1 Computational Logic 2008 - Dr G.Bellin

Solutions Coursework 7, Part 1.

1. Modal Logic. A sequent calculus for the classical modal system **S4** is given by the system **G3C** with in addition the following inference rules:

$$\frac{\Box\Gamma \Rightarrow A, \Diamond\Delta}{\Pi, \Box\Gamma \Rightarrow \Box A, \Diamond\Delta, \Lambda}\ \Box\text{-R} \qquad \frac{\Box\Gamma, A \Rightarrow \Diamond\Delta}{\Pi, \Diamond A, \Box\Gamma \Rightarrow \Diamond\Delta, \Lambda}\ \Diamond\text{-L}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\ \Box\text{-L} \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Diamond A, \Delta}\ \Diamond\text{-R}$$

Proofs in **S4**:

1:

$$\frac{P \Rightarrow P}{\Box P \Rightarrow P}\ \Box\text{-L}$$

2:

$$\frac{P \Rightarrow P}{P \Rightarrow \Diamond P}\ \Diamond\text{-R}$$

We can simplify proofs by allowing *non-atomic* axioms:

3:

$$\dfrac{\dfrac{\Box P \Rightarrow \Box P}{\Box P \Rightarrow \Diamond\Box P}\ \Diamond\text{-R}}{\Box P \Rightarrow \Box\Diamond\Box P}\ \Box\text{-R}$$

4:

$$\dfrac{\dfrac{\Diamond P \Rightarrow \Diamond P}{\Box\Diamond P \Rightarrow \Diamond P}\ \Box\text{-L}}{\Diamond\Box\Diamond P \Rightarrow \Diamond P}\ \Diamond\text{-L}$$

5:

$$\frac{\Diamond\Box P \Rightarrow \Diamond\Box P}{\Box\Diamond\Box P \Rightarrow \Diamond\Box P}\ \Box\text{-L}$$

8:

$$\frac{\Box\Diamond P \Rightarrow \Box\Diamond P}{\Box\Diamond P \Rightarrow \Diamond\Box\Diamond P}\ \Diamond\text{-R}$$

6:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P \Rightarrow P}{\Box P \Rightarrow P}\ \Box\text{-L}}{\Box P \Rightarrow \Diamond P}\ \Diamond\text{-R}}{\Box P \Rightarrow \Box\Diamond P}\ \Box\text{-R}}{\Box P \Rightarrow \Diamond\Box\Diamond P}\ \Diamond\text{-R}}{\Diamond\Box P \Rightarrow \Diamond\Box\Diamond P}\ \Diamond\text{-L}$$

7:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P \Rightarrow P}{\Box P \Rightarrow P}\ \Box\text{-L}}{\Box P \Rightarrow \Diamond P}\ \Diamond\text{-R}}{\Diamond\Box P \Rightarrow \Diamond P}\ \Diamond\text{-L}}{\Box\Diamond\Box P \Rightarrow \Diamond P}\ \Box\text{-L}}{\Box\Diamond\Box P \Rightarrow \Box\Diamond P}\ \Box\text{-R}$$

To show that the converses of 1-8 are false it suffices to exhibit Kripke models $(W, \leq, \Vdash)$ with $\leq$ *reflexive and transitive* where they are false. Since such models can be constructed as a result of a *semantic tableaux procedure* for **S4**, (see Dispense di logica modale, pp.40-41), here we sketch the semantic tableaux procedure and then indicate the Kripke models.
Notice that only the Kripke models are required for the coursework: the procedure may be useful as a *heuristic tool.*

1':

$$\dfrac{\text{open} \atop w_1 :\Rightarrow P}{w_0 : P \Rightarrow \Box P}\ \Box\text{-R}$$

Goal: (i) $w_0 \Vdash P$, (ii) $w_0 \nVdash \Box P$;
Stage 0: P is in the antecedent;
thus we can set $w_0 \Vdash P$ and (i) is OK.
Invert $\Box$-R. Set $w_0 \leq w_1$.
Stage 1: P is in the succedent only;
set $w_1 \nVdash P$, (ii) is OK.

2'. ;

$$\dfrac{\text{open} \atop w_1 : P \Rightarrow}{w_0 : \Diamond P \Rightarrow P}\ \Diamond\text{-L}$$

Goal: (i) $w_0 \Vdash \Diamond P$, (ii) $w_0 \nVdash P$;
Stage 0: P is in the succedent;
thus we can set $w_0 \nVdash P$ and (ii) is OK.
Invert $\Diamond$-L. Set $w_0 \leq w_1$.
Stage 1: P in the antecedent only;
set $w_1 \Vdash P$, (i) is OK.

Models:

$\mathcal{M} = \big(W, \leq, \Vdash\big)$ where
$W = \{w_0, w_1\}$,
$\leq = \mathbf{RTrCl}(w_0 \leq w_1)$,
$\Vdash$ satisfies $(w_0 \Vdash P, w_1 \nVdash P)$

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$\Vdash$ satisfies $(w_0 \nVdash P, w_1 \Vdash P)$

Here $\mathbf{RTrCl}(\{x \leq y\})$ is the reflexive and transitive closure of the set of accessibility conditions $\{x \leq y\}$.

3':

$$\dfrac{\dfrac{\dfrac{\text{loop} \atop w_1 : \Box\Diamond\Box P, \Diamond\Box P, \Box P, P \Rightarrow}{w_1 : \Box\Diamond\Box P, \Box P \Rightarrow}\ \Box\text{-L}}{w_0 : \Box\Diamond\Box P, \Diamond\Box P \Rightarrow}\ \Diamond\text{-L} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\text{loop} \atop w_3 : \Box\Diamond\Box P, \Diamond\Box P, \Box P, P \Rightarrow}{w_3 : \Box\Diamond\Box P, \Box P \Rightarrow}\ \Box\text{-L}}{w_2 : \Box\Diamond\Box P, \Diamond\Box P \Rightarrow P}\ \Diamond\text{-L}}{w_2 : \Box\Diamond\Box P \Rightarrow P}\ \Box\text{-L}}{w_0 : \Box\Diamond\Box P \Rightarrow \Box P}\ \Box\text{-L}}{\dfrac{w_0 : \Box\Diamond\Box P, \Diamond\Box P \Rightarrow \Box P}{w_0 : \Box\Diamond\Box P \Rightarrow \Box P}\ \Box\text{-L}}$$

- *Goal*: (i) $w_0 \Vdash \Box\Diamond\Box P$, (ii) $w_0 \nVdash \Box P$;

  Stage 0: Invert $\Box$-L; then we have two possibilities:

  (a) invert $\Diamond$-L, with principal formula $\Diamond\Box P$;

  (b) invert $\Box$-R, with principal formula $\Box P$.

- Set $w_0 \leq w_1$. *Subgoal* (a): $w_1 \Vdash \Box\Diamond\Box P$ and $w_1 \Vdash \Box P$.

  Stage 1: invert $\Box$-L: we obtain $\Box\Diamond\Box P, \Diamond\Box P, \Box P, P$ in the antecedent only; all modal formulas have been already considered in stages 0 and 1 when inverting rules; i.e., we have entered a *loop*: thus we *stop on this branch* letting $w_1 \leq w_1$ only.

We have P in the antecedent, hence $w_1 \Vdash P$; since $w_1 \leq w_1$ only, we have also $w_1 \Vdash \Box P$, $w_1 \Vdash \Diamond\Box P$,$w_1 \Vdash \Box\Diamond\Box P$. This satisfies Subgoal (a).

- Set $w_0 \leq w_2$. *Subgoal* (b)(i) $w_2 \Vdash \Box\Diamond\Box P$ and (b)(ii) $w_2 \nVdash P$.

  Stage 2: we have P in succedent, subgoal (b)(ii) is OK;

  invert $\Box$-L and invert $\Diamond$-L with principal formula $\Diamond\Box P$;

- Set $w_2 \leq w_3$. *Subgoal* (b)(iii) $w_3 \Vdash \Diamond\Box P$

  Stage 3: Invert $\Box$-R; we obtain $\Box\Diamond\Box P, \Diamond\Box P, \Box P, P$ in the antecedent only, and we enter a loop; *stop on this branch* with $w_3 \leq w_3$ only.

  We have P is in the antecedent, hence $w_1 \Vdash P$; since $w_3 \leq w_3$ only, we have $w_3 \Vdash \Box P$, $w_3 \Vdash \Diamond\Box P$,$w_3 \Vdash \Box\Diamond\Box P$, this satisfies subgoal (b)(i).

Models:

1. *From the procedure* we obtain the model $\mathcal{M} = \big(W, \leq, \Vdash\big)$ where

- $W = \{w_0, w_1, w_2, w_3\}$;
- $\leq = \mathbf{RTrCl}(w_0 \leq w_1, w_0 \leq w_2, w_2 \leq w_3)$;
- $\Vdash$ satisfies $(w_1 \Vdash P, w_3 \Vdash P, w_2 \nVdash P)$.

Since $w_0 \leq w_2$, $w_2 \nVdash P$, $w_0 \nVdash \Box P$; since $w_2 \leq w_3$ and $w_3 \leq w_3$ only, we have $w_3 \Vdash \Box P$, $w_3 \Vdash \Diamond\Box P$ and $w_2 \Vdash \Diamond\Box P$.
Similarly, $w_0 \leq w_1$, $w_1 \Vdash P$ and $w_1 \leq w_1$ only. Hence $w_1 \Vdash \Box P$, $w_1 \Vdash \Diamond\Box P$ and $w_0 \Vdash \Diamond\Box P$.
Since $w_0 \leq w_i$, for $i = 0, 1, 2, 3$ and for all i, $w_i \Vdash \Diamond\Box P$, we have $w_0 \Vdash \Box\Diamond\Box P$, as required.

2. *A simpler model is* $\mathcal{M} = \big(W, \leq, \Vdash\big)$ where

- $W = \{w_0, w_1\}$;
- $\leq = \mathbf{RTrCl}(w_0 \leq w_1)$;
- $\Vdash$ satisfies $(w_0 \nVdash P, w_1 \Vdash P)$

Check that this suffices.

(4'): $\mathcal{M} = \big(W, \leq, \Vdash\big)$ where

- $W = \{w_0, w_1\}$;
- $\leq = \mathbf{RTrCl}(w_0 \leq w_1)$;
- $\Vdash$ satisfies $(w_0 \Vdash P, w_1 \nVdash P)$

Here $w_0 \Vdash \Diamond P$ but $w_1 \nVdash \Diamond P$, hence neither $w_1 \nVdash \Box\Diamond P$ nor $w_0 \nVdash \Box\Diamond P$.

(5') and (8'): $\mathcal{M} = \big(W, \leq, \Vdash\big)$ where

- $W = \{w_0, w_1, w_2\}$;
- $\leq = \mathbf{RTrCl}(w_0 \leq w_1, w_0 \leq w_2)$;
- $\Vdash$ satisfies $(w_0 \nVdash P, w_1 \Vdash P, w_2 \nVdash P)$

Here $w_1 \Vdash \Box P$, hence $w_0 \Vdash \Diamond\Box P$; but $w_2 \nVdash \Diamond\Box P$, hence $w_0 \nVdash \Box\Diamond\Box P$ and (5') is falsified at w_0.

Also $w_1 \Vdash \Box\Diamond P$, hence $w_0 \Vdash \Diamond\Box\Diamond P$; but $w_2 \nVdash \Diamond P$, hence $w_0 \nVdash \Box\Diamond A$ and (8') is falsified at w_0.

(6') and (7'): $\mathcal{M} = \big(W, \leq, \Vdash\big)$ where

- $W = \{w_0, w_1\}$;
- $\leq = \mathbf{RTrCl}(w_0 \leq w_1, w_1 \leq w_0)$;
- $\Vdash$ satisfies $(w_0 \nVdash P, w_1 \Vdash P$

Here $w_0, w_1 \Vdash \Diamond P$, since $w_1 \Vdash p$ thus $w_0 \Vdash \Box\Diamond P$, as $w_0 \leq w_0, w_1$ only; but neither $w_0 \Vdash \Box P$ nor $w_1 \Vdash \Box P$, because $w_0 \nVdash P$ and $w_0, w_1 \leq w_0$. Hence $w_0 \nVdash \Diamond\Box P$, and (6') is falsified at w_0 as required.

For the same analysis we have $w_0 \Vdash \Box\Diamond P$, but neither $w_0 \Vdash \Box P$ nor $w_1 \Vdash \Box P$, hence $w_0 \nVdash \Diamond\Box P$, as $w_0 \leq w_0, w_1$ only; thus also $w_0 \nVdash \Box\Diamond\Box P$ and (7') is falsified at w_0.