1 Computational Logic 2008 - Dr G.Bellin

Pre-Examination

17th December 2008 - Time allowed: 2 hours

Answer the following four questions. Questions 1 and 4 carry 25 marks. Questions 2 and 3 carry 35 marks. Marks above 100 are bonus for the final mark.

QUESTION 1. Consider the language of modal logic

$$A \;:=\; \qquad P \mid \bot \mid A_1 \to A_2 \mid \Box A$$

Extend the sequent calculus system for classical logic **G3C** with the following rules for the modal system **S4**:

$$\frac{\Box\Gamma \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A, \Lambda}\ \Box\text{-R} \qquad\qquad \frac{A, \Box A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\ \Box\text{-L}$$

(a) Consider a Kripke model $\mathcal{M} = (W, R, \Vdash)$, where W is a set of possible worlds, $R \subseteq W \times W$ is the accessibility relation and $\Vdash \subseteq W \times \mathbf{Atoms}$. **Answer** the following questions:

a.1 What is the *frame* of $\mathcal{M}$?

1 mark

a.2 What does it mean to say that a sentence A is *valid in* (or *true in* $\mathcal{M}$?

2 marks

a.3 What does it mean to say that a sentence A is *valid in* **K**?

2 marks

a.4 What does it mean to say that a sentence A is *valid in* **S4**?

2 marks

Consider the following sequents:

(i) S_1: $\Rightarrow \Box\big(\Box(\Box A \to B) \to A\big) \to A$

(ii) S_2: $\Rightarrow \Box\big(\Box(\Box A \to B) \to A\big) \to \Box\neg\Box\neg A$

(a) **Answer**: are S_1 or S_2 valid in **S4**? (*yes* or *no* answer)

2 marks

(b) If the sequent S_1 or S_2 is falsifiable, **define** a Kripke model $(W, R, \Vdash)$ with a world $w \in W$ such that $w \not\Vdash S_i$. Otherwise, **write** a derivation of S_i in the sequent calculus for **S4**.

16 marks

TOTAL: 25 marks

QUESTION 2. (a) Consider the language of *classical logic* in the form:

$$A \ := \qquad P \mid \neg P \mid A_1 \wedge A_2 \mid A_1 \vee A_2$$

Consider the sequent calculus system for classical logic (one sided) **G3C** with the following axioms and rules:

STRUCTURAL RULE

$$\frac{\Rightarrow \Gamma, B, A, \Delta}{\Rightarrow \Gamma, A, B, \Delta}\ \textit{Exchange}$$

IDENTITY

$$\frac{\text{axiom}}{\Rightarrow \Gamma, A, \neg A}$$

LOGICAL RULES

$$\frac{\Rightarrow \Gamma, A \qquad \Rightarrow \Gamma, B}{\Rightarrow \Gamma, A \wedge B}\ \wedge\text{-R} \qquad\qquad \frac{\Rightarrow \Gamma, A, B}{\Rightarrow \Gamma, A \vee B}\ \vee\text{-R}$$

Consider the following sequents:

(iii) S_3: $\Rightarrow \neg A \vee (\neg B \wedge \neg C), (A \wedge B) \vee (A \wedge C)$;

(iv) S_4: $\Rightarrow \neg A \vee (\neg B \wedge \neg C), (A \wedge B) \vee C)$;

(v) S_5: $\Rightarrow (\neg A \vee \neg B) \wedge \neg C, (A \wedge B) \vee C$.

Are they derivable? If yes, **write** a derivation; otherwise, **write** a truth value assignment that makes the sequent false.

15 marks

(c) Consider the sequent calculus for classical logic (one sided) **G1C** (provably equivalent to **G3C**) with *explicit rules of Contraction and Weakening* and with *axioms* and *cut rule* of the following forms:

$$\frac{\text{axiom}}{\Rightarrow A, \neg A} \qquad\qquad \frac{\Rightarrow \Gamma, \neg A \qquad \Rightarrow A, \Delta}{\Rightarrow \Gamma, \Delta}\ \textit{cut}$$

Consider the derivation

$$\dfrac{\dfrac{\Rightarrow B, \neg B}{\Rightarrow B, \neg B, \neg A}\ \textit{weakening} \qquad \dfrac{\Rightarrow C, \neg C}{\Rightarrow A, C, \neg C}\ \textit{weakening}}{B, \neg B, C, \neg C}\ \textit{cut}$$

(b) How many ways are there to eliminate the indicated *cut*? **Write** all the cut-free derivations.

10 marks

(c) **Does** cut-elimination for **C1C** enjoy the Church-Rosser property? **Explain.**

10 marks

TOTAL: 35 marks

QUESTION 3. Consider the language of **MLL** classical *multiplicative linear logic* (without units):

$$A \ := \qquad P \mid P^{\perp} \mid A_1 \otimes A_2 \mid A_1 \wp A_2$$

Consider the sequent calculus system for classical **MLL** with the following rules:

STRUCTURAL RULE

$$\frac{\Rightarrow \Gamma, B, A, \Delta}{\Rightarrow \Gamma, A, B, \Delta}\ \textit{Exchange}$$

IDENTITY

$$\frac{\text{axiom}}{\Rightarrow A, A^{\perp}}$$

LOGICAL RULES

$$\frac{\Rightarrow \Gamma, A \qquad \Rightarrow \Delta, B}{\Rightarrow \Gamma, \Delta, A \otimes B}\ \otimes\text{-R} \qquad\qquad \frac{\Rightarrow \Gamma, A, B}{\Rightarrow \Gamma, A \wp B}\ \wp\text{-R}$$

Consider the following sequents:

(vi) S_6: $\Rightarrow A^{\perp} \wp (B^{\perp} \otimes C^{\perp}), (A \otimes B) \wp (A \otimes C)$

(vii) S_7: $\Rightarrow (A^{\perp} \wp A^{\perp}) \wp (B^{\perp} \otimes C^{\perp}), (A \otimes B) \wp (A \otimes C)$

(viii) S_8: $\Rightarrow A^{\perp} \wp (B^{\perp} \otimes C^{\perp}), (A \otimes B) \wp C$

(a) Are they derivable? For each of S_6-S_8 **answer** *yes or no.*

6 marks

(b) If any one of S_6-S_8 is derivable, **write** a derivation of it.

10 marks

(c) Consider the fragment of the above sequent calculus for (one sided) **MLL** containing only *axioms* and the *cut rule* in the following form:

$$\frac{\Rightarrow \Gamma, A^{\perp} \qquad \Rightarrow A, \Delta}{\Rightarrow \Gamma, \Delta}\ \textit{cut}$$

Does cut-elimination for this fragment enjoy the strong normalization and the Church Rosser property? **Explain** informally your answer.

9 marks

(d) Extend the fragment in (c) adding the new *structural rule* of MIX:

$$\frac{\Rightarrow \Gamma \qquad \Rightarrow \Delta}{\Rightarrow \Gamma, \Delta}\ \textit{MIX}$$

and also the rule of Exchange.

Does cut-elimination for this fragment enjoy the strong normalization and the Church Rosser property? **Explain** informally your answer.

10 marks

TOTAL: 35 marks

QUESTION 4 (a) What does it mean to say that a category $\mathcal{C}$ has binary products?

7 marks

(b) Verify that the collection **Pset** having sets as objects and *partial functions* as morphisms forms a category. [*Hint:* Notice that for any sets A and B there is a *totally undefined* partial function $\texttt{empty} : A \rightharpoonup B$. Can the identity id_A be partial?]

8 marks

(c) Does **Pset** have binary products? [*Hint:* Consider the pair of functions $f : C \to A$ and $\texttt{empty} : C \rightharpoonup \emptyset$, where f is total. What is $C \times \emptyset$? Can we have $f = \pi_0 \circ \langle f, \texttt{empty} \rangle$?]

10 marks
TOTAL: 25 marks

END OF PRE-EXAM