Homework Assignment 1 - Extended Hints

Due **Friday 23-01-04**

1. Let ϕ be a partial function from the set $\mathbf{N}$ of natural numbers *onto* the set A. Show that either there is a bijection $f : n \to A$ where $n \in \mathbf{N}$ or there is a bijection $f : \mathbf{N} \to A$. In other words, either A is finite or A has the same cardinality as $\mathbf{N}$.

Hint. Let ϕ in a partial function from $\mathbf{N}$ onto A. We distinguish two cases: A is empty or A is non-empty. If A is empty, what must ϕ be? ...
Identifying 0 with the empty set $\emptyset$, can we say that ϕ is already a bijection $f : \emptyset \to A$? ...

(*1 point.*)

If A is nonempty, let $a \in A$. Define a total surjective function $g : \mathbf{N} \to A$ as follows:

$$g(n) = \phi(n) \qquad \text{if } \phi(n) \text{ is defined,} \qquad g(n) = a \qquad \text{otherwise.}$$

Now we define the required function f by recursion. Simultaneously, we also define an auxiliary partial function ν from $\mathbf{N}$ to $\mathbf{N}$. Let $f(1) = g(1)$ and $\nu(1) = 1$.

Suppose we have defined $\nu(n)$ and $f(1) = a_1, \ldots, f(n) = a_n$ so that $a_i \neq a_j$ for all $i \neq j \leq n$; we would like to define

$$\nu(n+1) = \text{ the least } y > \nu(n) \text{ such that } \forall x \leq n. g(y) \neq f(x).$$

Now if there exists $y > \nu(n)$ such that $g(y) \neq f(x)$ for all $x \leq n$, then $\nu(n+1)$ is indeed defined, and we let

$$a_{n+1} = f(n+1) = g(\nu(n+1)).$$

Otherwise, $\nu(n+1)$ is undefined, and so is $f(n+1)$.

Now there are two cases:

First case: for some n, $\nu(n)$ is defined but $\nu(n+1)$ is undefined. Why can we say that in this case $A = \{a_1, \ldots, a_n\}$ and that $f : n \to A$ is a bijection? ...

(*2 points.*)

Second case: for all n, $\nu(n)$ is defined. Why can we say that in this case $A = \{a_1, a_2, \ldots\}$ and that $f : \mathbf{N} \to A$ is a bijection? ...

(*2 points.*)

2. Prove that for all k the sum of the first k positive integers $1 + 2 + \ldots + k = \dfrac{k(k+1)}{2}$.

Hint: One way to prove this is by induction on k: show that the equation holds for $k = 1$; then assuming that it holds for $k = n$, prove that it holds also for $k = n + 1$. (*5 points.*)

3. Let $\mathbf{N}$ be the set of the natural numbers (including 0). Show that the function $J : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ given by

$$J(m,n) = \frac{(m+n)(m+n+1)}{2} + m$$

is a bijection.

Hint. Using exercise 2, why can we say that the function $f(k) = \Sigma_{i \leq k} i$ is injective? ...

(*1 point.*)

Prove that if $m \leq k$, then $m + \Sigma_{i \leq k} i < \Sigma_{i \leq (k+1)} i$.

(*1 point.*)

Why can we say that the function $J(m,n)$ is injective? ...

(*1 point.*)

Define a function $G : \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ as

$$G(k) = (g_1(k), g_2(k))$$

where

- $s(k) =$ the least x such that $k < \Sigma_{i \leq (x+1)} i$ [thus $\Sigma_{i \leq s(k)} i \leq k$];
- $g_1(k) = k - \Sigma_{i \leq s(k)} i$

and

- $g_2(k) = s(k) - g_1(k)$.

Show that $J(G(k)) = k$ for all k, i.e., that J is surjective.

Hint to the Hint: To do this, verify that

$$J(G(k)) \; = \ldots = \; J(g_1(k), s(k) - g_1(k)) \; = \ldots = \; \Sigma_{i \leq s(k)} i + g_1(k) \; = \ldots = \; k.$$

(*2 points.*)

Alternative proof: prove not only that $J(G(k)) = k$ for all k, but also that $G(J(m,n)) = (m,n)$ for all pairs (m,n). By a result discussed in class, this suffices to show that J is a bijection, i.e., you do not need to prove that J is injective.