1 Computational Logic 2008 - Dr G. Bellin

Solution of Pre-Examination

17th December 2008

Answer the following four questions. Questions 1 and 4 carry 25 marks. Questions 2 and 3 carry 35 marks. Marks above 100 are bonus for the final mark.

QUESTION 1. Consider the language of modal logic

\[ A := P \mid \bot \mid A_1 \rightarrow A_2 \mid \Box A \]

Extend the sequent calculus system for classical logic \( \text{G3C} \) with the following rules for the modal system \( \text{S4} \):

\[
\begin{align*}
\frac{\Box \Gamma \Rightarrow A}{\Pi, \Box \Gamma \Rightarrow \Box A, \Lambda} & \quad \Box - \text{R} \\
\frac{A, \Box A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta} & \quad \Box - \text{L}
\end{align*}
\]

(a) Consider a Kripke model \( \mathcal{M} = (W, R, \models) \), where \( W \) is a set of possible worlds, \( R \subseteq W \times W \) is the accessibility relation and \( \models \subseteq W \times \text{Atoms} \).

Answer the following questions:

(a.1) What is the frame of \( \mathcal{M} \)?

Answer: The frame of \( \mathcal{M} \) is the structure \((W, R)\), the set of possible worlds and the accessibility relation.

1 mark

(a.2) What does it mean to say that a sentence \( A \) is valid in \( \mathcal{M} \)?

Answer: A is valid in \( \mathcal{M} \) if for all \( w \in W \) we have \( w \models A \) (equivalently, \( V(w, A) = T \), where \( V \) is a valuation relativized to the possible world \( w \)). [The relation \( w \) forces \( A \) is defined inductively as a propositional valuation \( V \) relativized to each possible world, setting \( w \models \Box B \) if and only if for all \( w' \in W \) such that \( wRw' \) we have \( w' \models B \).]

2 marks

(a.3) What does it mean to say that a sentence \( A \) is valid in \( K \)?

Answer: A is valid in \( K \) if for all models \( \mathcal{M} \) \( A \) is true in \( \mathcal{M} \).

2 marks
(a.4) What does it mean to say that a sentence $A$ is valid in $\textbf{S4}$?

Answer: $A$ is valid in $\textbf{S4}$ if $A$ is true in all models $\mathcal{M} = (W, R, \models)$ where $R$ is reflexive and transitive (a preorder).

Consider the following sequents:

(i) $S_1: \Rightarrow \Box(\Box A \rightarrow B) \rightarrow A$

(ii) $S_2: \Rightarrow \Box(\Box A \rightarrow B) \rightarrow \Box \neg \neg A$

(a) Question: are $S_1$ or $S_2$ valid in $\textbf{S4}$? (yes or no answer)

Answer: $S_1$ is not valid, $S_2$ is valid.

(b) If the sequent $S_1$ or $S_2$ is falsifiable, define a Kripke model $(W, R, \models)$ with a world $w \in W$ such that $w \not\models S_i$. Otherwise, write a derivation of $S_i$ in the sequent calculus for $\textbf{S4}$.

Answer: (i) We need a model $(W, R, \models)$ and a world $w_0 \in W$ such that $w_0 \not\models S_1$ and $w_0 \not\models A$.

Since $w_0 \not\models A$ and $R$ is reflexive, we have $w_0 \not\models \Box A$ hence $w_0 \models \Box A \rightarrow B$. Since we have $w_0 \not\models A$, the only possibility for $\Box(\Box A \rightarrow B) \rightarrow A$ to be true in $w_0$ is that in some world $w'$ accessible from $w_0$ we have $w' \not\models \Box A \rightarrow B$. But $w'$ cannot be $w_0$. Hence suppose there exists $w_1$ such that $w_0Rw_1$ and

(c) $w_1 \models \Box A$ and $w_1 \not\models B$.

Since $w_1Rw_1$, we must have

(d) $w_1 \models A$.

Thus we can let $(W, R, \models)$ where $W = \{w_0, w_1\}$, $R$ is the reflexive and transitive closure of $w_0Rw_1$ and $\models$ satisfies (b), (c) and (d).

Answer (ii): $\Rightarrow \Box(\Box A \rightarrow B) \rightarrow A \rightarrow \Box \neg \Box \neg A$ is derivable as follows:
\[
\begin{align*}
\text{axiom} & : \ A \Rightarrow A \\
\square (\square (\square A \rightarrow B) \rightarrow A), \neg A, \square \neg A, A, \square A \Rightarrow B & \quad \rightarrow \neg L \\
\square (\square (\square A \rightarrow B) \rightarrow A), \square \neg A, \square A \Rightarrow B & \quad \square \neg L \text{ twice} \\
\square (\square (\square A \rightarrow B) \rightarrow A), \square \neg A, \Rightarrow \square A \rightarrow B & \quad \rightarrow \neg R \\
\square (\square (\square A \rightarrow B) \rightarrow A), \square \neg A, \Rightarrow \square (\square A \rightarrow B) & \quad \square \rightarrow R \\
\Rightarrow \square (\square (\square A \rightarrow B) \rightarrow A), \neg A, \square \neg A \Rightarrow & \quad \neg \neg \rightarrow L \\
\square (\square (\square A \rightarrow B) \rightarrow A), \square \neg A \Rightarrow & \quad \neg \neg \neg \rightarrow R \\
\square (\square (\square A \rightarrow B) \rightarrow A), \Rightarrow \square \neg A \rightarrow & \quad \Rightarrow \square \neg A \rightarrow R
\end{align*}
\]

\text{TOTAL: 25 marks}

**QUESTION 2.** (a) Consider the language of \textit{classical logic} in the form:

\[ A := P | \neg P | A_1 \land A_2 | A_1 \lor A_2 \]

Consider the sequent calculus system for classical logic (one sided) \textbf{G3C} with the following axioms and rules:

\[
\begin{align*}
\text{STRUCTURAL RULE} & \quad \Rightarrow \Gamma, B, A, \Delta \\
\text{IDENTITY} & \quad \text{Exchange} \\
\Rightarrow \Gamma, A, B, \Delta \\
\text{LOGICAL RULES} & \quad \Rightarrow \Gamma, A, \neg A \\
\Rightarrow \Gamma, A \land B & \quad \land \neg R \\
\Rightarrow \Gamma, A \lor B & \quad \lor \neg R
\end{align*}
\]

Consider the following sequents:

(iii) \( S_3 : \Rightarrow \neg A \lor (\neg B \land \neg C), (A \land B) \lor (A \land C) \);

(iv) \( S_4 : \Rightarrow \neg A \lor (\neg B \land \neg C), (A \land B) \lor C \);

(v) \( S_5 : \Rightarrow (\neg A \lor \neg B) \land \neg C, (A \land B) \lor C \).

Are they derivable? If yes, **write** a derivation; otherwise, **write** a truth value assignment that makes the sequent false.
Answer: The following are proofs of (iii), (iv) and (v):

\[
\begin{align*}
\text{axiom} & \quad \Rightarrow \neg A, \neg B, A \land C, A \land B \\
\land-R & \quad \Rightarrow \neg A, \neg B, A \land C, A \land B \land C \\
\Rightarrow & \quad \neg A \lor (\neg B \land \neg C), (A \land B) \lor (A \land C) \\
\lor-R & \quad \text{twice}
\end{align*}
\]

Consider the sequent calculus for classical logic (one sided) \( \text{G1C} \) (provably equivalent to \( \text{G3C} \)) with explicit rules of Contraction and Weakening and with axioms and cut rule of the following forms:

\[
\begin{align*}
\text{axiom} & \quad \Rightarrow A, \neg A \\
\text{cut} & \quad \Rightarrow \Gamma, \neg A \quad \Rightarrow A, \Delta
\end{align*}
\]

Consider the derivation \( \mathcal{D} \):

\[
\begin{align*}
\Rightarrow B, \neg B & \quad \text{weakening} \\
\Rightarrow C, \neg C & \quad \text{weakening} \\
\Rightarrow A, C, \neg C & \quad \text{cut}
\end{align*}
\]

(b) Question: How many ways are there to eliminate the indicated cut? Write all the cut-free derivations.

Answer: [Remember that each step in the algorithm of cut-elimination consists either (i) in replacing a cut inference with cuts of lower complexity (logical reductions) or (ii) permuting a cut inference with the final inference \( I_i \) in one of the two derivations of the sequent-premises of the cut (permutative conversion), if a logical reduction cannot be applied because \( I_i \) does not introduce a cut-formula, or (iii) in eliminating the cut inference altogether, if
one of the premises of the cut is an axiom or the conclusion of a weakening. Notice also that steps (ii) and (iii) are non-deterministic when the lowermost inferences \( \mathcal{I} \) in both derivations of the cut premises do not introduce the cut formula, or are axioms or weakenings. Figuratively, we can say that in these cases we have a choice between pushing the cut up in the left or in the right sub-derivation.

In our example, we can either push the cut up in the left or in the right premise we obtain two distinct derivations:

\[
\begin{align*}
D_1 & : \text{ axiom } B, \neg B \\
     & \quad \text{ weakening twice } B, \neg B, C, \neg C
\end{align*}
\]

\[
\begin{align*}
D_2 & : \text{ axiom } C, \neg C \\
     & \quad \text{ weakening twice } B, \neg B, C, \neg C
\end{align*}
\]

Since we do not have a generalized weakening rule, introducing several formulas, derivation \( D_1 \) really corresponds to two derivations \( D_{1,1} \) and \( D_{1,2} \), depending on whether it is \( C \) or \( \neg C \) that is introced first, and similarly for \( D_2 \); thus in conclusion we do have four cut-free derivations.

The difference between \( D_{1,1} \) and \( D_{1,2} \) is relatively unimportant: it is about the order in which we introduce irrelevance. On the contrary, \( D_1 \) and \( D_2 \) are essentially diferent, as they differ for what is relevant, namely the two formulas whose connection defines a logical axiom, and what is irrelevant, i.e., introduced by weakening.

10 marks

(c) Does cut-elimination for \( \text{C1C} \) enjoy the Church-Rosser property? Explain.

Answer: [Notice that here the computation process is cut-elimination, as in the lambda calculus it is \( \beta \)-reduction.] To say that cut-elimination has the Church-Rosser property is to say that given a derivation \( D \) which reduces by cut-elimination to either \( D_1 \) and \( D_2 \), we can find a \( D_3 \) such that both \( D_1 \) and \( D_2 \) reduce to \( D_3 \) by cut-elimination. But for the derivations \( D, D_1 \) and \( D_2 \) considered above we have that \( D_1 \) and \( D_2 \) are cut-free and essentially diferent. Thus there can be no \( D_3 \) which \( D_1 \) and \( D_2 \) reduce to. This shows that the process of cut-elimination in classical logic \( \text{G1C} \) does not enjoy the Church-Rosser property.

10 marks

TOTAL: 35 marks
QUESTION 3. Consider the language of \textbf{MLL} classical \textit{multiplicative linear logic} (without units):

$$A := P | P^\perp | A_1 \otimes A_2 | A_1 \wp A_2$$

Consider the sequent calculus system for classical \textbf{MLL} with the following rules:

\begin{align*}
\text{STRUCTURAL RULE} & \quad \Rightarrow \Gamma, B, A, \Delta \\
& \quad \Rightarrow \Gamma, A, B, \Delta \\
\text{Exchange} & \\
\text{IDENTITY} & \quad \text{axiom} \\
& \quad \Rightarrow A, A^\perp \\
\text{LOGICAL RULES} & \quad \Rightarrow \Gamma, A \\
& \quad \Rightarrow \Delta, B \\
& \quad \Rightarrow \Gamma, \Delta, A \otimes B \\
& \quad \Rightarrow \Gamma, \Delta, A^\perp \otimes B, A \otimes C \\
& \quad \Rightarrow \Gamma, \Delta, A \otimes B, A \otimes C \\
& \quad \Rightarrow (A^\perp \wp A^\perp) \wp (B^\perp \otimes C^\perp), (A \otimes B) \wp (A \otimes C) \\
\text{\textcircled{$\perp$}-R} & \\
& \quad \Rightarrow \Gamma, A, B \\
& \quad \Rightarrow \Gamma, A \wp B \\
& \quad \Rightarrow \Gamma, A \wp B \\
& \quad \Rightarrow \Gamma, A \wp B \\
& \quad \Rightarrow A^\perp \wp (B^\perp \otimes C^\perp), (A \otimes B) \wp (A \otimes C) \\
\text{\textcircled{$\perp$}-R two times} & \\
\text{\textcircled{$\perp$}-R three times} & \\
\text{\textcircled{$\perp$}-R} & \\
\text{\textcircled{$\perp$}-R} &
\end{align*}

Consider the following sequents:

(i) \( S_6: \Rightarrow A^\perp \wp (B^\perp \otimes C^\perp), (A \otimes B) \wp (A \otimes C) \)

(ii) \( S_7: \Rightarrow (A^\perp \wp A^\perp) \wp (B^\perp \otimes C^\perp), (A \otimes B) \wp (A \otimes C) \)

(iii) \( S_8: \Rightarrow A^\perp \wp (B^\perp \otimes C^\perp), (A \otimes B) \wp C \)

(a) Are they derivable? For each of \( S_6-S_8 \) answer yes or no.

\textit{Answer:} \( S_6 \) is not derivable: this can be seen as a corollary of the cut-elimination theorem for \textbf{MLL}, but the proof is not required. \( S_7 \) and \( S_8 \) are derivable.

(b) If any one of \( S_6-S_8 \) is derivable, write a derivation of it.

\begin{align*}
\Rightarrow & \quad \text{axiom} \\
& \quad \Rightarrow A^\perp, A \\
& \quad \Rightarrow B^\perp, B \\
& \quad \Rightarrow A^\perp, B^\perp, A \otimes B \\
& \quad \Rightarrow A^\perp, B^\perp, A \otimes B \\
\text{\textcircled{$\perp$}-R} & \\
\Rightarrow & \quad \text{axiom} \\
& \quad \Rightarrow A^\perp, A \\
& \quad \Rightarrow C^\perp, C \\
& \quad \Rightarrow A^\perp, C^\perp, A \otimes C \\
& \quad \Rightarrow A^\perp, C^\perp, A \otimes C \\
\text{\textcircled{$\perp$}-R two times} & \\
\Rightarrow & \quad \text{axiom} \\
& \quad \Rightarrow A^\perp, A \\
& \quad \Rightarrow B^\perp, B \\
& \quad \Rightarrow A^\perp, B^\perp, A \otimes B \\
\text{\textcircled{$\perp$}-R} & \\
\Rightarrow & \quad \text{axiom} \\
& \quad \Rightarrow A^\perp, A \\
\text{\textcircled{$\perp$}-R} & \\
\Rightarrow & \quad \text{axiom} \\
& \quad \Rightarrow A^\perp, A \\
\text{\textcircled{$\perp$}-R twice} & \\
\Rightarrow & \quad \text{axiom} \\
& \quad \Rightarrow A^\perp, A \\
\text{\textcircled{$\perp$}-R twice} & \\
\Rightarrow & \quad \text{axiom} \\
& \quad \Rightarrow A^\perp, A \\
\text{\textcircled{$\perp$}-R three times} & \\
\Rightarrow & \quad \text{axiom} \\
& \quad \Rightarrow A^\perp, A \\
\text{\textcircled{$\perp$}-R three times} & \\
\Rightarrow & \quad \text{axiom} \\
& \quad \Rightarrow A^\perp, A \\
\text{\textcircled{$\perp$}-R} & \\
\Rightarrow & \quad \text{axiom} \\
& \quad \Rightarrow A^\perp, A \\
\text{\textcircled{$\perp$}-R} & \\
\Rightarrow & \quad \text{axiom} \\
& \quad \Rightarrow A^\perp, A \\
\text{\textcircled{$\perp$}-R twice} & \\
\Rightarrow & \quad \text{axiom} \\
& \quad \Rightarrow A^\perp, A \\
\text{\textcircled{$\perp$}-R twice} & \\
\Rightarrow & \quad \text{axiom} \\
& \quad \Rightarrow A^\perp, A \\
\text{\textcircled{$\perp$}-R} & \\
\Rightarrow & \quad \text{axiom} \\
& \quad \Rightarrow A^\perp, A \\
\text{\textcircled{$\perp$}-R three times} &
\end{align*}

6 marks

10 marks
(c) Consider the fragment of the above sequent calculus for (one sided) MLL containing only axioms and the cut rule in the following form:
\[
\frac{\Rightarrow \Gamma, A \perp}{\Rightarrow \Gamma, \Delta} \quad \text{cut}
\]
Does cut-elimination for this fragment enjoy the strong normalization and the Church Rosser property? Explain informally your answer.

Solution: We need a Lemma:

**Lemma 1:** Every derivation in the axiom-cut fragment of MLL (without units) consists of occurrences of the same sequent \( \Rightarrow A \perp, A \), for a given formula \( A \).

The proof is by induction on the proof-tree: in the base case, the derivation is an axiom and the result is clear. Assuming the Lemma true for the derivations of the premises \( \Rightarrow X \perp, X \) and \( \Rightarrow Y \perp, Y \) of a cut, since the cut-formulas must be one the linear negation of the other we must have \( X = Y \). Hence there is a formula \( A \) such that \( X = A = Y \) for all formula occurrences in the derivation.

From the Lemma it follows that any step of cut-elimination in the axiom-cut fragment reduces the number of sequents, and eventually the derivation reduces to exactly one sequent \( \Rightarrow A \perp, A \). Hence the Church-Rosser property follows.

9 marks

(d) Extend the fragment in (c) adding the new structural rule of MIX:
\[
\frac{\Rightarrow \Gamma \Rightarrow \Delta}{\Rightarrow \Gamma, \Delta} \quad \text{mix}
\]
and also the rule of Exchange.

Does cut-elimination for this fragment enjoy the strong normalization and the Church Rosser property? Explain informally your answer.

Solution: We need a Lemma:

**Lemma 2.** Every derivation in axiom-cut-mix-MLL can be transformed into one where all applications of mix occur below all applications of cut.

The proof is by induction on the number of mix inferences occurring above a cut. It is plain that the following commutation is permissible:
\[
\frac{D_1}{\Rightarrow \Gamma} \quad \frac{D_2}{\Rightarrow \Delta, A} \quad \frac{D_3}{\Rightarrow \Pi, A \perp} \quad \text{cut}
\]
reduces to
\[
\frac{\Rightarrow \Gamma, \Delta, \Pi}{\Rightarrow \Gamma, \Delta, A \perp} \quad \text{cut}
\]

7
A derivation $D$ resulting from an application of Lemma 2 is not necessarily unique: we can indeed permute also mix inferences with each others. Thus to obtain the Church-Rosser property we may need also several applications of the following Lemma:

**Lemma 3.** In a derivation in the fragment axiom-cut-mix-MLL any two applications of mix can be permuted with each other.

The proof is obvious, by iterating the following commutation:

$$
\begin{array}{c}
D_1 \xrightarrow{mix} \Gamma \\
D_2 \xrightarrow{\text{mix}} \Delta \\
D_3 \xrightarrow{\text{mix}} \Pi \\
\end{array}
$$

reduces to

$$
\begin{array}{c}
\Rightarrow \Gamma, \Delta, \Pi \\
\Rightarrow \Gamma, \Delta, \Pi \\
\Rightarrow \Gamma, \Delta, \Pi \\
\end{array}
$$

(A more elegant formulation of the same results is in term of proof nets.)

10 marks

TOTAL: 35 marks
QUESTION 4

(a) What does it mean to say that a category $C$ has binary products?

For an answer, look at the lecture notes on Categorical Logic.

7 marks

(b) Verify that the collection $\text{Pset}$ having sets as objects and partial functions as morphisms forms a category. [Hint: Notice that for any sets $A$ and $B$ there is a totally undefined partial function $\text{empty} : A \to B$. Can the identity $id_A$ be partial?]

Solution: Given sets $A$, $B$, $C$, the composition $g \circ f : A \to C$ of two partial functions $f : A \rightarrow B$ and $g : B \rightarrow C$ is defined as usual in set theory and is a partial function. Set-theoretic composition is associative for partial functions as well as for total functions. The total identity function is also a partial function and is the identity for composition:

$$id_B \circ f = f = f \circ id_A : A \to B.$$ 

8 marks

(c) Does $\text{Pset}$ have binary products? [Hint: Consider the pair of functions $f : C \to A$ and $\text{empty} : C \to \emptyset$, where $f$ is not totally undefined. What is $A \times \emptyset$? Can we have $f = \pi_0 \circ (f, \text{empty})$?]

Answer: No. In set theory, $A \times \emptyset = \emptyset$, and the only partial function $h : A \to \emptyset$ is $\text{empty}$. Hence

$$\pi_A \circ (f, \text{empty}) = \pi_A \circ \text{empty} = \text{empty} \neq f$$

Hence the cartesian product of two sets is not a categorical product in $\text{Pset}$.

10 marks

TOTAL: 25 marks

END OF PRE-EXAM