# 1 Computational Logic 2008 - Dr G.Bellin

**Coursework 7**

**1. Modal Logic.** A sequent calculus for the classical modal system **S4** is given by the system **G3C** with in addition the following inference rules:

$$\frac{\Box\Gamma \Rightarrow A, \Diamond\Delta}{\Pi, \Box\Gamma \Rightarrow \Box A, \Diamond\Delta, \Lambda}\ \Box\text{-R} \qquad \frac{\Box\Gamma, A \Rightarrow \Diamond\Delta}{\Pi, \Diamond A, \Box\Gamma \Rightarrow \Diamond\Delta, \Lambda}\ \Diamond\text{-L}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\ \Box\text{-L} \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Diamond A, \Delta}\ \Diamond\text{-R}$$

Prove the following sequents in **S4**:

| | |
|---|---|
| 1. $\Box P \Rightarrow P$; | 2. $P \Rightarrow \Diamond P$; |
| 3. $\Box P \Rightarrow \Box\Diamond\Box P$; | 4. $\Diamond\Box\Diamond P \Rightarrow \Diamond P$; |
| 5. $\Box\Diamond\Box P \Rightarrow \Diamond\Box P$; | 6. $\Diamond\Box P \Rightarrow \Diamond\Box\Diamond P$; |
| 7. $\Box\Diamond\Box P \Rightarrow \Box\Diamond P$; | 8. $\Box\Diamond P \Rightarrow \Diamond\Box\Diamond P$. |

Show that the converses of 1-8 are false by exhibiting Kripke models $(W, \leq, \Vdash)$ with $\leq$ *reflexive and transitive* where they are false.

| | |
|---|---|
| 1'. $P \Rightarrow \Box P$; | 2'. $\Diamond P \Rightarrow P$; |
| 3'. $\Box\Diamond\Box P \Rightarrow \Box P$; | 4'. $\Diamond P \Rightarrow \Diamond\Box\Diamond P$; |
| 5'. $\Diamond\Box P \Rightarrow \Box\Diamond\Box P$; | 6'. $\Diamond\Box\Diamond P \Rightarrow \Diamond\Box P$ |
| 7'. $\Box\Diamond P \Rightarrow \Box\Diamond\Box P$; | 8'. $\Diamond\Box\Diamond P \Rightarrow \Box\Diamond P$. |

**2. Category Theory.** (a) Let $\mathcal{M} = (M, \cdot_{\mathcal{M}}, 1_{\mathcal{M}})$ be a *monoid*. Show that $\mathcal{M}$ is a category (with only one object $M$ and with arrows the elements $m \in M$). [Remember that the multiplication operation of $\mathcal{M}$ is associative and commutative and has $1_{\mathcal{M}}$ as the identity.]

(b) Given a monoid $\mathcal{M}$, an $\mathcal{M}$-set is a set $A$ with a mapping $M \times A \to A$, written as $(m, a) \mapsto ma$, such that $(1, a) \mapsto a$ and $(m \cdot m', a) \mapsto m(m'a)$. Show that an $\mathcal{M}$-set may be regarded as a functor from $\mathcal{M}$ to **Set** the category of sets. [**Note**: $ma$ and $m(m'a)$ *are elements of* $A$*!*]

(c) Let $\mathcal{M}$ and $\mathcal{M}'$ be monoids regarded as categories as in (a). Let $(A, F_{\mathcal{M}})$ be an $\mathcal{M}$-set and let $(A, F_{\mathcal{M}'})$ be an $\mathcal{M}'$ set with $F_{\mathcal{M}} : M \times A \to A$ and $F_{\mathcal{M}'} : M' \times A \to A$ regarded as functors as in (b). Show that any homomorphism of monoids $h : \mathcal{M} \to \mathcal{M}'$ is a natural transformation between $F_{\mathcal{M}}$ and $F_{\mathcal{M}'}$. [Remember that an homomorphism $h : \mathcal{M} \to \mathcal{M}'$ of monoids has the property that $h(1_{\mathcal{M}}) = 1_{\mathcal{M}'}$ and that $h(m_1 \cdot_{\mathcal{M}} m_2) = h(m_1) \cdot_{\mathcal{M}'} h(m_2)$.]