Computational Logic
Coursework 4

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Exercise 1. For the first part, "write a derivation in $\mathbf{NJ}^{\to\wedge\bot}$ of $((A \to B) \to A) \to \neg\neg A$", consider the following proof tree:

$$
\dfrac{\dfrac{\dfrac{\overset{\{1\}}{A \to \bot} \qquad \dfrac{\overset{\{2\}}{(A \to B) \to A} \qquad \dfrac{\dfrac{\dfrac{\overset{\{1\}}{A \to \bot} \qquad \overset{\{3\}}{A}}{\bot}\,\scriptstyle E\to}{B}\,\scriptstyle I\bot}{A \to B}\,\scriptstyle I\to 3}{A}\,\scriptstyle E\to}{\bot}\,\scriptstyle E\to}{(A \to \bot) \to \bot}\,\scriptstyle I\to 1}{((A \to (B \to \bot)) \to A) \to ((A \to \bot) \to \bot)}\,\scriptstyle I\to 2
$$

For the second part "every derivation d in $\mathbf{NJ}^{\to\wedge\bot}$ can be transformed into a derivation d' where all conclusions A of a $\bot$ -int rule are atomic", remember the following definitions. Let $c(A)$ be the *l*ogical complexity of a formula A: $c(p) = 0$ if p is atomic, $c(\bot) = 1$, $c(A \to B) = c(A \wedge B) = max\{c(a), c(b)\} + 1$. Given a derivation d in $\mathbf{NJ}^{\to\wedge\bot}$, let $m(d)$ be the maximum logical complexity $c(A)$ of the formulas A that are conclusions of some $\bot$-I in d, $m(d) = 0$ if there is no application of $\bot$-I. Let $n(d)$ be the number of $\bot$-I inferences in d whose conclusion has maximal logical complexity $m(d)$.

If the conclusion of a $\bot$-I is $A = A_0 \to A_1$ then the sub-derivation $\underline{d}$ of d, rooted at $A_0 \to A_1$:

$$
\dfrac{\overset{d^+}{\bot}}{A_0 \to A_1}\,\scriptstyle I\bot \qquad \text{is replaced with the sub-derivation } \underline{d}^-\text{:} \qquad \dfrac{\dfrac{\overset{d^+}{\bot}}{A_1}\,\scriptstyle I\bot}{A_0 \to A_1}\,\scriptstyle I\to
$$

Let d' be the resulting derivation. Notice that d' is obtained from d by:

(i) removing one occurrence of the $\bot$-introduction rule of complexity $c(A_0 \to A_1)$;

(ii) adding one occurrence of the $\perp$-introduction rule of complexity $c(A_1)$.

But no other rules are changed or added and from the definition of complexity of terms we have $c(A_1) < c(A_0 \to A_1)$.
Now suppose that $c(A_0 \to A_1) = \mathrm{m(d)}$ is the maximum complexity in d of any conclusion of $\perp$-I. Then we can conclude that:

(i) if in d there are *o*ther inferences $\perp$-I of maximal complexity m(d), then $m(d') = m(d)$, but $n(d') < n(d)$.

(ii) otherwise $\mathrm{n(d)} = 1$ and $m(d') < m(d)$.

In either cases we have:

$$\langle m(d'), n(d')\rangle < \langle m(d), n(d)\rangle.$$

Therefore we can apply the inductive hypothesis on d' and conclude the thesis on d.