1 Computational Logic 2008 - Dr G. Bellin

Midterm

17th November 2008 - Time allowed: 2 hours

Answer the following four questions. Each question carries 25 marks.

QUESTION 1. (a) Write a derivation tree of
\[ g : A \to A, y : A \vdash ((2) \lambda z^A. (g) z) y : A \]
in the simply typed \( \lambda \)-calculus. Here as usual \( 2 \) is \( \lambda f \lambda x. (f)((f)x) \).
(At each line of the derivation please indicate both the term and its type).

(b) Write a reduction sequence starting with the derivation of
\[ g : A \to A, y : A \vdash ((2) \lambda z^A. (g) z) y : A \]
given in part (a).
(At each line of each derivation indicate both the term and its type. Do not reduce a lambda term if you do not reduce the tree).

(c) In each derivation written in part (b) count all the redexes present in it (not only the one you are reducing).

TOTAL: 25 marks

QUESTION 2. Consider the following derivation \( D \) in \( \text{NJ}^- \) of \( \neg A, A \vdash B \):

\[
\begin{array}{c}
A \to \bot \\
\hline
A \to \bot \\
\hline
A \to B \\
\hline
\bot \text{-int}
\end{array}
\]

\[
\begin{array}{c}
A \to B \\
\hline
\bot \text{-int}
\end{array}
\]

Answer the following questions:

(a) Write the definition of Maximal Formula in \( \text{NJ}^- \). Is \( D \) normal with respect to of \( \to \) reductions (i.e., elimination of maximal formulas of the form \( X \to Y \))?

(b) Write the definition of the Subformula Property. Explain why \( D \) does not enjoy the subformula property.

(c) Briefly describe and implement a procedure that given \( D \) yields a derivation \( D^* \) of \( \neg A, A \vdash B \) with the subformula property.

TOTAL: 25 marks
**QUESTION 3** Prove the *Weak Normalization Theorem* for intuitionistic implicative logic $\text{NJ}^-$:

> There exists a reduction strategy that transforms every derivation in $\text{NJ}^-$ into a normal derivation.

**TOTAL: 25 marks**

**QUESTION 4.** The relation $\rho$ between $\lambda$-terms *(modulo $\alpha$ conversion)* is the smallest $\lambda$-compatible binary relation on $\Lambda$ such that

\[ t \rho t', u \rho u' \Rightarrow (\lambda x.u)t \rho u'[t'/x] \]

Give the definition of the Church-Rosser property and prove that $\rho$ enjoys it.

You may assume Lemma 11:

(i) If $x \rho t'$, where $x$ is a variable, then $x = t'$.

(ii) If $\lambda x.u \rho t'$, then $t' \equiv \lambda x.u'$ and $u \rho u'$

(iii) If $(v)u \rho t'$ then

(a) either $t' \equiv (v')u'$ with $v \rho v'$ and $u \rho u'$;

(b) or $v \equiv \lambda w$ and $t' \equiv w'[u'/x]$ with $u \rho u'$ and $w \rho w'$.

and also Lemma 12:
If $t \rho t'$ and $u \rho u'$ then $u[t/x] \rho u'[t'/u']$.

Proceed by induction on the complexity of $t_0$ such that $t_0 \rho t_1$ and $t_0 \rho t_2$

**TOTAL: 25 marks**

END OF MIDTERM