Computational Logic
Coursework 3

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Exercise 1 (Typed Lambda Calculus)**.** First some notational conventions. Let x be a variable and u, v be λ-terms; then we write $\lambda x\, u$ the functional abstraction and $(u)v$ the functional application. Also, we assume that the generic λ-terms m, n, f, x have the following types:

λ-term	*Type*
x	A^0
f	$A^0 \to A^0$
m, n	$(A \to A^0) \to (A \to A^0)$

To further simplify the notation of types we introduce the following convention:

$$\begin{cases} A^0 = A, \\ A^{n+1} = (A^n) \to (A^n). \end{cases}$$

For example, with this convention we denote A^1 the type of f and A^2 the type of m and n.

$$\mathbf{4} = \lambda f\, \lambda x\, (f)(f)(f)(f)x.$$

$$\dfrac{\dfrac{\dfrac{\{1\}\ f : A^1 \qquad \dfrac{\{1\}\ f : A^1 \qquad \dfrac{\{1\}\ f : A^1 \qquad \dfrac{\{1\}\ f : A^1 \qquad \{2\}\ x : A^0}{(f)x : A^0}\,\mathsf{E}{\to}}{(f)(f)x : A^0}\,\mathsf{E}{\to}}{(f)(f)(f)x : A^0}\,\mathsf{E}{\to}}{(f)(f)(f)(f)x : A^0}\,\mathsf{E}{\to}}{\dfrac{\lambda x\, (f)(f)(f)(f)x : A^1}{\lambda f\, \lambda x\, (f)(f)(f)(f)x : A^2}\,\mathsf{I}{\to}1}\,\mathsf{I}{\to}2$$

$$\mathbf{suc} = \lambda n\, \lambda f\, \lambda x\, ((n)f)(f)x.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\{1\}\ n : A^2 \qquad \{2\}\ f : A^1}{(n)f : A^1}\,\mathsf{E}{\to} \qquad \dfrac{\{2\}\ f : A^1 \qquad \{3\}\ x : A^0}{(f)x : A^0}\,\mathsf{E}{\to}}{((n)f)(f)x : A^0}\,\mathsf{E}{\to}}{\lambda x\, ((n)f)(f)x : A^1}\,\mathsf{I}{\to}3}{\lambda f\, \lambda x\, ((n)f)(f)x : A^2}\,\mathsf{I}{\to}2}{\lambda n\, \lambda f\, \lambda x\, ((n)f)(f)x : A^3}\,\mathsf{I}{\to}1$$

$$\mathbf{plus} = \lambda m\, \lambda n\, \lambda f\, \lambda x\, ((m)f)((n)f)x.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\{1\}\ m : A^2 \quad \{2\}\ f : A^1}{(m)f : A^1}\,\mathrm{E}{\to} \qquad \dfrac{\dfrac{\{3\}\ n : A^2 \quad \{2\}\ f : A^1}{(n)f : A^1}\,\mathrm{E}{\to} \quad \{4\}\ x : A^0}{((n)f)x : A^0}\,\mathrm{E}{\to}}{((m)f)((n)f)x : A^0}\,\mathrm{E}{\to}}{\lambda x\, ((m)f)((n)f)x : A^1}\,\mathrm{I}{\to}4}{\lambda f\, \lambda x\, ((m)f)((n)f)x : A^2}\,\mathrm{I}{\to}2}{\lambda n\, \lambda f\, \lambda x\, ((m)f)((n)f)x : A^3}\,\mathrm{I}{\to}3}{\lambda m\, \lambda n\, \lambda f\, \lambda x\, ((m)f)((n)f)x : A^2 \to A^3}\,\mathrm{I}{\to}1$$

$$\mathbf{times} = \lambda m\, \lambda n\, \lambda f\, (m)(n)f.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\{1\}\ m : A^2 \qquad \dfrac{\{2\}\ n : A^2 \quad \{3\}\ f : A^1}{(n)f : A^1}\,\mathrm{E}{\to}}{(m)(n)f : A^1}\,\mathrm{E}{\to}}{\lambda f\, (m)(n)f : A^2}\,\mathrm{I}{\to}3}{\lambda n\, \lambda f\, (m)(n)f : A^3}\,\mathrm{I}{\to}2}{\lambda m\, \lambda n\, \lambda f\, (m)(n)f : A^2 \to A^3}\,\mathrm{I}{\to}1$$

Consider the term:

$$\begin{aligned}
(\mathbf{suc})\mathbf{1} &= (\lambda n\, \lambda f\, \lambda x\, ((n)f)(f)x)\ \mathbf{1} \\
&\overset{\beta}{\rightsquigarrow} \lambda f\, \lambda x\, ((\mathbf{1})f)(f)x \\
&= \lambda f\, \lambda x\, ((\lambda f\, \lambda x\, (f)x)\ f)(f)x \\
&\overset{\beta}{\rightsquigarrow} \lambda f\, \lambda x\, (\lambda x\, (f)x)\ (f)x \\
&\overset{\beta}{\rightsquigarrow} \lambda f\, \lambda x\, (f)(f)x = \mathbf{2}.
\end{aligned}$$

This sequence of β-reductions corresponds to the following sequence of trees:

$$\dfrac{\begin{array}{c}\cdots\\ \mathbf{suc} : A^3 \\ \lambda n\, \lambda f\, \lambda x\, ((n)f)(f)x\end{array} \qquad \begin{array}{c}\cdots\\ \mathbf{1} : A^2 \\ \lambda f\, \lambda x\, (f)x\end{array}}{(\lambda n\, \lambda f\, \lambda x\, ((n)f)(f)x)\mathbf{1} : A^2}\,\mathrm{E}{\to}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\cdots\\ \mathbf{1} : A^2 \\ \lambda f\, \lambda x\, (f)x\end{array} \quad \{1\}\ f : A^1}{(\mathbf{1})f : A^1}\,\mathrm{E}{\to} \qquad \dfrac{\{1\}\ f : A^1 \quad \{2\}\ x : A^0}{(f)x : A^0}\,\mathrm{E}{\to}}{((\mathbf{1})f)(f)x : A^0}\,\mathrm{E}{\to}}{\lambda f\, \lambda x\, ((\mathbf{1})f)(f)x : A^2}\,\mathrm{I}{\to}1,2$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\{1\}\ f : A^1 \quad \{2\}\ x : A^0}{(f)x : A^0}\,\mathrm{E}{\to}}{\lambda x\, (f)x : A^1}\,\mathrm{I}{\to}2 \qquad \dfrac{\{1\}\ f : A^1 \quad \{2\}\ x : A^0}{(f)x : A^0}\,\mathrm{E}{\to}}{(\lambda x\, (f)x)(f)x : A^0}\,\mathrm{E}{\to}}{\lambda f\, \lambda x\, (\lambda x\, (f)x)(f)x : A^2}\,\mathrm{I}{\to}1,2$$

$$\dfrac{\dfrac{\{1\}\ f : A^1 \qquad \dfrac{\{1\}\ f : A^1 \qquad \{2\}\ x : A^0}{(f)x : A^0}\,\mathrm{E}{\to}}{(f)(f)x : A^0}\,\mathrm{E}{\to}}{\lambda f\,\lambda x\,(f)(f)x : A^2}\,\mathrm{I}{\to}1,2$$

Consider the term:

$$\begin{aligned}
((\mathbf{plus})\mathbf{2})\mathbf{1} &= ((\lambda m\,\lambda n\,\lambda f\,\lambda x\,((m)f)((n)f)x)\,\mathbf{2})\mathbf{1}\\
&\overset{\beta}{\leadsto} (\lambda n\,\lambda f\,\lambda x\,((\mathbf{2})f)((n)f)x)\,\mathbf{1}\\
&\overset{\beta}{\leadsto} \lambda f\,\lambda x\,((\mathbf{2})f)((\mathbf{1})f)x\\
&= \lambda f\,\lambda x\,((\lambda f\,\lambda x\,(f)(f)x)\,f)((\lambda f\,\lambda x\,(f)x)\,f)x\\
&\overset{\beta}{\leadsto} \lambda f\,\lambda x\,(\lambda x\,(f)(f)x)\,(\lambda x\,(f)x)\,x\\
&\overset{\beta}{\leadsto} \lambda f\,\lambda x\,(\lambda x\,(f)(f)x)\,(f)x\\
&\overset{\beta}{\leadsto} \lambda f\,\lambda x\,(f)(f)(f)x = \mathbf{3}.
\end{aligned}$$

This sequence of β-reductions corresponds to the following sequence of trees:

$$\dfrac{\dfrac{\begin{array}{c}\cdots\\ \mathbf{plus} : A^2 \to A^3\\ \lambda m\,\lambda n\,\lambda f\,\lambda x\,((m)f)((n)f)x\end{array} \qquad \begin{array}{c}\cdots\\ \mathbf{2} : A^2\\ \lambda f\,\lambda x\,(f)(f)x\end{array}}{(\lambda m\,\lambda n\,\lambda f\,\lambda x\,((m)f)((n)f)x)\mathbf{2} : A^3}\,\mathrm{E}{\to} \qquad \begin{array}{c}\cdots\\ \mathbf{1} : A^2\\ \lambda f\,\lambda x\,(f)x\end{array}}{((\lambda m\,\lambda n\,\lambda f\,\lambda x\,((m)f)((n)f)x)\mathbf{2})\mathbf{1} : A^2}\,\mathrm{E}{\to}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\cdots\\ \mathbf{2} : A^2\\ \lambda f\,\lambda x\,(f)(f)x\end{array} \quad \{1\}\ f : A^1}{(\mathbf{2})f : A^1}\,\mathrm{E}{\to} \qquad \dfrac{\dfrac{\{2\}\ n : A^2 \quad \{1\}\ f : A^1}{(n)f : A^1}\,\mathrm{E}{\to} \qquad \{3\}\ x : A^0}{((n)f)x : A^0}\,\mathrm{E}{\to}}{((\mathbf{2})f)((n)f) : A^0}\,\mathrm{E}{\to}}{\lambda n\,\lambda f\,\lambda x\,((\mathbf{2})f)((n)f) : A^3}\,\mathrm{I}{\to}1,2,3 \qquad \begin{array}{c}\cdots\\ \mathbf{1} : A^2\\ \lambda f\,\lambda x\,(f)x\end{array}}{(\lambda n\,\lambda f\,\lambda x\,((\mathbf{2})f)((n)f))\mathbf{1} : A^1}\,\mathrm{E}{\to}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\cdots\\ \mathbf{2} : A^2\\ \lambda f\,\lambda x\,(f)(f)x\end{array} \quad \{1\}\ f : A^1}{(\mathbf{2})f : A^1}\,\mathrm{E}{\to} \qquad \dfrac{\dfrac{\begin{array}{c}\cdots\\ \mathbf{1} : A^2\\ \lambda f\,\lambda x\,(f)x\end{array} \quad \{1\}\ f : A^1}{(\mathbf{1})f : A^1}\,\mathrm{E}{\to} \qquad \{3\}\ x : A^0}{((\mathbf{1})f)x : A^0}\,\mathrm{E}{\to}}{((\mathbf{2})f)((\mathbf{1})f) : A^0}\,\mathrm{E}{\to}}{\lambda f\,\lambda x\,((\mathbf{2})f)((\mathbf{1})f) : A^2}\,\mathrm{I}{\to}1,3$$

$$\dfrac{\dfrac{\dfrac{\{1\}\ f : A^1 \quad \{2\}\ x : A^0}{(f)x : A^0}\,\mathrm{E}{\to}}{\lambda f\,\lambda x\,(f)x : A^2}\,\mathrm{I}{\to}1,2 \qquad \{1\}\ f : A^1}{\lambda x\,(f)x : A^1}\,\mathrm{E}{\to} \quad \Rightarrow \quad \dfrac{\dfrac{\{1\}\ f : A^1 \quad \{2\}\ x : A^0}{(f)x : A^0}\,\mathrm{E}{\to}}{\lambda x\,(f)x : A^1}\,\mathrm{I}{\to}1$$

$$\dfrac{\dfrac{\{1\}\ f : A^1 \qquad \dfrac{\{1\}\ f : A^1 \quad \{2\}\ x : A^0}{(f)x : A^0}\,\mathrm{E}{\to}}{\dfrac{(f)(f)x : A^0}{\lambda x\,(f)(f)x : A^2}\,\mathrm{I}{\to}1}\,\mathrm{E}{\to} \qquad \dfrac{\{1\}\ f : A^1 \quad \{2\}\ x : A^0}{(f)x : A^0}\,\mathrm{E}{\to}}{(\lambda x\,(f)(f)x)(f)x : A^0}\,\mathrm{E}{\to} \quad \Rightarrow$$

$$\dfrac{\dfrac{\{1\}\ f : A^1 \qquad \dfrac{\{1\}\ f : A^1 \qquad \dfrac{\{1\}\ f : A^1 \quad \{2\}\ x : A^0}{(f)x : A^0}\,\mathrm{E}{\to}}{(f)(f)x : A^0}\,\mathrm{E}{\to}}{(f)(f)(f)x : A^0}\,\mathrm{E}{\to}}{\lambda f\,\lambda x\,(f)(f)(f)x : A^0}\,\mathrm{I}{\to}1,2$$

Consider the term:

$$\begin{aligned}
((\mathbf{times})\mathbf{2})\mathbf{3} &= ((\lambda m\,\lambda n\,\lambda f\,(m)(n)f)\,\mathbf{2})\mathbf{3}\\
&\overset{\beta}{\rightsquigarrow} (\lambda n\,\lambda f\,(\mathbf{2})(n)f)\,\mathbf{3}\\
&\overset{\beta}{\rightsquigarrow} \lambda f\,(\mathbf{2})(\mathbf{3})f\\
&= \lambda f\,(\lambda f\,\lambda x\,(f)(f)x)\,(\lambda f\,\lambda x\,(f)(f)(f)x)\,f\\
&\overset{\beta}{\rightsquigarrow} \lambda f\,(\lambda f\,\lambda x\,(f)(f)x)\,\lambda x\,(f)(f)(f)x\\
&\overset{\beta}{\rightsquigarrow} \lambda f\,\lambda x\,(\lambda x\,(f)(f)(f)x)\,(\lambda x\,(f)(f)(f)x)\,x\\
&\overset{\beta}{\rightsquigarrow} \lambda f\,\lambda x\,(\lambda x\,(f)(f)(f)x)\,(f)(f)(f)x\\
&\overset{\beta}{\rightsquigarrow} \lambda f\,\lambda x\,(f)(f)(f)(f)(f)(f)x = \mathbf{6}.
\end{aligned}$$

This sequence of β-reductions corresponds to the following sequence of trees:

$$\dfrac{\dfrac{\dfrac{\dfrac{\{1\}\ m : A^2 \qquad \dfrac{\{2\}\ n : A^2 \quad \{3\}\ f : A^1}{(n)f : A^1}\,\mathrm{E}{\to}}{(m)(n)f : A^1}\,\mathrm{E}{\to}}{\lambda m\,\lambda n\,\lambda f\,(m)(n)f : A^3 \to A^2}\,\mathrm{I}{\to}1,2,3 \qquad \dfrac{\dots}{\dfrac{\mathbf{2} : A^2}{\lambda f\,\lambda x\,(f)(f)x}}}{(\lambda m\,\lambda n\,\lambda f\,(m)(n)f)\mathbf{2} : A^3}\,\mathrm{E}{\to} \qquad \dfrac{\dots}{\dfrac{\mathbf{3} : A^2}{\lambda f\,\lambda x\,(f)(f)(f)x}}}{((\lambda m\,\lambda n\,\lambda f\,(m)(n)f)\mathbf{2})\mathbf{3} : A^2}\,\mathrm{E}{\to}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dots}{\dfrac{\mathbf{2} : A^2}{\lambda f\,\lambda x\,(f)(f)x}} \qquad \dfrac{\{1\}\ n : A^2 \quad \{2\}\ f : A^1}{(n)f : A^1}\,\mathrm{E}{\to}}{(\mathbf{2})(n)f : A^1}\,\mathrm{E}{\to}}{\lambda n\,\lambda f\,(\mathbf{2})(n)f : A^3}\,\mathrm{I}{\to}1,2 \qquad \dfrac{\dots}{\dfrac{\mathbf{3} : A^2}{\lambda f\,\lambda x\,(f)(f)(f)x}}}{(\lambda n\,\lambda f\,(\mathbf{2})(n)f)\mathbf{3} : A^2}\,\mathrm{E}{\to}$$

$$\dfrac{\dfrac{\dfrac{\dots}{\dfrac{\mathbf{2} : A^2}{\lambda f\,\lambda x\,(f)(f)x}} \qquad \dfrac{\dfrac{\dots}{\dfrac{\mathbf{3} : A^2}{\lambda f\,\lambda x\,(f)(f)(f)x}} \quad \{1\}\ f : A^1}{(\mathbf{3})f : A^1}\,\mathrm{E}{\to}}{(\mathbf{2})(\mathbf{3})f : A^1}\,\mathrm{E}{\to}}{\lambda f\,(\mathbf{2})(\mathbf{3})f : A^2}\,\mathrm{I}{\to}1$$

$$\dfrac{\dfrac{\{1\}\ f : A^1 \qquad \dfrac{\{1\}\ f : A^1 \qquad \{2\}\ x : A^0}{(f)x : A^0}\,\mathsf{E}{\to}}{\dfrac{(f)(f)x : A^0}{\lambda f\, \lambda x\, (f)(f)x : A^2}\,\mathsf{I}{\to}1,2}\,\mathsf{E}{\to} \qquad \begin{matrix}\cdots\\ : A^1\\ \lambda x\,(f)(f)(f)x\end{matrix}}{\dfrac{(\lambda f\, \lambda x\, (f)(f)x)\lambda x\, (f)(f)(f)x : A^1}{\lambda f\, (\lambda f\, \lambda x\, (f)(f)x)\lambda x\, (f)(f)(f)x : A^2}\,\mathsf{I}{\to}1}\,\mathsf{E}{\to}$$

$$\dfrac{\begin{matrix}\cdots\\ : A^1\\ \lambda x\,(f)(f)(f)x\end{matrix} \qquad \dfrac{\begin{matrix}\cdots\\ : A^1\\ \lambda x\,(f)(f)(f)x\end{matrix} \qquad \{2\}\ x : A^0}{(\lambda x\,(f)(f)(f)x)x : A^0}\,\mathsf{E}{\to}}{\dfrac{(\lambda x\,(f)(f)(f)x)(\lambda x\,(f)(f)(f)x)x : A^0}{\lambda f\, \lambda x\,(\lambda x\,(f)(f)(f)x)(\lambda x\,(f)(f)(f)x)x : A^2}\,\mathsf{I}{\to}}\,\mathsf{E}{\to}$$

$$\dfrac{\dfrac{\{1\}\ f : A^1 \qquad \dfrac{\{1\}\ f : A^1 \qquad \dfrac{\{1\}\ f : A^1 \quad \{2\}\ x : A^0}{(f)x : A^0}\,\mathsf{E}{\to}}{(f)(f)x : A^0}\,\mathsf{E}{\to}}{\dfrac{(f)(f)(f)x : A^0}{\lambda x\,(f)(f)(f)x : A^1}\,\mathsf{I}{\to}2}\,\mathsf{E}{\to} \qquad \begin{matrix}\cdots\\ : A^0\\ (f)(f)(f)x\end{matrix}}{\dfrac{(\lambda x\,(f)(f)(f)x)(f)(f)(f)x : A^0}{\lambda f\, \lambda x\,(\lambda x\,(f)(f)(f)x)(f)(f)(f)x : A^2}\,\mathsf{I}{\to}1,2}\,\mathsf{E}{\to}$$

$$\dfrac{\dfrac{\{1\}\ f : A^1 \qquad \dfrac{\{1\}\ f : A^1 \qquad \dfrac{\{1\}\ f : A^1 \quad \begin{matrix}\cdots\\ : A^0\\ (f)(f)(f)x\end{matrix}}{(f)(f)(f)(f)x : A^0}\,\mathsf{E}{\to}}{(f)(f)(f)(f)(f)x : A^0}\,\mathsf{E}{\to}}{(f)(f)(f)(f)(f)(f)x : A^0}\,\mathsf{E}{\to}}{\lambda f\, \lambda x\,(f)(f)(f)(f)(f)(f)x : A^2}\,\mathsf{I}{\to}1,\ldots$$

Exercise 2 (Strong Normalization)**.** In what follows we denote $\mathrm{NJ}^{\to}$ the theory of natural deduction restricted to the use of the implication only; we write $\mathcal{F}$ to refer to the set of the formulas in $\mathrm{NJ}^{\to}$.

For convention of notation, we improperly say that a proof D in $\mathrm{NJ}^{\to}$ contains a maximum γ, where $\gamma \in \mathcal{F}$, to mean that D contains at least one occurrence of γ and at least one of these occurrences is a maximum. When not ambiguous, we identify the occurrence that is the maximum with the formula.

Definition 1 (Maximum)**.** Let D be a proof in $\mathrm{NJ}^{\to}$. We call *maximum* of D every occurrence of a formula in D such that:

(i) it is the conclusion of an implication introduction step;

(ii) it is the major premise of an implication elimination step.

Notice that from this definition it follows that every maximum δ of D:

(i) is of the form $\alpha \to \beta$ and the premise of the implication introduction step is β;

(ii) being the major premise of an implication elimination step we have that the minor premise is α and the conclusion is β.

For convenience of notation we write:

- D_α to represent the subtree of D rooted at the minor premise of the elimination step;
- D_β to represent the subtree of D rooted at the premise of the introduction step;
- D'_β to represent the subtree of D rooted at the conclusion of the elimination step.

With this notation, it is possible to graphically represent a maximum as follows:

$$\dfrac{\dfrac{\begin{array}{c}[\alpha]\\ D_\beta\\ \beta\end{array}}{\alpha\to\beta}\,\mathsf{I}{\to} \qquad \begin{array}{c}D_\alpha\\ \alpha\end{array}}{\beta}\,\mathsf{E}{\to}$$

Definition 2 (Reduction Step)**.** Let D be a proof of $\mathsf{NJ}^{\to}$. We call *reduction step* the transformation that, with the above notation, returns a new proof G obtained from D as follows.

(i) Let D''_β be the result of the replacement of the leaves of D_β that are discharged by the removed implication introduction step (which contains the formula α) with the tree D_α.

(i) The subtree D'_β of D is replaced with the subtree D''_β defined in the previous point.

Definition 3 (Linear Deduction Tree)**.** Let D be a deduction tree in $\mathsf{NJ}^{\to}$. We say that D is *linear* if in D every occurrence of the implication introduction rule discharges exactly one leaf.

To prove that in linear natural deduction strong normalization holds, we need to show the following facts:

$(\mathbf{1})$ *every deduction tree resulting from a linear deduction tree by a reduction step is linear*;

Moreover we need to find a measure $s(\mathcal{D})$ of derivations such that

$(\mathbf{2})$ *e*very reduction step reduces the measure s.

To prove $(\mathbf{1})$ it is enough to notice that no new implication introduction is created by a reduction, hence if $\mathcal{D}'$ results from a reduction step from $\mathcal{D}$ and $\mathcal{D}$ is linear, then so is $\mathcal{D}'$.

To prove $(\mathbf{2})$, consider that in deduction tree may be may be regarded as a tree whose nodes are labelled with formulae and edges with inference rules, or, dually, we may label edges with formulae and nodes with inferences. Taking the first viewpoint, notice that in a reduction step applied to the proof in the figure above *o*ne node is cancelled and two pairs of nodes are identified:

- the node labelled with the maximum formula $\alpha\to\beta$ is erased;
- the premise β of the implication introduction is identified with the conclusion of the implication elimination imediately below it;

- the minor premise α of the implication elimination is identified with the only assumption discharged by the implication introduction above it.

Let $s(\mathcal{D})$ be the number of nodes in $\mathcal{D}$. It is clear that if $\mathcal{D}$ reduces in one step to $\mathcal{D}'$, then $s(\mathcal{D}') = s(\mathcal{D}) - 3$. It follows that no matter which reduction strategy is applied, the reduction process will eventually terminate (there is no infinite descending sequence of natural numbers).