1 Computational Logic 2008 - Dr G. Bellin

Pre-Examination
17th December 2008 - Time allowed: 2 hours

Answer the following four questions. Questions 1 and 4 carry 25 marks.
Questions 2 and 3 carry 35 marks. Marks above 100 are bonus for the final mark.

QUESTION 1. Consider the language of modal logic

\[ A := P \mid \bot \mid A_1 \rightarrow A_2 \mid \square A \]

Extend the sequent calculus system for classical logic G3C with the following
rules for the modal system S4:

\[
\frac{\square \Gamma \Rightarrow A}{\Pi, \square \Gamma \Rightarrow \square A, \Delta} \quad \square \text{-R}
\]

\[
\frac{A, \square A, \Gamma \Rightarrow \Delta}{\square A, \Gamma \Rightarrow \Delta} \quad \square \text{-L}
\]

(a) Consider a Kripke model \( M = (W, R, \models) \), where \( W \) is a set of possible worlds, \( R \subseteq W \times W \) is the accessibility relation and \( \models \subseteq W \times \text{Atoms} \).

Answer the following questions:

a.1 What is the \textit{frame} of \( M \)?  

\[ 1 \text{ mark} \]

a.2 What does it mean to say that a sentence \( A \) is \textit{valid} in \( M \)?

\[ 2 \text{ marks} \]

a.3 What does it mean to say that a sentence \( A \) is \textit{valid in K}?  

\[ 2 \text{ marks} \]

a.4 What does it mean to say that a sentence \( A \) is \textit{valid in S4}?  

\[ 2 \text{ marks} \]

Consider the following sequents:

(i) \( S_1: \Rightarrow \square (\square A \rightarrow B) \rightarrow A \)  
(ii) \( S_2: \Rightarrow \square (\square A \rightarrow B) \rightarrow \square \neg \square \neg A \)

(a) \textbf{Answer}: are \( S_1 \) or \( S_2 \) valid in \( S4 \)? (\textit{yes} or \textit{no} answer)  

\[ 2 \text{ marks} \]

(b) If the sequent \( S_1 \) or \( S_2 \) is falsifiable, \textbf{define} a Kripke model \( (W, R, \models) \) with a world \( w \in W \) such that \( w \not\models S_i \). Otherwise, \textbf{write} a derivation of \( S_i \) in the sequent calculus for \( S4 \).

\[ 16 \text{ marks} \]

\textit{TOTAL:} 25 marks
QUESTION 2. (a) Consider the language of classical logic in the form:

\[ A := P \mid \neg P \mid A_1 \land A_2 \mid A_1 \lor A_2 \]

Consider the sequent calculus system for classical logic (one sided) \( G_3C \) with the following axioms and rules:

**STRUCTURAL RULE**

\[
\frac{\Rightarrow \Gamma, B, A, \Delta}{\Rightarrow \Gamma, A, B, \Delta} \quad \text{Exchange}
\]

**IDENTITY**

\[
\text{axiom} \quad \Rightarrow \Gamma, A, \neg A
\]

**LOGICAL RULES**

\[
\frac{\Rightarrow \Gamma, A}{\Rightarrow \Gamma, A \land B} \quad \land -R
\]

\[
\frac{\Rightarrow \Gamma, A}{\Rightarrow \Gamma, A \lor B} \quad \lor -R
\]

Consider the following sequents:

(iii) \( S_3: \Rightarrow \neg A \lor (\neg B \land \neg C), (A \land B) \lor (A \land C) \);

(iv) \( S_4: \Rightarrow \neg A \lor (\neg B \land \neg C), (A \land B) \lor C \);

(v) \( S_5: \Rightarrow (\neg A \lor \neg B) \land \neg C, (A \land B) \lor C \).

Are they derivable? If yes, write a derivation; otherwise, write a truth value assignment that makes the sequent false.

(c) Consider the sequent calculus for classical logic (one sided) \( G_1C \) (provably equivalent to \( G_3C \)) with explicit rules of Contraction and Weakening and with axioms and cut rule of the following forms:

\[
\frac{\text{axiom}}{\Rightarrow A, \neg A}
\]

\[
\frac{\Rightarrow \Gamma, \neg A}{\Rightarrow \Gamma, A, \Delta} \quad \text{cut}
\]

Consider the derivation

\[
\Rightarrow B, \neg B \quad \Rightarrow C, \neg C
\]

\[
\Rightarrow B, \neg B, \neg A \quad \Rightarrow A, C, \neg C
\]

\[
\Rightarrow B, \neg B, C, \neg C
\]

(b) How many ways are there to eliminate the indicated cut? Write all the cut-free derivations.

(c) Does cut-elimination for \( C_1C \) enjoy the Church-Rosser property? Explain.
QUESTION 3. Consider the language of **MLL** classical *multiplicative linear logic* (without units):

\[ A := P \mid P^\perp \mid A_1 \otimes A_2 \mid A_1 \wp A_2 \]

Consider the sequent calculus system for classical **MLL** with the following rules:

**STRUCTURAL RULE**

\[ \Rightarrow \Gamma, B, A, \Delta \quad \text{Exchange} \]
\[ \Rightarrow \Gamma, A, B, \Delta \]

**IDENTITY**

\[ \text{axiom} \quad \Rightarrow A, A^\perp \]

**LOGICAL RULES**

\[ \Rightarrow \Gamma, A \Rightarrow \Delta, B \quad \otimes\text{-R} \]
\[ \Rightarrow \Gamma, \Delta, A \otimes B \]
\[ \Rightarrow \Gamma, A, B \quad \wp\text{-R} \]
\[ \Rightarrow \Gamma, A \wp B \]

Consider the following sequents:

(vi) \( S_6: \Rightarrow A^\perp \wp (B^\perp \otimes C^\perp), (A \otimes B) \wp (A \otimes C) \)

(vii) \( S_7: \Rightarrow (A^\perp \wp A^\perp) \wp (B^\perp \otimes C^\perp), (A \otimes B) \wp (A \otimes C) \)

(viii) \( S_8: \Rightarrow A^\perp \wp (B^\perp \otimes C^\perp), (A \otimes B) \wp C \)

(a) Are they derivable? For each of \( S_6 - S_8 \) answer yes or no.

(b) If any one of \( S_6 - S_8 \) is derivable, write a derivation of it.

(c) Consider the fragment of the above sequent calculus for (one sided) **MLL** containing only *axioms* and the *cut rule* in the following form:

\[ \Rightarrow \Gamma, A^\perp \Rightarrow A, \Delta \quad \text{cut} \]

Does cut-elimination for this fragment enjoy the strong normalization and the Church Rosser property? Explain informally your answer.

(d) Extend the fragment in (c) adding the new *structural rule* of MIX:

\[ \Rightarrow \Gamma, \Delta \quad \text{MIX} \]

and also the rule of Exchange.

Does cut-elimination for this fragment enjoy the strong normalization and the Church Rosser property? Explain informally your answer.

**TOTAL: 35 marks**
QUESTION 4  (a) What does it mean to say that a category $\mathcal{C}$ has binary products?

7 marks

(b) Verify that the collection $\text{Pset}$ having sets as objects and partial functions as morphisms forms a category. [Hint: Notice that for any sets $A$ and $B$ there is a totally undefined partial function $\text{empty} : A \to B$. Can the identity $id_A$ be partial?]

8 marks

(c) Does $\text{Pset}$ have binary products? [Hint: Consider the pair of functions $f : C \to A$ and $\text{empty} : C \to \emptyset$, where $f$ is total. What is $C \times \emptyset$? Can we have $f = \pi_0 \circ (f, \text{empty})$? ]

10 marks

TOTAL: 25 marks

END OF PRE-EXAM