1 Computational Logic 2008 - Dr G.Bellin

Solution of Pre-Examination

17th December 2008

Answer the following four questions. Questions 1 and 4 carry 25 marks. Questions 2 and 3 carry 35 marks. Marks above 100 are bonus for the final mark.

QUESTION 1. Consider the language of modal logic

$$A \ := \qquad P \mid \bot \mid A_1 \to A_2 \mid \Box A$$

Extend the sequent calculus system for classical logic **G3C** with the following rules for the modal system **S4**:

$$\frac{\Box\Gamma \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A, \Lambda}\ \Box\text{-R} \qquad \frac{A, \Box A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\ \Box\text{-L}$$

(a) Consider a Kripke model $\mathcal{M} = (W, R, \Vdash)$, where W is a set of possible worlds, $R \subseteq W \times W$ is the accessibility relation and $\Vdash \subseteq W \times \textbf{Atoms}$. **Answer** the following questions:

(a.1) What is the *frame* of $\mathcal{M}$?

1 mark

Answer: The frame of $\mathcal{M}$ is the structure (W, R), the set of possible worlds and the accessibility relation.

(a.2) What does it mean to say that a sentence A is *valid in* (or *true in*) $\mathcal{M}$?

2 marks

Answer: A is *valid in* (or *true in*) $\mathcal{M}$ if for all $w \in W$ we have $w \Vdash A$ (equivalently, $\mathcal{V}(w, A) = T$, where $\mathcal{V}$ is a valuation relatiized to the possible world w). [The relation w *forces* A is defined inductively as a propositional valuation $\mathcal{V}$ relativized to each possible world, setting $w \Vdash \Box B$ if and only if for all $w' \in W$ such that wRw' we have $w' \Vdash B$.]

(a.3) What does it mean to say that a sentence A is *valid in* **K**?

2 marks

Answer: A is *valid in* **K** if for all models $\mathcal{M}$ A is true in $\mathcal{M}$.

(a.4) What does it mean to say that a sentence A is *valid in* **S4**?

2 marks

Answer: A is *valid in* **S4** if A is true in all models $\mathcal{M} = (W, R, \Vdash)$ where R is reflexive and transitive (a *preorder*).

Consider the following sequents:

(i) S_1: $\Rightarrow \Box\big(\Box(\Box A \to B) \to A\big) \to A$

(ii) S_2: $\Rightarrow \Box\big(\Box(\Box A \to B) \to A\big) \to \Box\neg\Box\neg A$

(a) **Question**: are S_1 or S_2 valid in **S4**? (*yes* or *no* answer)

2 marks

Answer: S_1 is not valid, S_2 is valid.

(b) If the sequent S_1 or S_2 is falsifiable, **define** a Kripke model $(W, R, \Vdash)$ with a world $w \in W$ such that $w \nVdash S_i$. Otherwise, **write** a derivation of S_i in the sequent calculus for **S4**.

16 marks

Answer: (i) We need a model $(W, R, \Vdash)$ and a world $w_0 \in W$ such that

(a) $w_0 \Vdash \Box\big(\Box(\Box A \to B) \to A\big)$ and

(b) $w_0 \nVdash A$.

Since $w_0 \nVdash A$ and R is reflexive, we have $w_0 \nVdash \Box A$ hence $w_0 \Vdash \Box A \to B$. Since we have $w_0 \nVdash A$, the only possibility for $\Box(\Box A \to B) \to A$ to be true in w_0 is that in some world w' accessible from w_0 we have $w' \nVdash \Box A \to B$. But w' cannot be w_0. Hence suppose there exists w_1 such that $w_0 R w_1$ and

(c) $w_1 \Vdash \Box A$ and $w_1 \nVdash B$.

Since $w_1 R w_1$, we must have

(d) $w_1 \Vdash A$.

Thus we can let $(W, R, \Vdash)$ where $W = \{w_0, w_1\}$, R is the reflexive and transitive closure of $w_0 R w_1$ and $\Vdash$ satisfies (b), (c) and (d).

Answer (ii): $\Rightarrow \Box\big(\Box(\Box A \to B) \to A\big) \to \Box\neg\Box\neg A$ is derivable as follows:

$$
\begin{prooftree}
\AxiomC{\textit{axiom}}
\noLine
\UnaryInfC{$\ldots A \Rightarrow A \ldots$}
\RightLabel{$\neg$-L}
\UnaryInfC{$\Box\big(\Box(\Box A \to B) \to A\big), \neg A, \Box\neg A, A, \Box A \Rightarrow B$}
\RightLabel{$\Box$-L twice}
\UnaryInfC{$\Box\big(\Box(\Box A \to B) \to A\big), \Box\neg A, \Box A \Rightarrow B$}
\RightLabel{$\to$-R}
\UnaryInfC{$\Box\big(\Box(\Box A \to B) \to A\big), \Box\neg A, \Rightarrow \Box A \to B$}
\RightLabel{$\Box$-R}
\UnaryInfC{$\Box\big(\Box(\Box A \to B) \to A\big), \Box\neg A, \Rightarrow \Box(\Box A \to B)$}
\AxiomC{\textit{axiom}}
\noLine
\UnaryInfC{$A \Rightarrow A$}
\UnaryInfC{$A, \neg A \Rightarrow$}
\RightLabel{$\to$-L}
\BinaryInfC{$\Box(\Box A \to B) \to A, \Box\big(\Box(\Box A \to B) \to A\big), \neg A, \Box\neg A \Rightarrow$}
\RightLabel{$\Box$-L twice}
\UnaryInfC{$\Box\big(\Box(\Box A \to B) \to A\big), \Box\neg A \Rightarrow$}
\RightLabel{$\neg$-R}
\UnaryInfC{$\Box\big(\Box(\Box A \to B) \to A\big) \Rightarrow \neg\Box\neg A$}
\RightLabel{$\Box$-R}
\UnaryInfC{$\Box\big(\Box(\Box A \to B) \to A\big) \Rightarrow \Box\neg\Box\neg A$}
\RightLabel{$\to$-R}
\UnaryInfC{$\Rightarrow \Box\big(\Box(\Box A \to B) \to A\big) \to \Box\neg\Box\neg A$}
\end{prooftree}
$$

TOTAL: 25 marks

QUESTION 2. (a) Consider the language of *classical logic* in the form:

$$A \;:=\; \quad P \mid \neg P \mid A_1 \wedge A_2 \mid A_1 \vee A_2$$

Consider the sequent calculus system for classical logic (one sided) **G3C** with the following axioms and rules:

STRUCTURAL RULE

$$\frac{\Rightarrow \Gamma, B, A, \Delta}{\Rightarrow \Gamma, A, B, \Delta}\ \textit{Exchange}$$

IDENTITY

$$\frac{\text{axiom}}{\Rightarrow \Gamma, A, \neg A}$$

LOGICAL RULES

$$\frac{\Rightarrow \Gamma, A \qquad \Rightarrow \Gamma, B}{\Rightarrow \Gamma, A \wedge B}\ \wedge\text{-R} \qquad\qquad \frac{\Rightarrow \Gamma, A, B}{\Rightarrow \Gamma, A \vee B}\ \vee\text{-R}$$

Consider the following sequents:

(iii) S_3: $\Rightarrow \neg A \vee (\neg B \wedge \neg C), (A \wedge B) \vee (A \wedge C)$;

(iv) S_4: $\Rightarrow \neg A \vee (\neg B \wedge \neg C), (A \wedge B) \vee C)$;

(v) S_5: $\Rightarrow (\neg A \vee \neg B) \wedge \neg C, (A \wedge B) \vee C$.

Are they derivable? If yes, **write** a derivation; otherwise, **write** a truth value assignment that makes the sequent false.

Answer: The following are proofs of (iii), (iv) and (v):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{axiom}}{\Rightarrow \neg A, \ldots, A} \qquad \overset{\text{axiom}}{\Rightarrow \neg B, \ldots, B}}{\Rightarrow \neg A, \neg B, A \wedge C, A \wedge B}\ \wedge\text{-R} \qquad \dfrac{\overset{\text{axiom}}{\Rightarrow \neg A, \ldots, A} \qquad \overset{\text{axiom}}{\Rightarrow \neg C, \ldots, C}}{\Rightarrow \neg A, \neg C, A \wedge B, A \wedge C}\ \wedge\text{-R}}{\Rightarrow \neg A, \neg B \wedge \neg C, A \wedge B, A \wedge C}\ \wedge\text{-R}}{\Rightarrow \neg A \vee (\neg B \wedge \neg C), (A \wedge B) \vee (A \wedge C)}\ \vee\text{-R twice}}{}
$$

$$
\dfrac{\dfrac{\dfrac{\overset{\text{axiom}}{\Rightarrow \neg A, \neg B, A, C} \qquad \overset{\text{axiom}}{\Rightarrow \neg A, \neg B, B, C}}{\Rightarrow \neg A, \neg B, A \wedge B, C}\ \wedge\text{-R} \qquad \overset{\text{axiom}}{\Rightarrow \neg A, \neg C, A \wedge B, A}}{\Rightarrow \neg A, \neg B \wedge \neg C, A \wedge B, C}\ \wedge\text{-R}}{\Rightarrow \neg A \vee (\neg B \wedge \neg C), (A \wedge B) \vee C}\ \vee\text{-R twice}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{axiom}}{\Rightarrow \neg A, \neg B, A, C} \qquad \overset{\text{axiom}}{\Rightarrow \neg A, \neg B, B, C}}{\Rightarrow \neg A, \neg B, A \wedge B, C}\ \wedge\text{-R}}{\Rightarrow \neg A \vee \neg B, A \wedge B, C}\ \vee\text{-R} \qquad \overset{\text{axiom}}{\Rightarrow \neg C, A \wedge B, C}}{\Rightarrow (\neg A \vee \neg B) \wedge C, A \wedge B, C}\ \wedge\text{-R}}{\Rightarrow (\neg A \vee \neg B) \wedge C, (A \wedge B) \vee C}\ \vee\text{-R}
$$

15 marks

Consider the sequent calculus for classical logic (one sided) **G1C** (provably equivalent to **G3C**) with *explicit rules of Contraction and Weakening* and with *axioms* and *cut rule* of the following forms:

$$
\dfrac{\text{axiom}}{\Rightarrow A, \neg A} \qquad\qquad \dfrac{\Rightarrow \Gamma, \neg A \qquad \Rightarrow A, \Delta}{\Rightarrow \Gamma, \Delta}\ \textit{cut}
$$

Consider the derivation $\mathcal{D}$:

$$
\dfrac{\dfrac{\Rightarrow B, \neg B}{\Rightarrow B, \neg B, \neg A}\ \textit{weakening} \qquad \dfrac{\Rightarrow C, \neg C}{\Rightarrow A, C, \neg C}\ \textit{weakening}}{B, \neg B, C, \neg C}\ \textit{cut}
$$

(b) **Question:** How many ways are there to eliminate the indicated *cut*? **Write** all the cut-free derivations.

Answer: [Remember that each step in the algorithm of cut-elimination consists either (i) in replacing a *cut* inference with *cuts* of lower complexity (*logical reductions*) or (ii) permuting a *cut* inference with the final inference $\mathcal{I}_i$ in one of the two derivations of the sequent-premises of the *cut* (*permutative conversion*), if a logical reduction cannot be applied because $\mathcal{I}_i$ does not introduce a cut-formula, or (iii) in eliminating the *cut* inference altogether, if one of the premises of the *cut* is an axiom or the conclusion of a *weakening*. Notice also that steps (ii) and (iii) are *non-deterministic* when the lowermost inferences $\mathcal{I}$ in both derivations of the *cut* premises do not introduce the cut formula, or are axioms or *weakenings*. Figuratively, we can say that in these cases we have a choice between *pushing the cut up in the left or in the right sub-derivation.*]

In our example, we can either *push the cut up* in the left or in the right premise we obtain two distinct derivations:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \textit{axiom}\\ B, \neg B\end{array}}{B, \neg B, C, \neg C}\ \textit{weakening}\text{ twice} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \textit{axiom}\\ C, \neg C\end{array}}{B, \neg B, C, \neg C}\ \textit{weakening}\text{ twice}$$

Since we do not have a *generalized weakening rule*, introducing several formulas, derivation $\mathcal{D}_1$ really corresponds to two derivations $\mathcal{D}_{1,1}$ and $\mathcal{D}_{1,2}$, depending on whether it is C or $\neg C$ that is introced first, and similarly for $\mathcal{D}_2$: thus in conclusion we do have four cut-free derivations.

The difference between $\mathcal{D}_{1,1}$ and $\mathcal{D}_{1,2}$ is relatively unimportant: it is about the order in which we introduce irrelevance. On the contrary, $\mathcal{D}_1$ and $\mathcal{D}_2$ are essentially diferent, as they differ for what is relevant, namely the two formulas whose connection defines a logical axiom, and what is irrelevant, i.e., introduced by *weakening*.

10 marks

(c) **Does** cut-elimination for **C1C** enjoy the Church-Rosser property? **Explain.**

Answer: [Notice that here the computation process is *cut-elimination*, as in the lambda calculus it is β-reduction.] To say that cut-elimination has the Church-Rosser property is to say that given a derivation $\mathcal{D}$ which reduces by cut-elimination to either $\mathcal{D}_1$ and $\mathcal{D}_2$, we can find a $\mathcal{D}_3$ such that both $\mathcal{D}_1$ and $\mathcal{D}_2$ reduce to $\mathcal{D}_3$ by cut-elimination. But for the derivations $\mathcal{D}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ considered above we have that $\mathcal{D}_1$ and $\mathcal{D}_2$ are cut-free and essentially different. Thus there can be no $\mathcal{D}_3$ which $\mathcal{D}_1$ and $\mathcal{D}_2$ reduce to. This shows that the process of cut-elimination in classical logic **G1C** does not enjoy the Church-Rosser property.

10 marks

TOTAL: 35 marks

QUESTION 3. Consider the language of **MLL** classical *multiplicative linear logic* (without units):

$$A \ := \qquad P \mid P^{\perp} \mid A_1 \otimes A_2 \mid A_1 \wp A_2$$

Consider the sequent calculus system for classical **MLL** with the following rules:

STRUCTURAL RULE

$$\frac{\Rightarrow \Gamma, B, A, \Delta}{\Rightarrow \Gamma, A, B, \Delta}\ \textit{Exchange}$$

IDENTITY

$$\frac{\text{axiom}}{\Rightarrow A, A^{\perp}}$$

LOGICAL RULES

$$\frac{\Rightarrow \Gamma, A \qquad \Rightarrow \Delta, B}{\Rightarrow \Gamma, \Delta, A \otimes B}\ \otimes\text{-R} \qquad\qquad \frac{\Rightarrow \Gamma, A, B}{\Rightarrow \Gamma, A \wp B}\ \wp\text{-R}$$

Consider the following sequents:

(vi) S_6: $\Rightarrow A^{\perp} \wp (B^{\perp} \otimes C^{\perp}), (A \otimes B) \wp (A \otimes C)$

(vii) S_7: $\Rightarrow (A^{\perp} \wp A^{\perp}) \wp (B^{\perp} \otimes C^{\perp}), (A \otimes B) \wp (A \otimes C)$

(viii) S_8: $\Rightarrow A^{\perp} \wp (B^{\perp} \otimes C^{\perp}), (A \otimes B) \wp C$

(a) Are they derivable? For each of S_6-S_8 **answer** *yes or no.*

Answer: S_6 is not derivable: this can be seen as a corollary of the cut-elimination theorem for **MLL**, but the proof is not required. S_7 and S_8 are derivable.

6 marks

(b) If any one of S_6-S_8 is derivable, **write** a derivation of it.

$$\dfrac{\dfrac{\dfrac{\overset{\text{axiom}}{\Rightarrow A^{\perp}, A} \qquad \overset{\text{axiom}}{\Rightarrow B^{\perp}, B}}{\Rightarrow A^{\perp}, B^{\perp}, A \otimes B}\ \otimes\text{-R} \qquad \dfrac{\overset{\text{axiom}}{\Rightarrow A^{\perp}, A} \qquad \overset{\text{axiom}}{\Rightarrow C^{\perp}, C}}{\Rightarrow A^{\perp}, C^{\perp}, A \otimes C}\ \otimes\text{-R}}{\Rightarrow A^{\perp}, A^{\perp}, B^{\perp} \otimes C^{\perp}, A \otimes B, A \otimes C}\ \otimes\text{-R}}{\Rightarrow (A^{\perp} \wp A^{\perp}) \wp (B^{\perp} \otimes C^{\perp}), (A \otimes B) \wp (A \otimes C)}\ \wp\text{-R three times}$$

$$\dfrac{\dfrac{\dfrac{\overset{\text{axiom}}{\Rightarrow A^{\perp}, A} \qquad \overset{\text{axiom}}{\Rightarrow B^{\perp}, B}}{\Rightarrow A^{\perp}, B^{\perp}, A \otimes B}\ \otimes\text{-R} \qquad \overset{\text{axiom}}{\Rightarrow A^{\perp}, A}}{\Rightarrow A^{\perp}, B^{\perp} \otimes C^{\perp}, A \otimes B, C}\ \otimes\text{-R}}{\Rightarrow A^{\perp} \wp (B^{\perp} \otimes C^{\perp}), (A \otimes B) \wp C}\ \wp\text{-R twice}$$

10 marks

(c) Consider the fragment of the above sequent calculus for (one sided) **MLL** containing only *axioms* and the *cut rule* in the following form:

$$\frac{\Rightarrow \Gamma, A^{\perp} \qquad \Rightarrow A, \Delta}{\Rightarrow \Gamma, \Delta}\, cut$$

Does cut-elimination for this fragment enjoy the strong normalization and the Church Rosser property? **Explain** informally your answer.

Solution: We need a Lemma:

Lemma 1: *Every derivation in the* axiom-cut *fragment of* **MLL** (without units) *consists of occurrences of the same sequent* $\Rightarrow A^{\perp}, A$*, for a given formula* A.

The proof is by induction on the proof-tree: in the base case, the derivation is an axiom and the result is clear. Assuming the Lemma true for the derivations of the premises $\Rightarrow X^{\perp}, X$ and $\Rightarrow Y^{\perp}, Y$ of a cut, since the cut-formulas must be one the linear negation of the other we must have $X = Y$. Hence there is a formula A such that $X = A = Y$ for all formula occurrences in the derivation.

From the Lemma it follows that any step of cut-elimination in the *axiom-cut* fragment reduces the number of sequents, and eventually the derivation reduces to exactly one sequent $\Rightarrow A^{\perp}, A$. Hence the Church-Rosser property follows.

9 marks

(d) Extend the fragment in (c) adding the new *structural rule* of MIX:

$$\frac{\Rightarrow \Gamma \qquad \Rightarrow \Delta}{\Rightarrow \Gamma, \Delta}\, mix$$

and also the rule of Exchange.

Does cut-elimination for this fragment enjoy the strong normalization and the Church Rosser property? **Explain** informally your answer.

Solution: We need a Lemma:

Lemma 2. *Every derivation in* axiom-cut-mix-**MLL** *can be transformed into one where all applications of* mix *occur below all applications of* cut.

The proof is by induction on the number of *mix* inferences occurring above a *cut*. It is plain that the following commutation is permissible:

$$mix\,\frac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Rightarrow \Gamma\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \Rightarrow \Delta, A\end{matrix}}{\Rightarrow \Gamma, \Delta, A} \qquad \begin{matrix}\mathcal{D}_3\\ \Rightarrow \Pi, A^{\perp}\end{matrix}}{\Rightarrow \Gamma, \Delta, \Pi}\, cut \qquad \text{reduces to}$$

$$mix\,\frac{\begin{matrix}\mathcal{D}_1\\ \Rightarrow \Gamma\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_2\\ \Rightarrow \Delta, A\end{matrix} \qquad \begin{matrix}\mathcal{D}_3\\ \Rightarrow \Pi, A^{\perp}\end{matrix}}{\Rightarrow \Delta, \Pi}\, cut}{\Rightarrow \Gamma, \Delta, \Pi}$$

A derivation $\mathcal{D}$ resulting from an application of Lemma 2 is not necessarily unique: we can indeed permute also *mix* inferences with each others. Thus to obtain the Church-Rosser property we may need also several applications of the following Lemma:

Lemma 3. *In a derivation in the fragment* axiom-cut-mix-**MLL** *any two applications of* mix *can be permuted with each other.*

The proof is obvious, by iterating the following commutation:

$$\dfrac{\mathit{mix}\,\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \Rightarrow \Gamma & \Rightarrow \Delta\end{array}}{\Rightarrow \Gamma, \Delta} \qquad \begin{array}{c}\mathcal{D}_3\\ \Rightarrow \Pi\end{array}}{\Rightarrow \Gamma, \Delta, \Pi}\,\mathit{mix} \qquad \text{reduces to}$$

$$\mathit{mix}\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Rightarrow \Gamma\end{array} \qquad \dfrac{\begin{array}{cc}\mathcal{D}_2 & \mathcal{D}_3\\ \Rightarrow \Delta & \Rightarrow \Pi\end{array}}{\Rightarrow \Delta, \Pi}\,\mathit{cut}}{\Rightarrow \Gamma, \Delta, \Pi}$$

(A more elegant formulation of the same results is in term of *proof nets*.)

10 marks

TOTAL: 35 marks

QUESTION 4 (a) What does it mean to say that a category $\mathcal{C}$ has binary products?

For an answer, look at the lecture notes on Categorical Logic.

7 marks

(b) **Verify** that the collection **Pset** having sets as objects and *partial functions* as morphisms forms a category. [*Hint:* Notice that for any sets A and B there is a *totally undefined* partial function $\texttt{empty} : A \rightharpoonup B$. Can the identity id_A be partial?]

Solution: Given sets A, B, C, the composition $g \circ f : A \rightharpoonup C$ of two partial functions $f : A \rightharpoonup B$ and $g : B \rightharpoonup C$ is defined as usual in set theory and is a partial function. Set-theoretic composition is associative for partial functions as well as for total functions. The total identity function is also a partial function and is the identity for composition:

$$id_B \circ f = f = f \circ id_A : A \rightharpoonup B.$$

8 marks

(c) Does **Pset** have binary products? [*Hint:* Consider the pair of functions $f : C \to A$ and $\texttt{empty} : C \rightharpoonup \emptyset$, where f is not totally undefined. What is $A \times \emptyset$? Can we have $f = \pi_0 \circ \langle f, \texttt{empty} \rangle$?]

Answer: No. In set theory, $A \times \emptyset = \emptyset$, and the only partial function $h : A \rightharpoonup \emptyset$ is $\texttt{empty}$. Hence

$$\pi_A \circ \langle f, \texttt{empty} \rangle = \pi_A \circ \texttt{empty} = \texttt{empty} \neq f$$

Hence the cartesian product of two sets is not a categorical product in **Pset**.

10 marks

TOTAL: 25 marks

END OF PRE-EXAM