1 Computational Logic 2008 - Dr G.Bellin

Solutions Coursework 7, Part 1.

1. Modal Logic. A sequent calculus for the classical modal system S4 is given by the system G3C with in addition the following inference rules:

\[ \Gamma \Rightarrow A, \Delta \]
\[ \Pi, \Delta \Rightarrow \Box A, \Box \Delta, \Lambda \]
\[ \Box A, \Gamma \Rightarrow \Delta \]

\[ \Pi, \Delta \Rightarrow \Box A, \Box \Delta, \Lambda \]
\[ \Gamma \Rightarrow \Delta \]
\[ \Gamma \Rightarrow \Box A, \Delta \]

\[ \Box A \Rightarrow \Gamma \]
\[ \Pi, \Box A, \Delta \Rightarrow \Box \Delta, \Lambda \]
\[ \Gamma \Rightarrow \Box A \]

Proofs in S4:

1:

\[ P \Rightarrow P \]
\[ \Box P \Rightarrow P \]
\[ 1: P \Rightarrow P \]

2:

\[ P \Rightarrow P \]
\[ P \Rightarrow \Diamond P \]
\[ 2: P \Rightarrow P \]

We can simplify proofs by allowing non-atomic axioms:

3:

\[ \Box P \Rightarrow \Box P \]
\[ \Box P \Rightarrow \Diamond P \]
\[ \Diamond P \Rightarrow \Box \Diamond P \]
\[ \Box P \Rightarrow \Box \Diamond P \]
\[ \Diamond \Diamond P \Rightarrow \Diamond P \]
\[ 3: \Box P \Rightarrow \Box P \]

4:

\[ \Diamond P \Rightarrow \Diamond P \]
\[ \Box \Diamond P \Rightarrow \Diamond P \]
\[ \Diamond \Box \Diamond P \Rightarrow \Diamond P \]
\[ \Box \Diamond P \Rightarrow \Box \Diamond P \]
\[ \Diamond \Box P \Rightarrow \Diamond P \]
\[ 4: \Diamond P \Rightarrow \Diamond P \]

5:

\[ \Diamond \Diamond P \Rightarrow \Diamond P \]
\[ \Box \Diamond \Diamond P \Rightarrow \Diamond P \]
\[ \Diamond \Box \Diamond P \Rightarrow \Diamond P \]
\[ \Box \Diamond \Diamond P \Rightarrow \Diamond P \]
\[ \Diamond \Diamond \Diamond P \Rightarrow \Diamond \Diamond P \]
\[ 5: \Diamond \Diamond P \Rightarrow \Diamond P \]

6:

\[ P \Rightarrow P \]
\[ \Box P \Rightarrow P \]
\[ \Diamond P \Rightarrow \Box P \]
\[ \Box \Diamond P \Rightarrow \Diamond P \]
\[ \Diamond \Box \Diamond P \Rightarrow \Diamond P \]
\[ 6: P \Rightarrow P \]

To show that the converses of 1-8 are false it suffices to exhibit Kripke models \((W, \leq, \models)\) with \(\leq\) reflexive and transitive where they are false. Since such models can be constructed as a result of a semantic tableaux procedure for S4, (see Dispense di logica modale, pp.40-41), here we sketch the semantic tableaux procedure and then indicate the Kripke models.

Notice that only the Kripke models are required for the coursework: the procedure may be useful as a heuristic tool.
1':  open
\[ w_1 : \square \Rightarrow P \]  \[ w_0 : P \Rightarrow \square P \] \[ \Box - R \]

Goal: (i) \( w_0 \models P \), (ii) \( w_0 \not\models \square P \);
Stage 0: \( P \) is in the antecedent;
thus we can set \( w_0 \models P \) and (i) is OK.
Invert \( \Box - R \). Set \( w_0 \leq w_1 \).
Stage 1: \( P \) is in the succedent only;
set \( w_1 \not\models P \), (ii) is OK.

Models:
\( M = (W, \leq, \models) \) where
\( W = \{ w_0, w_1 \} \),
\[ \leq = \text{RT} \text{Cl}(w_0 \leq w_1), \]
\( \models \) satisfies \( (w_0 \models P, w_1 \not\models P) \)

Here \( \text{RT} \text{Cl}(\{x \leq y\}) \) is the reflexive and transitive closure of the set of accessibility conditions \( \{x \leq y\} \).

2':  open
\[ w_1 : P \Rightarrow \Diamond P \]  \[ w_0 : \Diamond P \Rightarrow P \] \[ \Diamond - L \]

Goal: (i) \( w_0 \models \Diamond P \), (ii) \( w_0 \not\models P \);
Stage 0: \( P \) is in the succedent;
thus we can set \( w_0 \not\models P \) and (ii) is OK.
Invert \( \Diamond - L \). Set \( w_0 \leq w_1 \).
Stage 1: \( P \) in the antecedent only;
set \( w_1 \models P \), (i) is OK.

Stage 0: Invert \( \Box - L \); then we have two possibilities:
(a) invert \( \Diamond - L \), with principal formula \( \Diamond P \);
(b) invert \( \Box - R \), with principal formula \( \Box P \).
Set \( w_0 \leq w_1 \). Subgoal (a): \( w_1 \models \Box \Diamond P \) and \( w_1 \models \Box P \).

Stage 1: invert \( \Box - L \); we obtain \( \Box \Diamond P, \Box \Diamond P, \Box P \) in the antecedent only; all modal formulas have been already considered in stages 0 and 1 when inverting rules; i.e., we have entered a loop: thus we stop on this branch letting \( w_1 \leq w_1 \) only.
We have $P$ in the antecedent, hence $w_1 \vdash P$; since $w_1 \leq w_1$ only, we have also $w_1 \vdash \Box P, w_1 \vdash \Diamond P, w_1 \vdash \Box \Diamond P$. This satisfies Subgoal (a).

- Set $w_0 \leq w_2$. Subgoal (b)(i) $w_2 \vdash \Box \Diamond P$ and (b)(ii) $w_2 \nmid P$.

**Stage 2:** we have $P$ in succedent, subgoal (b)(ii) is OK; invert $\Box$-L and invert $\Diamond$-L with principal formula $\Diamond P$;

- Set $w_2 \leq w_3$. Subgoal (b)(iii) $w_3 \vdash \Diamond P$

**Stage 3:** Invert $\Box$-R; we obtain $\Box \Diamond P, \Diamond P, \Box P, P$ in the antecedent only, and we enter a loop; *stop on this branch* with $w_3 \leq w_3$ only.

We have $P$ is in the antecedent, hence $w_1 \vdash P$; since $w_3 \leq w_3$ only, we have $w_3 \vdash \Box P, w_3 \vdash \Diamond P, w_3 \vdash \Box \Diamond P$, this satisfies subgoal (b)(i).

**Models:**

1. *From the procedure* we obtain the model $M = (W, \leq, \vdash)$ where

   - $W = \{w_0, w_1, w_2, w_3\}$;
   - $\leq = \text{RT}\text{rCl}(w_0 \leq w_1, w_0 \leq w_2, w_2 \leq w_3)$;
   - $\vdash$ satisfies ($w_1 \vdash P, w_3 \vdash P, w_2 \nmid P$).

   Since $w_0 \leq w_2, w_2 \nmid P, w_0 \nmid \Box P$; since $w_2 \leq w_3$ and $w_3 \leq w_3$ only, we have $w_3 \vdash \Box P, w_3 \vdash \Diamond P$ and $w_2 \vdash \Diamond P$.

   Similarly, $w_0 \leq w_1, w_1 \vdash P$ and $w_1 \leq w_1$ only. Hence $w_1 \vdash \Box P, w_1 \vdash \Diamond P$ and $w_0 \vdash \Diamond P$.

   Since $w_0 \leq w_i$, for $i = 0, 1, 2, 3$ and for all $i, w_i \vdash \Diamond P$, we have $w_0 \vdash \Box \Diamond P$, as required.

2. *A simpler model is* $M = (W, \leq, \vdash)$ where

   - $W = \{w_0, w_1\}$;
   - $\leq = \text{RT}\text{rCl}(w_0 \leq w_1)$;
   - $\vdash$ satisfies ($w_0 \nmid P, w_1 \vdash P$)

Check that this suffices.
(4') \( \mathcal{M} = (W, \leq, \models) \) where
- \( W = \{w_0, w_1\} \);
- \( \leq = \text{RTrCl}(w_0 \leq w_1) \);
- \( \models \) satisfies \( w_0 \models P, w_1 \not\models P \)

Here \( w_0 \models \Diamond P \) but \( w_1 \not\models \Diamond P \), hence neither \( w_1 \not\models \Box \Diamond P \) nor \( w_0 \not\models \Box \Diamond P \).

(5') and (8') \( \mathcal{M} = (W, \leq, \models) \) where
- \( W = \{w_0, w_1, w_2\} \);
- \( \leq = \text{RTrCl}(w_0 \leq w_1, w_0 \leq w_2) \);
- \( \models \) satisfies \( w_0 \not\models P, w_1 \models P, w_2 \not\models P \)

Here \( w_1 \models \Box P \), hence \( w_0 \models \Diamond \Box P \); but \( w_2 \not\models \Diamond \Box P \), hence \( w_0 \not\models \Box \Diamond \Box P \) and (5') is falsified at \( w_0 \).

Also \( w_1 \models \Box \Diamond P \), hence \( w_0 \models \Diamond \Box \Diamond P \); but \( w_2 \not\models \Diamond \Box P \), hence \( w_0 \not\models \Box \Diamond A \) and (8') is falsified at \( w_0 \).

(6') and (7') \( \mathcal{M} = (W, \leq, \models) \) where
- \( W = \{w_0, w_1\} \);
- \( \leq = \text{RTrCl}(w_0 \leq w_1, w_1 \leq w_0) \);
- \( \models \) satisfies \( w_0 \not\models P, w_1 \models P \)

Here \( w_0, w_1 \models \Diamond P \), since \( w_1 \models p \) thus \( w_0 \models \Box \Diamond P \), as \( w_0 \leq w_0, w_1 \) only; but neither \( w_0 \models \Box P \) nor \( w_1 \models \Box P \), because \( w_0 \not\models P \) and \( w_0, w_1 \leq w_0 \). Hence \( w_0 \not\models \Diamond \Box P \), and (6') is falsified at \( w_0 \) as required.

For the same analysis we have \( w_0 \models \Box \Diamond P \), but neither \( w_0 \models \Box P \) nor \( w_1 \models \Box P \), hence \( w_0 \not\models \Diamond \Box P \), as \( w_0 \leq w_0, w_1 \) only; thus also \( w_0 \not\models \Box \Diamond \Box P \) and (7') is falsified at \( w_0 \).