C$$

tail-cut:

$$\frac{x : E \;\Rightarrow\; \pi_1 : \bullet \mid \overline{\ell} : \varUpsilon_1 \;;\; t : C \qquad y : C \;\Rightarrow\; \pi_2 : \bullet \mid \overline{\kappa} : \varUpsilon_2 \;;\; u : D}{x : E \;\Rightarrow\; \pi_1, \pi_2\{y := t\} : \bullet \mid \overline{\ell} : \varUpsilon_1, \overline{\kappa}\{y := t\} : \varUpsilon_2 \;;\; u\{y := t\} : D}$$

central-cut:

$$\frac{x : E \;\Rightarrow\; \pi_1 : \bullet \mid \overline{\ell} : \varUpsilon_1, \ell : C \;; \qquad y : C \;\Rightarrow\; \pi_2 : \bullet \mid \overline{\kappa} : \varUpsilon_2;}{x : E \;\Rightarrow\; \pi_1, \pi_2\{y := \ell\} : \bullet \mid \overline{\ell} : \varUpsilon_1, \overline{\kappa}\{y := \ell\} : \varUpsilon_2 \;;}$$

structural rules

contraction:

$$\frac{x : E \;\Rightarrow\; \pi : \bullet \mid \overline{\ell} : \varUpsilon, \ell : C, \kappa : C \;; u : D}{x : E \;\Rightarrow\; \pi : \bullet \mid \overline{\ell} : \varUpsilon, \ell * \kappa : C \;; u : D}$$

weakening:

$$\frac{x : E \;\Rightarrow\; \pi : \bullet \mid \ell : \varUpsilon \;;\; u : D}{x : E \;\Rightarrow\; \pi : \bullet \mid \ell : \varUpsilon, () : C \;;\; u : D}$$

dereliction:

$$\frac{x : E \;\Rightarrow\; \pi : \bullet \mid \ell : \varUpsilon \;;\; u : D}{x : E \;\Rightarrow\; \pi : \bullet \mid \ell : \varUpsilon, (u) : D \;;}$$

logical rules

$\smallsetminus$ *right:*

$$\frac{x : E \;\Rightarrow\; \pi_1 : \bullet \mid \overline{\ell} : \varUpsilon_1 \;; t : C \qquad y : D \;\Rightarrow\; \pi_2 : \bullet \mid \overline{\kappa} : \varUpsilon_2 \;;}{E \;\Rightarrow\; \pi_1, \pi_2\{y := \mathtt{y}(x)\} : \bullet \mid \overline{\ell} : \varUpsilon_1, \overline{\kappa}\{y := \mathtt{y}(x)\} : \varUpsilon_2 \;;\; \mathtt{mkc}(t, \mathtt{y}) : C \smallsetminus D}$$

$\smallsetminus$ *left*:

$$\frac{x : C \;\Rightarrow\; \pi_1 : \bullet \mid \overline{\ell} : \varUpsilon, \ell : D \;;}{y : C \smallsetminus D \;\Rightarrow\; \pi_1 : \bullet, \mathtt{postp}(\mathtt{x} \mapsto \ell\{x := \mathtt{x}(y)\}, y) : \bullet \mid \overline{\ell}\{x := \mathtt{x}(y)\} : \varUpsilon \;;}$$

$\curlyvee_0$ *right*:

$$\frac{x : E \;\Rightarrow\; \pi : \bullet \mid \overline{\ell} : \varUpsilon \;;\; t : C_0}{x : E \;\Rightarrow\; \pi : \bullet \mid \overline{\ell} : \varUpsilon \;;\; \mathtt{inl}(t) : C_0 \curlyvee C_1}$$

$\curlyvee_1$ *right*:

$$\frac{x : E \;\Rightarrow\; \pi : \bullet \mid \overline{\ell} : \varUpsilon \;;\; t : C_1}{x : E \;\Rightarrow\; \pi : \bullet \mid \overline{\ell} : \varUpsilon \;;\; \mathtt{inr}(t) : C_0 \curlyvee C_1}$$

$\curlyvee$ *left*:

$$\frac{x : C_0 \;\Rightarrow\; \pi_0 : \bullet \mid \overline{\ell}_0 : \varUpsilon_0 \;; \qquad y : C_1 \;\Rightarrow\; \pi_1 : \bullet \mid \overline{\ell}_1 : \varUpsilon_1}{z : C_0 \curlyvee C_1 \;\Rightarrow\; \pi_0', \pi_1' \;: \bullet \mid \overline{\ell}_0' : \varUpsilon_0, \overline{\ell}_1' : \varUpsilon_1}$$

where $\pi_0' = \pi_0\{x := \mathtt{casel}(z)\}$, $\pi_1' = \pi_1\{y := \mathtt{caser}(z)\}$

$\overline{\ell}_0'\{x := \mathtt{casel}(z)\}$, $\overline{\ell}_1' = \overline{\ell}_1\{\{y := \mathtt{caser}(z)\}$.

Table 6. The sequent calculus q-**LJ**$^{\smallsetminus\curlyvee}$

Sequent Calculus t-LJ$^{\supset\cap}$

identity rules

logical axiom:

$$x : A \;;\; \Rightarrow \;\; x : A$$

head-cut:

$$\frac{x : C \;;\; \overline{x} : \Theta_1 \;\Rightarrow t : A \qquad y : A \;;\; \overline{y} : \Theta_2 \;\Rightarrow\; u : B}{x : C \;;\; \overline{x} : \Theta_1, \overline{y} : \Theta_2 \;\Rightarrow\; u\{y := t\} : B}$$

central-cut:

$$\frac{;\; \overline{x} : \Theta_1 \;\Rightarrow t : A \qquad ;\; y : A, \overline{y} : \Theta_2 \;\Rightarrow\; u : B}{;\; \overline{x} : \Theta_1, \overline{y} : \Theta_2 \;\Rightarrow\; u\{y := t\} : B}$$

structural rules

contraction:

$$\frac{y : C \;;\; \overline{x} : \Theta, x_1 : A, x_2 : A \;\Rightarrow\; t : B}{y : C \;;\; \overline{x} : \Theta, x : A \;\Rightarrow\; t\{x_1 := x, x_2 := x\} : B}$$

weakening:

$$\frac{y : C \;;\; \overline{x} : \Theta, \;\Rightarrow\; t : B}{y : C \;;\; \overline{x} : \Theta, x : A \;\Rightarrow\; t : B}$$

dereliction:

$$\frac{x : A \;;\; \overline{x} : \Theta \;\Rightarrow\; t : B}{;\; x : A, \overline{x} : \Theta \;\Rightarrow\; t : B}$$

logical rules

$\supset$ *right*:

$$\frac{;\; \overline{x} : \Theta, x : A \;\Rightarrow\; t : C}{;\; \overline{x} : \Theta \;\Rightarrow\; \lambda x.t : A \supset B}$$

$\supset$ *left:*

$$\frac{;\; \overline{x} : \Theta_0 \;\Rightarrow\; t : A \qquad y : B \;;\; \overline{y} : \Theta_1 \;\Rightarrow\; u : C}{f : A \supset B \;;\; \overline{x} : \Theta_0, \overline{y} : \Theta_1 \;\Rightarrow\; u\{y := f(t)\} : C}$$

$\cap$ *right*:

$$\frac{;\; \overline{x} : \Theta_0 \;\Rightarrow\; t_0 : A_0 \qquad\qquad ;\; \overline{y} : \Theta_1 \;\Rightarrow\; t_1 : A_1}{;\; \overline{x} : \Theta_0, \overline{y} : \Theta_1 \;\Rightarrow\; \langle t_0, t_1 \rangle : A_0 \cap A_1}$$

$\cap_0$ *left*:

$$\frac{x : A_0 \;;\; \overline{x} : \Theta \;\Rightarrow\; t : C}{z : A_0 \cap A_1 \;;\; \overline{x} : \Theta \;\Rightarrow\; t[\pi_0(z)/x] : C}$$

$\cap_1$ *left*:

$$\frac{x : A_1 \;;\; \overline{x} : \Theta \;\Rightarrow\; t : C}{z : A_0 \cap A_1 \;;\; \overline{x} : \Theta \;\Rightarrow\; t\{x := \pi_1(z)\} : C}$$

Table 7. The sequent calculus t-**LJ**$^{\supset\cap}$