# Computational Logic 

## Project: Categorical Semantics

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## 1 Preliminaries

We provide the formal definitions of some important categorical concepts.
Definition 1.1 Let $\mathbb{C}$ and $\mathbb{D}$ be categories. A functor $\mathcal{F}: \mathbb{C} \longrightarrow \mathbb{D}$ is a map that takes an object X of $\mathbb{C}$ to an object $\mathcal{F}(\mathrm{X})$ of $\mathbb{D}$ and a morphism $\mathrm{f}: \mathrm{X} \longrightarrow \mathrm{Y}$ of $\mathbb{C}$ to a morphism $\mathcal{F}(\mathrm{f}): \mathcal{F}(\mathrm{X}) \longrightarrow \mathcal{F}(\mathrm{Y})$ such that, for all objects X of $\mathbb{C}$ and all morphisms $\mathrm{f}: \mathrm{X} \longrightarrow \mathrm{Y}$ and $\mathrm{g}: \mathrm{Y} \longrightarrow \mathrm{Z}$ of $\mathbb{C}, \mathcal{F}\left(\mathrm{id}_{\mathrm{X}}\right)=\mathrm{id}_{\mathcal{F}(\mathrm{X})}$ and $\mathcal{F}(\mathrm{g} \circ \mathrm{f})=\mathcal{F}(\mathrm{g}) \circ \mathcal{F}(\mathrm{f})$.

Definition 1.2 Let $\mathbb{C}$ and $\mathbb{D}$ be categories, and $\mathcal{F}: \mathbb{C} \longrightarrow \mathbb{D}$ and $\mathcal{G}: \mathbb{C} \longrightarrow \mathbb{D}$ functors from $\mathbb{C}$ to $\mathbb{D}$. A natural transformation $\nu$ from $\mathcal{F}$ to $\mathcal{G}$, written $\nu: \mathcal{F} \longrightarrow \mathcal{G}$, is a function that assigns to every object X of $\mathbb{C}$ a morphism $\nu_{\mathrm{X}}: \mathcal{F}(\mathrm{X}) \longrightarrow \mathcal{G}(\mathrm{X})$ of $\mathbb{D}$ such that, for any morphism $\mathrm{f}: \mathrm{X} \longrightarrow \mathrm{Y}$ of $\mathbb{C}$, the equation $\mathcal{G}(\mathrm{f}) \circ \nu_{\mathrm{X}}=\nu_{\mathrm{Y}} \circ \mathcal{F}(\mathrm{f})$ holds in $\mathbb{D}$. [EXERCISE: write the equation as a commuting diagram.]

Definition 1.3 A symmetric monoidal category $(\mathbb{M}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau)$ consists of a category $\mathbb{M}$ together with a bifunctor $\otimes: \mathbb{M} \times \mathbb{M} \longrightarrow \mathbb{M}$, an object I of $\mathbb{M}$, and four natural isomorphisms $\alpha_{\mathrm{X}, \mathrm{Y}, \mathrm{Z}}: \mathrm{X} \otimes(\mathrm{Y} \otimes \mathrm{Z}) \longrightarrow(\mathrm{X} \otimes \mathrm{Y}) \otimes \mathrm{Z}$, $\lambda_{\mathrm{X}}: \mathrm{I} \otimes \mathrm{X} \longrightarrow \mathrm{X}, \rho_{\mathrm{X}}: \mathrm{X} \otimes \mathrm{I} \longrightarrow \mathrm{X}, \tau_{\mathrm{X}, \mathrm{Y}}: \mathrm{X} \otimes \mathrm{Y} \longrightarrow \mathrm{Y} \otimes \mathrm{X}$ such that $\lambda_{\mathrm{I}}=$ $\rho_{\mathrm{I}}: \mathrm{I} \otimes \mathrm{I} \longrightarrow \mathrm{I}$ and the following five diagrams commute for all objects $\mathrm{X}, \mathrm{Y}$, $Z$, and $U$ of $\mathbb{M}$.



Remark 1.4 It seems worth to provide some explanations at this point.

1. A bifunctor $\mathcal{F}: \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{D}$ is a functor $\mathcal{F}$ that maps a pair of objects ( $\mathrm{X}, \mathrm{Y}$ ) of $\mathbb{C}$ to an object $\mathcal{F}(\mathrm{X}, \mathrm{Y})$ of $\mathbb{D}$ and a pair of morphisms ( $\mathrm{f}, \mathrm{g}$ ) of $\mathbb{C}$ to a morphism $\mathcal{F}(\mathrm{f}, \mathrm{g})$ of $\mathbb{D}$. Note that we shall write $\mathrm{X} \otimes \mathrm{Y}$ and $\mathrm{f} \otimes \mathrm{g}$ instead of $\otimes(\mathrm{X}, \mathrm{Y})$ and $\otimes(\mathrm{f}, \mathrm{g})$.
2. A natural isomorphism is a natural transformations $\nu: \mathcal{F} \longrightarrow \mathcal{G}$ that comes endowed with an inverse natural transformation $\bar{\nu}: \mathcal{G} \xrightarrow{\longrightarrow} \mathcal{F}$ such that both $\nu_{\mathrm{X}} \circ \bar{\nu}_{\mathrm{X}}=\mathrm{id}_{\mathrm{X}}=\bar{\nu}_{\mathrm{X}} \circ \nu_{\mathrm{X}}$.
Example 1.5 Every cartesian category is a symmetric monoidal category.
Definition 1.6 A symmetric monoidal functor $(\mathcal{F}, \mu, \mathrm{u}): \mathbb{M} \longrightarrow \mathbb{M}^{\prime}$ between two symmetric monoidal categories $\mathbb{M}=(\mathbb{M}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau)$ and $\mathbb{M}^{\prime}=$ $\left(\mathbb{M}^{\prime}, \otimes^{\prime}, \mathrm{I}^{\prime}, \alpha^{\prime}, \lambda^{\prime}, \rho^{\prime}, \tau^{\prime}\right)$ consists of a functor $\mathcal{F}: \mathbb{M} \longrightarrow \mathbb{M} \mathbb{M}^{\prime}$ together with a natural transformation $\mu_{\mathrm{X}, \mathrm{Y}}: \mathcal{F}(\mathrm{X}) \otimes^{\prime} \mathcal{F}(\mathrm{Y}) \longrightarrow \mathcal{F}(\mathrm{X} \otimes \mathrm{Y})$ and a morphism $\mathrm{u}: \mathrm{I}^{\prime} \longrightarrow \mathcal{F}(\mathrm{I})$ such that, for all objects $\mathrm{X}, \mathrm{Y}$, and Z of $\mathbb{M}$, the following four diagrams commute in $\mathbb{M}^{\prime}$.



Definition 1.7 A symmetric monoidal category $(\mathbb{M}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau)$ is closed if, for all objects X and Y of $\mathbb{M}$, there exists an object $[\mathrm{X} \multimap \mathrm{Y}]$ of $\mathbb{M}$ together with a morphism app: $[\mathrm{X} \multimap \mathrm{Y}] \otimes \mathrm{X} \longrightarrow \mathrm{Y}$ of $\mathbb{M}$ such that, for every object Z of $\mathbb{M}$ and every morphism $\mathrm{f}: \mathrm{Z} \otimes \mathrm{X} \longrightarrow \mathrm{Y}$ of $\mathbb{M}$, there exists a unique morphism $\operatorname{cur}(\mathrm{f}): \mathrm{Z} \longrightarrow[\mathrm{X} \multimap \mathrm{Y}]$ of $\mathbb{M}$ that satisfies app $\circ\left(\operatorname{cur}(\mathrm{f}) \otimes \mathrm{id}_{\mathrm{X}}\right)=\mathrm{f}$.

## 2 Project proposal

We provide a collection of problems about topics in categorical semantics.

### 2.1 Modal logic

The aim of this exercise is to prove soundness and completeness results for the propositional intuitionistic modal logic IK. Formulae or types $A$ of IK are defined by the grammar $A::=\top|A \wedge A| A \rightarrow A \mid \square A$. The sound (a.k.a. valid) and complete categorical semantics for the corresponding extension of the simply typed $\lambda$-calculus is given by a cartesian closed category $\mathbb{C}$ that comes equipped with a monoidal endofunctor $(\mathcal{F}, \eta, \mathrm{u})$, i.e. a monoidal functor $(\mathcal{F}, \mu, \mathrm{u})$ from $\mathbb{C}$ to $\mathbb{C}$. The extended interpretation function $\llbracket-\rrbracket_{\mathbb{C}}: \lambda \rightarrow \mathbb{C}$ is defined as expected, thus assigning an object $\llbracket \square A \rrbracket_{\mathbb{C}}=\mathcal{F}\left(\llbracket A \rrbracket_{\mathbb{C}}\right)$ of $\mathbb{C}$ to the type $\square A$. The other missing details are provided in the following sections.

### 2.1.1 Term formation rule

$$
\frac{\Gamma \vdash s_{1}: \square A_{1} \quad \cdots \quad \Gamma \vdash s_{n}: \square A_{n} \quad x_{1}: A_{1}, \ldots, x_{n}: A_{n} \vdash s: A}{\Gamma \vdash \operatorname{box} s \text { with }\left(s_{1}, \ldots, s_{n}\right) \text { for }\left(x_{1}, \ldots, x_{n}\right): \square A} \square
$$

### 2.1.2 Equations in context

$$
\frac{\left\{\begin{array}{cccc}
\Gamma \vdash r_{1}: \square B_{1} & \cdots & \Gamma \vdash r_{m}: \square B_{m} & y_{1}: B_{1}, \ldots, y_{m}: B_{m} \vdash r: B \\
\Gamma \vdash s_{1}: \square A_{1} & \cdots & \Gamma \vdash s_{n}: \square A_{n} & x_{1}: A_{1}, \ldots, x_{n}: A_{n}, x: B \vdash s: A
\end{array}\right\}}{\begin{array}{l}
\Gamma \vdash \operatorname{box} s \text { with }(\vec{s} \text {, box } r \text { with } \vec{r} \text { for } \vec{y}) \text { for }(\vec{x}, x) \\
=\operatorname{box} s[r / x] \text { with }(\vec{s}, \vec{r}) \text { for }(\vec{x}, \vec{y}): \square A
\end{array}}
$$

$$
\frac{\Gamma \vdash s: \square A}{\Gamma \vdash \text { box } x \text { with } s \text { for } x=s: \square A} \eta_{\square}
$$

### 2.1.3 Interpretation of term

$$
\begin{aligned}
\llbracket \Gamma \vdash \text { box } s \text { with } & \left(s_{1}, \ldots, s_{n}\right) \text { for }\left(x_{1}, \ldots, x_{n}\right): \square A \rrbracket \mathbb{C} \\
= & \mathcal{F}\left(\llbracket x_{1}: A_{1}, \ldots, x_{n}: A_{n} \vdash s: A \rrbracket \mathbb{C}\right) \circ \hat{\mu} \\
& \circ\left\langle\llbracket \Gamma \vdash s_{1}: \square A_{1} \rrbracket \mathbb{C}, \ldots, \llbracket \Gamma \vdash s_{n}: \square A_{n} \rrbracket_{\mathbb{C}}\right\rangle \\
{\left[\hat{\mu}: \mathcal{F}\left(\llbracket A_{1} \rrbracket_{\mathbb{C}}\right) \times\right.} & \left.\cdots \times \mathcal{F}\left(\llbracket A_{n} \rrbracket_{\mathbb{C}}\right) \longrightarrow \mathcal{F}\left(\llbracket A_{1} \rrbracket_{\mathbb{C}}^{\square} \times \cdots \times \llbracket A_{n} \rrbracket_{\mathbb{C}}\right)\right]
\end{aligned}
$$

Task 2.1 State the obvious soundness and complete theorems and provide proofs of them. In particular, note that you are not required to repeat the parts of the proofs for the $\square$-free fragment already discussed in class.

### 2.2 Linear logic

The aim of this exercise is to prove soundness and completeness results for the multiplicative fragment of intuitionistic linear logic IMLL. Formulae or types $A$ of IMLL are defined by the grammar $A::=I|A \otimes A| A \multimap A$. The sound (a.k.a. valid) and complete categorical semantics for the corresponding variant of the simply typed $\lambda$-calculus is given by a symmetric monoidal closed category $\mathbb{M}$. The interpretation function $\llbracket-\rrbracket_{\mathbb{M}}: \lambda \rightarrow \mathbb{M}$, also written as $\llbracket-\rrbracket$, is defined as expected, thus assigning objects $\llbracket I \rrbracket=\mathrm{I}, \llbracket A \otimes B \rrbracket=$ $\llbracket A \rrbracket \otimes \llbracket B \rrbracket$, and $\llbracket A \multimap B \rrbracket=[\llbracket A \rrbracket \multimap \llbracket B \rrbracket]$ of $\mathbb{M}$ to the types $I, A \otimes B$, and $A \multimap B$. Contexts are interpreted as follows: $\llbracket \emptyset \rrbracket=\mathrm{I} ; \llbracket \Gamma, x: A \rrbracket=\llbracket \Gamma \rrbracket \otimes \llbracket A \rrbracket$. All the other missing details are provided in the following sections.

### 2.2.1 Term formation rules

$$
\begin{gathered}
\frac{\Gamma: A \vdash x: A}{x} a x \\
\frac{\Gamma \vdash s: A}{\vdash, \Delta \vdash} I_{i} \frac{\Gamma \vdash s: I \quad \Delta \vdash r: A}{\Delta, \Gamma \vdash \text { let } s \text { be } * \text { in } r: A} I_{e} \\
\frac{\Delta \vdash r: B}{\Gamma, \Delta \otimes B} \otimes_{i} \\
\frac{\Gamma \vdash s: A_{1} \otimes A_{2} \quad \Delta, x_{1}: A_{1}, x_{2}: A_{2} \vdash r: B}{\Delta \vdash, \Gamma \vdash \operatorname{let} s \text { be } x_{1} \otimes x_{2} \text { in } r: B} \otimes_{e} \\
\multimap_{i} \quad \frac{\Gamma \vdash s: A \multimap B \quad \Delta \vdash r: A}{\Gamma, \Delta \vdash(s) r: B} \multimap_{e}
\end{gathered}
$$

### 2.2.2 Equations in context

$$
\begin{gathered}
\frac{\Gamma \vdash s: A}{\Gamma \vdash \operatorname{let} * \text { be } * \text { in } r=r: A} \beta_{I} \\
\frac{\Gamma \vdash s: I \quad \Delta, x: I \vdash r: B}{\Delta, \Gamma \vdash \operatorname{let} s \text { be } * \text { in }(r[* / x])=r[s / x]: B} \eta_{I} \\
\frac{\Gamma_{1} \vdash s_{1}: A_{1} \quad \Gamma_{2} \vdash s_{2}: A_{2} \quad \Delta, x_{1}: A_{1}, x_{2}: A_{2} \vdash r: B}{\Delta, \Gamma_{1}, \Gamma_{2} \vdash \operatorname{let} s_{1} \otimes s_{2} \text { be } x_{1} \otimes x_{2} \text { in } r=r\left[s_{1} / x_{1}, s_{2} / x_{2}\right]: B} \beta_{\otimes} \\
\frac{\Gamma \vdash s: A_{1} \otimes A_{2} \quad \Delta, x: A_{1} \otimes A_{2} \vdash r: B}{\Delta, \Gamma \vdash \operatorname{let} s \text { be } x_{1} \otimes x_{2} \text { in }\left(r\left[x_{1} \otimes x_{2} / x\right]\right)=r[s / x]: B} \eta_{\otimes} \\
\frac{\Gamma, x: A, \vdash s: B \quad \Delta \vdash r: A}{\Gamma, \Delta \vdash(\lambda x . s) r=s[r / x]: B} \beta_{-} \quad \frac{\Gamma \vdash s: A \multimap B}{\Gamma \vdash \lambda x .(s) x: A \multimap B} \eta_{\multimap}
\end{gathered}
$$

### 2.2.3 Interpretation of terms

$$
\begin{aligned}
& \llbracket x: A \vdash x: A \rrbracket=\operatorname{id}_{\llbracket A \rrbracket} \\
& \llbracket \vdash *: I \rrbracket=\mathrm{id}_{\llbracket I \rrbracket}=\mathrm{id}_{I} \\
& \llbracket \Delta, \Gamma \vdash \text { let } s \text { be } * \text { in } r: A \rrbracket=\llbracket \Delta \vdash r: A \rrbracket \circ \rho_{\llbracket \Delta \rrbracket} \\
& \circ\left(\mathrm{id}_{\llbracket \Delta \rrbracket} \otimes \llbracket \Gamma \vdash s: I \rrbracket\right) \\
& \llbracket \Gamma, \Delta \vdash s \otimes r: A \otimes B \rrbracket=\llbracket \Gamma \vdash s: A \rrbracket \otimes \llbracket \Delta \vdash r: B \rrbracket \\
& \llbracket \Delta, \Gamma \vdash \text { let } s \text { be } x_{1} \otimes x_{2} \text { in } r: B \rrbracket=\llbracket \Delta, x_{1}: A_{1}, x_{2}: A_{s} \vdash r: B \rrbracket \\
& \circ\left(\mathrm{id}_{\llbracket \Delta \rrbracket} \otimes \llbracket \Gamma \vdash s: A_{1} \otimes A_{2} \rrbracket\right) \\
& \llbracket \Gamma \vdash \lambda x . s: A \multimap B \rrbracket=\operatorname{cur}(\llbracket \Gamma, x: A \vdash s: B \rrbracket) \\
& \llbracket \Gamma, \Delta \vdash(s) r: A \multimap B \rrbracket=\operatorname{app} \circ(\llbracket \Gamma \vdash s: A \multimap B \rrbracket \otimes \llbracket \Delta \vdash r: A \rrbracket)
\end{aligned}
$$

Task 2.2 State the obvious soundness and complete theorems and provide proofs of them. In particular, note that: (a) in the proof of the soundness theorem you will have to make use of the substitution lemma given below; (b) since the proof of the completeness theorem needs quite a lot of routine calculations, it suffices two provide a concise description of all the possible cases that need to be considered together with some sample calculations.

Lemma 2.3 (substitution) If $\Gamma \vdash s: A$ and $\Delta, x: A \vdash r: B$ are derivable according to the rules given in section 2.2.1 above then $\llbracket \Delta, \Gamma \vdash r[s / x]: B \rrbracket=$ $\left(\mathrm{id}_{\llbracket \Delta \rrbracket} \otimes \llbracket \Gamma \vdash s: A \rrbracket\right) \circ \llbracket \Delta, x: A \vdash r: B \rrbracket$.

### 2.3 Adjointness

The aim of this exercise is to show that the connection between implication and conjunction can be expressed categorically in terms of an adjunction. In NJ, we have that

$$
\frac{C \wedge A \vdash B}{\overline{C \vdash A \rightarrow B}}
$$

where the double bar indicates that one can derive the conclusion from the premise and vice versa. In terms of cartesian closed categories this may also be reformulated as

$$
\frac{\mathrm{Z} \times \mathrm{X} \longrightarrow \mathrm{Y}}{\overline{\mathrm{Z} \longrightarrow[\mathrm{X} \rightarrow \mathrm{Y}]}}
$$

which is equivalent to say that implication is the right adjoint of conjunction.
Definition 2.4 An adjunction consists of a pair of categories $\mathbb{C}$ and $\mathbb{D}$, a pair of functors $\mathcal{F}: \mathbb{C} \longrightarrow \mathbb{D}$ and $\mathcal{G}: \mathbb{D} \longrightarrow \mathbb{C}$, and a natural transformation $\epsilon:(\mathcal{F} \circ \mathcal{G}) \longrightarrow \mathcal{I}_{\mathbb{D}}\left(\right.$ where $\mathcal{I}_{\mathbb{D}}: \mathbb{D} \longrightarrow \mathbb{D}$ is the identity functor on $\left.\mathbb{D}\right)$ such that, for each morphism $\mathrm{g}: \mathcal{F}(\mathrm{X}) \longrightarrow \mathrm{Y}$ in $\mathbb{D}$ there is a unique morphism $\mathrm{g}^{*}: \mathrm{X} \longrightarrow \mathcal{G}(\mathrm{Y})$ for which the equation $\mathrm{g}=\epsilon_{\mathrm{Y}} \circ \mathcal{F}\left(\mathrm{g}^{*}\right)$ holds in $\mathbb{D} .(\mathcal{F}, \mathcal{G})$ is called an adjoint pair; $\mathcal{F}$ is the left adjoint of $\mathcal{G}$, and $\mathcal{G}$ is the right adjoint of $\mathcal{F}$. The natural transformation $\epsilon$ is called the counit of the adjunction.

Task 2.5 Define a pair of functors $\mathcal{F}$ and $\mathcal{G}$, and a natural transformation $\epsilon$ that show that implication is the right adjoint of conjunction. In particular, provide proofs of the facts that $\mathcal{F}$ and $\mathcal{G}$ are functors and that $\epsilon$ is a natural transformation. Can the above claim be modified in such a way that it also applies to symmetric monoidal closed categories? Justify your answer.

