1 Computational Logic 2008 - Dr G.Bellin

Midterm

17th November 2008 - Time allowed: 2 hours

Answer the following four questions. Each question carries 25 marks.

QUESTION 1. (a) **Write** a derivation tree of

$$g : A \to A, y : A \vdash ((\mathbf{2})\lambda z^A.(g)z)y \ : A$$

in the simply typed λ-calculus. Here as usual $\mathbf{2}$ is $\lambda f \lambda x.(f)((f)x)$.
(At each line of the derivation please indicate both the term and its type).

5 marks

(b) **Write a reduction sequence** starting with the derivation of
$g : A \to A, y : A \vdash ((\mathbf{2})\lambda z^A.(g)z)y : A$ given in part (a).
(At each line of each derivation indicate both the term and its type. Do not reduce a lambda term if you do not reduce the tree).

15 marks

(c) In each derivation written in part (b) **count all the redexes** present in it (not only the one you are reducing).

5 marks
TOTAL: 25 marks

QUESTION 2. Consider the following derivation $\mathcal{D}$ in $\mathbf{NJ}^{\to\bot}$ of $\neg A, A \vdash B$:

$$\dfrac{\dfrac{\dfrac{A \to \bot \qquad A}{\bot}\,{\to}\text{-E}}{A \to B}\,\bot\text{-int} \qquad\qquad A}{B}\,{\to}\text{-E}$$

Answer the following questions:

(a) **Write** the definition of *Maximal Formula* in $\mathbf{NJ}^{\to}$. Is $\mathcal{D}$ normal with respect to of $\to$ reductions (i.e., elimination of maximal formulas of the form $X \to Y$)?

5 marks

(b) **Write** the definition of the *Subformula Property*. **Explain** why $\mathcal{D}$ does not enjoy the subformula property.

5 marks

(c) **Briefly describe** and **implement** a procedure that given $\mathcal{D}$ yields a derivation $\mathcal{D}^*$ of $\neg A, A \vdash B$ with the subformula property.

15 marks
TOTAL: 25 marks

QUESTION 3 Prove the *Weak Normalization Theorem* for intuitionistic implicative logic $\mathbf{NJ}^{\to}$:

> *There exists a reduction strategy that transforms every derivation in* $\mathbf{NJ}^{\to}$ *into a normal derivation.*

TOTAL: 25 marks

QUESTION 4. The relation ρ between λ-terms (*modulo* α conversion) is the smallest λ-compatible binary relation on Λ such that

$$t \ \rho \ t', u \ \rho \ u' \Rightarrow (\lambda x.u)t \ \rho \ u'[t'/x]$$

Give the definition of the Church-Rosser property and **prove** that ρ enjoys it.

You may assume **Lemma 11:**

(i) If $x \ \rho \ t'$, where x is a variable, then $x = t'$.

(ii) If $\lambda x.u \ \rho \ t'$, then $t' \equiv \lambda x.u'$ and $u \ \rho \ u'$

(iii) If $(v)u \ \rho \ t'$ then

  (a) either $t' \equiv (v')u'$ with $v \ \rho \ v'$ and $u \ \rho \ u'$;

  (b) or $v \equiv \lambda w$ and $t' \equiv w'[u'/x]$ with $u \ \rho \ u'$ and $w \ \rho \ w'$.

and also **Lemma 12:**
If $t \ \rho \ t'$ and $u \ \rho \ u'$ then $u[t/x] \ \rho \ u'[t'/u']$.
Proceed by induction on the complexity of t_0 such that $t_0 \ \rho \ t_1$ and $t_0 \ \rho \ t_2$

TOTAL: 25 marks

END OF MIDTERM