# A TERM ASSIGNMENT FOR DUAL INTUITIONISTIC LOGIC.

GIANLUIGI BELLIN

**Abstract.** We study the proof-theory of *co-Heyting algebras* and present a *calculus of continuations* typed in the *disjunctive–subtractive* fragment of *dual intuitionistic logic*. We give a *single-assumption multiple-conclusions* Natural Deduction system $\mathbf{NJ}^{\smallsetminus\curlyvee}$ for this logic: unlike the best-known treatments of multiple-conclusion systems (e.g., Parigot's $\lambda-\mu$ calculus, or Urban and Bierman's term-calculus) here the term-assignment is *distributed* to all conclusions, and exhibits several features of calculi for concurrency, such as remote capture of variable and remote substitution. The present construction can be regarded as the construction of a free co-Cartesian closed category, *dual* to the familiar construction of a free Cartesian-closed category from the syntax of positive intuitionistic logic. Here duality is extended from formulas to proofs and it is shown that every computation in our calculus of continuations is isomorphic to a computation in the simply typed $\lambda$-calculus. An informal interpretation of this system in the framework of the *logic for pragmatics* is suggested as a *calculus of refutations* in the *logic of conjectures*.

**§1. Preface.** This paper is a contribution to the proof-theory of co-Heyting algebras.[1] A *co-Heyting algebra* is a (distributive) lattice $\mathcal{C}$ such that its opposite $\mathcal{C}^{op}$ is a Heyting algebra. In $\mathcal{C}^{op}$ the operation of Heyting implication $B \to A$ is defined by the familiar adjunction, thus in the co-Heyting algebra $\mathcal{C}$ *subtraction* (i.e., *co-implication*) $A \smallsetminus B$ is defined dually

$$\frac{C \wedge B \ \leq \ A}{C \ \leq \ B \to A} \qquad\qquad \frac{A \ \leq \ B \vee C}{A \smallsetminus B \ \leq \ C}$$

Thus we have an intuitionistic propositional formal system, where formulas are built from atoms and a constant for *falsity* using *disjunction* and *subtraction*, which is the dual of a fragment of *minimal* (*positive intuitionistic*) logic. Proofs can be represented in a sequent calculus $\mathbf{LJ}^{\smallsetminus\curlyvee}$, where sequents are restricted to a single formula in the *antecedent*; here the rules for subtraction are precisely dual to the rules for implication

[1]Thanks to Stefano Berardi, Corrado Biasi, Tristan Crolard, Arnaud Fleury, Nicola Gambino, Maria Emilia Maietti, Kurt Ranalter, Edmund Robinson and Graham White for their help and cooperation at various stages of the project.

$$\frac{E \Rightarrow \Upsilon, C \qquad D \Rightarrow \Upsilon}{E \Rightarrow \Upsilon, C \smallsetminus D} \smallsetminus -R \qquad\qquad \frac{C \Rightarrow D, \Upsilon}{C \smallsetminus D \Rightarrow \Upsilon} \smallsetminus -L$$

$$\frac{\Theta \Rightarrow A \qquad B, \Theta \Rightarrow F}{A \supset B, \Theta \Rightarrow F} \supset -R \qquad\qquad \frac{A, \Theta \Rightarrow B}{\Theta \Rightarrow A \supset B} \supset -R$$

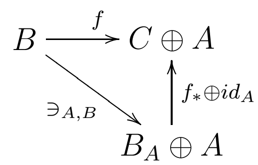
and similarly for disjunction. The rules of the sequent calculi $\mathbf{LJ}^{\supset\cap}$ and $\mathbf{LJ}^{\smallsetminus\curlyvee}$ in a **G3** system [21] are given in the Appendix, Table 2.

We present a corresponding *natural deduction system* $\mathbf{NJ}^{\smallsetminus\curlyvee}$ for this logic: as Natural Deduction $\mathbf{NJ}^{\supset\cap}$ for minimal logic is a *multiple-assumptions single-conclusion* system, so ours is *single-assumption multiple-conclusions*. Some subtle points need to be addressed in the definition of this formalism, to be discussed below.

From the graphs of Natural Deduction derivations for minimal logic, decorated with lambda terms, one constructs the free Cartesian-closed category generated by the syntax ([16], pp. 55) in a familiar way. We would like to dualize this construction, in particular, to define a term assignment suitable for the construction of free co-Cartesian closed categories. However, the following fact (spelt out in [10], Proposition 1.15) reminds us that not every mathematical model is suitable for such a dualization:

*Fact. In the category of sets, the coexponent $B_A$ of two sets $A$ and $B$ is defined if and only if $A = \emptyset$ or $B = \emptyset$.*

It is instructive to glance at the proof. The coexponent of $A$ and $B$ is an object $B_A$ together with an arrow $\ni_{A,B}\colon B \to B_A \oplus A$ such that for any arrow $f : B \to C \oplus B$ there exists a unique $f_* : B_A \to C$ such that the following diagram commutes:




In **Sets** the coexponents $B_\emptyset$ and $\emptyset_A$ are defined trivially. Conversely, suppose that $A \neq \emptyset \neq B$ and that $B_A$ is defined. For each $b \in B$, the (total) function $\ni_{A,B}$ chooses an element in the *disjoint union* $B_A \oplus A$, i.e., $\ni_{A,B}$ chooses either the left or the right *side* of $B_A \oplus A$ and moreover $f_* \oplus id_A$ preserves the side. To find a counterexample it suffices to choose $f : B \to C \oplus B$ and $b \in B$ such that the side of $f(b)$ is different from that of $(f_* \oplus id_A) \circ \ni_{A,B} (b)$.

Thus we shall not look for models in **Sets**. On the other hand, models in **Rel** are readily available if we translate $\mathbf{NJ}^{\smallsetminus,\curlyvee}$ into linear logic and then look at the coherent semantics of the translation. The following seems a suitable translation $(\ )^\circ$

$$
\begin{array}{rl}
(p)^\circ = & p \\
(\bot)^\circ = & \mathbf{0} \\
(C \smallsetminus D)^\circ = & C^\circ \otimes (?D^\circ)^\bot = C^\circ \otimes !(D^{\circ\bot}) \\
(C \curlyvee D)^\circ = & ?(C^\circ \oplus D^\circ) = ?C^\circ \wp ?D^\circ \\
(E \vdash C_1, \dots, C_n)^\circ = & ?E^\circ \vdash ?C_1^\circ, \dots, ?C_n^\circ \\
 & (\textit{namely, } ?E^\circ \vdash ?C_1^\circ \wp \dots \wp ?C_n^\circ)
\end{array}
$$

Spelling out the term assignment for the linearization of $\mathbf{LJ}^{\smallsetminus\curlyvee}$ and the categorical term model means to dualize the construction of Bierman's thesis [7] and then extract the term model for $\mathbf{LJ}^{\smallsetminus\curlyvee}$ itself; we shall not do this here. We simply outline the definition of the term assignment which appears to provide the required duality with the simply typed lambda calculus. Namely, given the dual diagrams[2]

$$
\begin{array}{ccc}
\Theta \times \Pi \times \Gamma & \qquad & F^\bot \\
\downarrow{\scriptstyle 1\times t\times 1} & & \downarrow{\scriptstyle v^\bot} \\
\Theta \times A \times \Gamma & & B^\bot + \Gamma^\bot \\
\downarrow{\scriptstyle u\times 1} & & \downarrow{\scriptstyle u^\bot + 1} \\
B \times \Gamma & & \Theta^\bot + A^\bot + \Gamma^\bot \\
\downarrow{\scriptstyle v} & & \downarrow{\scriptstyle 1 + t^\bot + 1} \\
F & & \Theta^\bot + \Pi^\bot + \Gamma^\bot
\end{array}
$$

we need to define the following data:

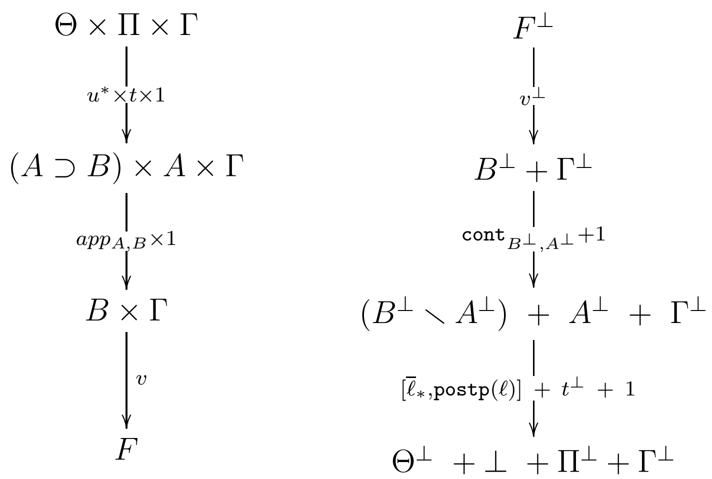
(i) terms $\overline{\ell} : B^\bot \to \Theta^\bot$ and $\ell : B^\bot \to A^\bot$ such that $u^\bot = [\overline{\ell}, \ell]$;

(ii) "continuation" terms $\texttt{cont}_{B^\bot, A^\bot} : B^\bot \to (B^\bot \smallsetminus A^\bot) + A^\bot$ and $\texttt{postp}(\ell) : (B^\bot \smallsetminus A^\bot) \to \bot$;

(iii) a binding operation $(\ )_*$ such that $\overline{\ell}_* : (B^\bot \smallsetminus A^\bot) \to \Theta^\bot$

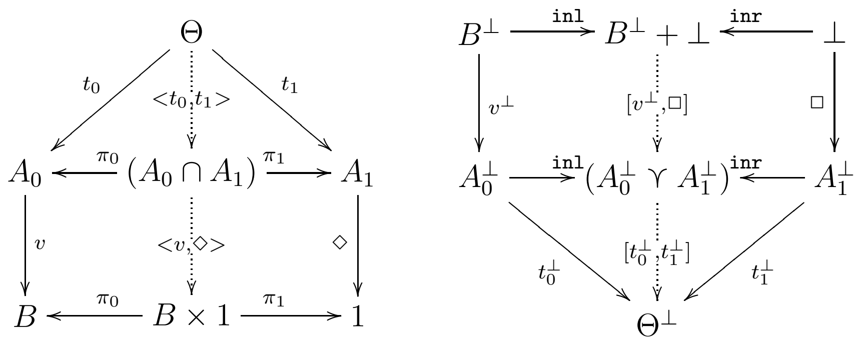
[2]Here we write "+" instead of "⊕" for the coproduct, in order to avoid confusions with Girard's *plus*, mentioned above.

in order to produce the following dual diagram to the right.




The possibility of finding the required $\overline{\ell}$ and $\ell$ could be argued informally as follows. By definition our terms code the *paths* of the dual *proof-graph* and also the overall structure of the dual proof-graph can be reconstructed from the set of terms. For instance, the term $u^\perp : B^\perp \to \Theta^\perp + A^\perp$ is in fact a *list* $\ell$ of terms, an element of $\ell$ for each summand $X$ in $\Theta^\perp + A^\perp$. As the decomposition of $u^\perp$ in $\overline{\ell}$ and $\ell$ identifies subgraphs of the *graph*, such decomposition must be possible also in the category freely generated from the graph.

The co-Cartesian structure of our logic is more standard (since we have implicitly assumed Girard's isomorphism $?(C \oplus D) \equiv ?C \wp ?D$). E.g., we need to implement the dual diagrams




Here the multiple conclusion structure of derivations allows us to assign *distributed* terms $\mathtt{casel} : A_0^\perp \curlyvee A_1^\perp \to A_0^\perp$ and $\mathtt{caser} : A_0^\perp \curlyvee A_1^\perp \to A_1^\perp$ to the conclusions of an application of $\curlyvee$-E such that

$\texttt{inl};\texttt{casel} = 1_{A_0^\perp}$ and $\texttt{inr};\texttt{caser} = 1_{A_1^\perp}$. We stipulate that a sequence $\texttt{inl};\texttt{caser}$ annihilates (i.e., reduces to `nil`) and that a substitution of `nil` for $x$ in $t$ annihilates $t$ as well. Thus `nil` acts as the identity of the coproduct and we use it to decorate terms introduced by Weakening-R. However we distinguish between Weakening-R and the logical rule $\perp$-E; for the latter we introduce terms of the form $\texttt{false}_C : \perp \longrightarrow C$.

The aim of this paper is to spell out the analogue of the Curry-Howard correspondence[3] for $\mathbf{NJ}^{\smallsetminus\curlyvee}$; but a formal categorical definition is not given. The main focus is rather on the definition of the natural deduction derivations and on the behaviour of the operation $(\ )_*$ and of the terms $\texttt{cont}_{B^\perp,A^\perp}$ and `postp` that decorate the rules for subtraction.[4]

**1.1. Main result.**

THEOREM 1. *There exists a duality* $(\ )^\perp$ *between the propositions, the natural deduction derivations and the term assignments of the systems* $\mathbf{NJ}^{\supset\cap}$ *and* $\mathbf{NJ}^{\smallsetminus\curlyvee}$ *such that*

1. *if* $\overline{x} : \Theta \ \rhd\ t : A$ *is a typing derivation in* $\mathbf{NJ}^{\supset\cap}$*, then*
$$(\overline{x} : \Theta \ \rhd\ t : A)^\perp = a : A^\perp \ \rhd\ t^\perp : \Theta^\perp$$
*is a typing derivation in* $\mathbf{NJ}^{\smallsetminus\curlyvee}$*, and conversely;*
2. (isomorphism) *if* $\overline{x} : \Theta \ \rhd\ t_0 : A$ *and* $t_0$ *$\beta$-reduces to* $t_1$*, then* $t_0^\perp$ *$\beta$-reduces to* $t_1^\perp$*, and conversely.*

We shall need a main lemma:

[3]Namely, the correspondence between propositions of intuitionistic logic and types, on one hand, and between derivations in **NJ** and $\lambda$-terms, on the other, resulting in the abstract characterization of the Curry-Howard correspondence in terms of Cartesian closed categories.

[4]In [11] Tristan Crolard presents a Natural Deduction system and a term assignment for Subtractive Logic, called $\lambda\mu^{\rightarrow+\times-}$-*calculus*, in the tradition of Parigot's $\lambda\mu$-calculus for classical Natural Deduction. Restrictions on the implication-introduction and subtraction-elimination rules are introduced to define a *constructive* system of Subtractive Logic and its term calculus. Clearly we work in a much more restricted environment, in which a duality result with the simply typed lambda calculus holds. But even extending our logic as in [4] and introducing negations, the crucial difference of our approach is that our terms are distributed over the conclusions (with the possible exception of `postpone` terms), and therefore there is no focalization nor defocalization and no obvious way to introduce a $\mu$-operator. It is an interesting problem for future research to find a more general framework in which *both* our intuitionistic systems and classical proof-theory could be fully expressed.

Lemma 1. (substitution) *If*

$$(\Theta_2 \rhd u : A)^\perp = a : A^\perp \rhd u^\perp : \Theta_2^\perp$$

*and*

$$(x : A, \Theta_1 \rhd t : B)^\perp = b : B^\perp \rhd \overline{\ell} : \Theta_1^\perp, \ell : A^\perp$$

*then*

$$(\Theta_2, \Theta_1 \rhd t[u/x] : B)^\perp = b : B^\perp \rhd \mathtt{l}(t^\perp) : \Theta_1^\perp, u^\perp[\ell/a] : \Theta_2^\perp$$

*and simmetrically.*

Notice that if $\Theta \rhd t : A$ then $t^\perp$ is *not* typed with $A^\perp$, but with the $\Theta^\perp$, i.e., with the dual of the *context* of the original type derivation, and $t^\perp$ is a list of terms distributed over $\Theta^\perp$. This remark seems related to the following fact: while in the lambda calculus a reduction operates *within a redex*, by substitution of a subterm in another subterm, in the dual calculus reductions involving `cont` and `postp` operate *outside the redex*, by remote substitutions in the context.

**§2. Informal interpretation.** As the proof theory of co-Heyting algebras formalizes a *logic*, we must specify which areas of *informal reasoning* such a logic is about. In fact we do have a "story to tell" about our technical results, as our work in dual intuitionistic logic is also part of investigations on the *logic for pragmatics*. This research project, introduced by Dalla Pozza and Garola in [12, 13] and developed in [4, 6, 5], aims at a formal characterization of the logical properties of *illocutionary operators*: it is concerned, e.g., with the operations by which we performs the act of *asserting* a proposition as true, either on the basis of a mathematical proof or by conclusive empirical evidence or by the recognition of physical necessity, or the act of taking a proposition as an *obligation*, either on the basis of a moral principle or by inference within a normative system.

The following is a rough account of the viewpoint in Dalla Pozza and Garola [12]. There is a logic of *propositions* and a logic of *judgements*. Propositions are entities which can be *true* of *false*, judgements are acts which can be *justified* or *unjustified*. The logic of propositions is about *truth* according to classical semantics. The logic of judgements gives conditions for the *justification* of acts of judgements. An instance of an elementary act of judgement is the *assertion* of a proposition $\alpha$, which is justified by the capacity to exhibit a *proof* of it, if $\alpha$ is a mathematical proposition, or conclusive *empirical evidence* if $\alpha$ is about states of affairs. It is then claimed that the justification of complex acts of judgement must be in terms of *Heyting's interpretation of intuitionistic connectives*: for instance, a *conditional judgement* where the assertion of $\beta$ depends on the assertibility of $\alpha$ is justified by a method that transforms any justification for the assertion of $\alpha$ into a justification for the assertion of $\beta$.

Which criteria shall we follow in extending the logic for pragmatics to a *logic of conjectures*? First of all, such a logic cannot deal with positive justifications of acts of conjecture, e.g., in terms of the *likelihood* of their propositional content being true: such a task would require probabilistic techniques not available here. Second, a characterization of the relations between acts of assertion and acts of conjecture may be based upon the similarity between what counts as a justification of the assertion that $\alpha$ is true, on one hand, and what counts as a refutation of the conjecture that $\alpha$ is false, on the other. Certainly in Dalla Pozza and Garola's approach, *proving the truth* of the proposition $\alpha$ is very close to *refuting the truth* of $\neg\alpha$. [5] Thus a formal treatment of the logic of conjectures could have the form of a *calculus of refutations* and the overall system should axiomatize a notion of *duality* between assertions and conjectures. Third, pragmatic connectives are regarded here as operations which express ways of building up complex acts of assertion or of conjecture from elementary acts of assertion and conjecture. The justification of a complex act depends on the justification of the component acts, possibly through intensional operations.

We are aware that there are notions of *conjecturing* in natural language and in scientific discourse which are not dual to *asserting*. In [4] the issue has been briefly discussed with reference to a (possibly) related philosophical issue, the atemporal existence of mathematical proofs. However a detailed philosophical investigation about notions of conjectures and assertions is more significant in the desing of a formal system *extending* both $\mathbf{LJ}^{\supset\cap}$ and $\mathbf{LJ}^{\smallsetminus\curlyvee}$, as in [4], indeed it is essential in making choices about such a system. Here for the

[5] In the formula $\mathcal{H}\,\neg\alpha$, the negation is *classical negation*, not the intuitionistic one: e.g., the conjecture $\mathcal{H}\,\neg\alpha$ may be refuted also by evidence that a certain state of affairs $\alpha$ does not obtain, not necessarily by a proof that there would be a contradiction assuming that $\alpha$ obtains.

time being we assume that there are structural similarities in common sense reasoning between the processes of justifying an assertion and of refuting a conjecture; we will appeal to these similarities in informal explanations of the *conjectural* connectives of $\mathbf{LJ}^{\smallsetminus\curlyvee}$ by comparison with the corresponding *assertive* ones of $\mathbf{LJ}^{\supset\cap}$.

Therefore our language for the logic of pragmatics deals with *act-types of assertion* $\vartheta$ and *act-types of conjecture* $\upsilon$.

Definition 1. (i) The languages $\mathcal{L}_\vartheta$ and $\mathcal{L}_\upsilon$ are generated from an infinite list of atomic propositions $p_i$ by the following grammars:

$$\alpha \;:=\; p \mid \neg p$$

$$\vartheta \;:=\; \vdash \alpha \mid \bigvee \mid \vartheta \supset \vartheta \mid \vartheta \cap \vartheta$$

$$\upsilon \;:=\; \mathcal{H}\,\alpha \mid \bigwedge \mid \upsilon \smallsetminus \upsilon \mid \upsilon \curlyvee \upsilon$$

(ii) We define the duality $(\;)^\perp$ between $\mathcal{L}_\vartheta^-$ and $\mathcal{L}_\upsilon^-$ inductively thus:

$$\begin{array}{rlrl}
(\vdash p)^\perp & = \mathcal{H}\neg p & (\mathcal{H} p)^\perp & = \vdash\neg p \\
(\vdash \neg p)^\perp & = \mathcal{H} p & (\mathcal{H}\neg p)^\perp & = \vdash p \\
(\bigvee)^\perp & = \bigwedge & (\bigwedge)^\perp & = \bigvee \\
(\vartheta_0 \supset \vartheta_1)^\perp & = \vartheta_1^\perp \smallsetminus \vartheta_0^\perp & (\upsilon_0 \smallsetminus \upsilon_1)^\perp & = \upsilon_1^\perp \supset \upsilon_0^\perp \\
(\vartheta_0 \cap \vartheta_1)^\perp & = \vartheta_0^\perp \curlyvee \vartheta_1^\perp & (\upsilon_0 \curlyvee \upsilon_1)^\perp & = \upsilon_0^\perp \cap \upsilon_1^\perp
\end{array}$$

**§3. Natural Deduction.** Natural deduction derivations in **NJ** represent the intuitionistic entailment relation $\Gamma \vdash A$ which holds between a set of hypotheses $\Gamma$ and a conclusion $A$. The representation of derivations as directed trees is deceptively simple: already in $\mathbf{NJ}^{\supset\cap}$ of **NJ** proof-graphs are trees with complex *additional structure* of logical, computational and geometric significance. A massive amount of research in proof theory, type theory and categorical logic, also stimulated by the Curry and Howard correpondence, has revisited this subject and thus Natural Deduction for minimal logic is well understood. However it often happens that investigations extending the scope of Gentzen's methods are forced to go back to the basic definitions.

It is important to notice, not just as an historical point but also in order to put current research in perspective, that in Prawitz's thesis [18] *two* distinct representations of Natural Deduction derivations are suggested, which may be called *sequent-style* (or *type-theoretic*) and *graphical*, respectively. On one hand, natural deductions may be regarded as trees whose edges are labelled with *sequents*, the formulas in the antecedent referring to the *assumption classes* which remain open at that edge; correct derivations are inductively generated from assumptions of the form $A, \Gamma \vdash A$ according to Prawitz's *deduction rules*. This view is common in type theory (see, e.g., Krivine [15], p. 34) where the assignment of variables to assumptions makes the discharging operations completely explicit (otherwise, the use of *labelled formulas* to characterize assumption classes is necessary).

On the other hand, we may regard proof-trees as directed labelled graphs, whose vertices are labelled with rules of inference and edges are labelled with formulas; *discharging functions* are pointers from leaves representing discharged assumptions to vertices representing discharging inferences ($\supset$-I and $\vee$-E in the propositional fragment). Discharging functions must satisfy a *correctness condition*: with each leaf in their domain they must associate a vertex occurring *below* that leaf in the tree-ordering; this correctness condition is in fact suggested by Prawitz's notation of the rules of inference $\rightarrow$-I and $\vee$-E.

Fundamental is the treatment of the tree transformations corresponding to the structural rules of Weakening and Contraction in the sequent calculus. (The consideration of Exchange, leading to *braided* systems, is beyond the scope of our investigation.) In the type-theoretical (sequent-style) representation, irrelevant assumptions may appear in the sequents whose variable do not occur in the term and an assumptions class is represented by labelling different leaves with a single variable, therefore Contraction can be represented by renaming of variables. In the graphical representation, in order to fix the identity of the derivation represented by a given correct graph some notation must be added to represent

(a) the partition of leaves into assumption and
(b) irrelevant assumptions that appear in the intended entailment relation but not in the graph;[6]

also the discharging function must be compatible with the partition in (a).

[6]In Prawitz [18] proof-graphs do not convey the information in (b), thus a correct proof-graph identifies a minimal set $\Gamma$ of hypotheses such that $\Gamma \vdash A$ but represents an infinite set of entailments, namely, all the $\Pi \vdash A$ such that $\Gamma \subseteq \Pi$. Since in [18] the &-I rule is *multiplicative* but the &-E rules are *additive* (and dually for disjunction), in the normalization process some actual dependencies may become vacuous; thus without representation of vacuous dependencies the entailment relation represented by the proof-graph changes with normalization.

Prawitz's thesis fills in the gaps of both representations by combining them: we may say that a proof-graph with a discharging function is a correct representation of (an infinite class of) Natural Deduction derivations if it can be inductively generated using the global rules of deduction.More recent developments (as Girard's proof nets for linear logic and further generalizations to affine, intuitionistic, non-commutative and braided variants of linear logic) show that the interplay between the two representations (as well between the sequent calculus and natural deduction) is a fruitful research tool. Although we are concerned mostly with natural deduction as a tool for typing derivations, we find it conceptually convenient to follow Prawitz and retain both representations[7].

In a directed proof-tree, the direction from the leaves to the root, may be called *main orientation*. Prawitz's analysis of *branches* in normal deductions for the fragment $\mathbf{NJ}^{\supset\cap}$ ([18] p. 41) identifies an *elimination part*, where vertices are $\supset$-E (*applications*) or $\cap$-E (*projections*), followed by an *introduction part*, where vertices are $\supset$-I ($\lambda$-*abstractions*) or $\cap$-I (*pairings*). Branches are connected at an application vertex $\supset$-E: the child branch terminates at the minor premise (*argument position*) while the parent branch continues from the major premise (*function position*) to the conclusion. This analysis identifies a "*flow of information*", from the elimination part of a branch to its introduction part, and from a branch to its parent, which may be called the *input-output orientation*[8]. It is a remarkable feature of Natural Deduction for $\mathbf{NJ}^{\supset\cap}$ that the input-output orientation and the main orientation *coincide* in a deduction tree. In the sequent calculus **LJ** the input-output orientation is "contravariant" in the antecedent and "covariant" in the succedent, namely, it runs from a formula to its ancestors in the antecedent and conversely in the succedent.

[7]We shall use the symbol "$\Rightarrow$" for the consequence relation in the sequent calculus, "$\vdash$" in the deduction rules of Natural Deduction systems and "$\rhd$" in type derivations.

[8]The terminology comes from research communities working on process calculi and on linear logic in the early 1990s. Prawitz' analysis of branches is robust and has clear counterparts in the lambda-calculus, in game semantics and in other representations of proofs. See also the applications of this concept to proof-nets in [2] and in F. Lamarche's *essential nets*, fruitful both for the abstract theory [3] and for applications to complexity theory [17].

In the full system **NJ**, which includes the (ordinary assertive) disjunction $\vee$, the analysis of branches is subsumed in that of *paths* ([18], pp. 52-3). Paths extend branches but in addition they go from the major premise of a $\vee$-elimination $\mathcal{I}$ to any one of the assumptions in the classes discharged by the inference $\mathcal{I}$. Thus in the case of $\vee$-elimination the main orientation diverges from input-output orientation: here the tree structure of the derivation performs a *control function*, namely, the verification that the minor premises of the inference coincide.

**3.1. Natural Deduction for conjectural reasoning.** As in $\mathbf{NJ}^{\supset\cap}$, also in $\mathbf{NJ}^{\smallsetminus\curlyvee}$ two representations of proofs are possible, one "sequent-style" (type-theoretic) and the other "graphical"; also fundamental is the representation of Weakening and Contraction, here *on the conclusions*. The graphical representation of $\mathbf{NJ}^{\smallsetminus\curlyvee}$ derivations merely relabels a proof-graph for $\mathbf{NJ}^{\supset\cap}$ after "turning it upside down": additional notation is introduced to specify *conclusion classes* and to represent irrelevant conclusions; assumption-discharging function are replaced by conclusion-discharging ones. Prawitz's branches are connected at a $\smallsetminus$-I vertex, the child branch beginning at the *minor conclusion*. The "flow of information" still goes from the elimination part to the introduction part of a branch, but from the introduction part of a parent branch it goes to the elimination part of a child branch. In our graphical representation of rules of inference, edges belonging to the elimination (*input*) part or introduction part (*output*) part are marked with **I** or **O**, respectively. This yields inference rules for *subtraction* perfectly symmetric to those for *implication*: $\smallsetminus$-I is dual to $\supset$-E:

$$\begin{array}{c}\vdots\\ \dfrac{\upsilon_1^{\mathbf{O}}}{\upsilon_2^{\mathbf{I}} \qquad (\upsilon_1 \smallsetminus \upsilon_2)^{\mathbf{O}}}\ \smallsetminus\text{-I}\\ \vdots \qquad\qquad \vdots\end{array} \qquad\qquad \begin{array}{c}\vdots \qquad\qquad \vdots\\ \dfrac{(\vartheta_1 \supset \vartheta_2)^{\mathbf{I}} \qquad \vartheta_1^{\mathbf{O}}}{\vartheta_2^{\mathbf{I}}}\ \supset\text{-E}\\ \vdots\end{array}$$

$\smallsetminus$-E is dual to $\supset$-I:

$$\begin{array}{c}\vdots\\ \dfrac{(\upsilon_2 \smallsetminus \upsilon_1)^{\mathbf{I}}}{\upsilon_2^{\mathbf{I}}}\ \smallsetminus\text{-E}\\ \vdots\\ [\upsilon_1^{\mathbf{O}} \ldots \upsilon_1^{\mathbf{O}}]\end{array} \qquad\qquad \begin{array}{c}[\vartheta_1]^{\mathbf{I}}\\ \vdots\\ \dfrac{\vartheta_2^{\mathbf{O}}}{(\vartheta_1 \supset \vartheta_2)^{\mathbf{O}}}\ \supset\text{-I}\\ \vdots\end{array}$$

For the sequent-style and type theoretic representation, there is no problem in giving multiple-conclusion deduction rules to be used as clauses for the inductive definition of derivations, which follows the *main orientation* of the derivation, *in all cases except for* $\smallsetminus$-E: here indeed (looking at the graphical representation) the inductive step must reverse the main orientation and, taking the the subderivation starting from the *conclusion* as inductive clause, extend it upwards to the premise. The problem, of course, is solved by adopting the familiar form of elimination rules and taking a multi-premises rule. This suggests also an alternative graphical representation of the elimination rule:

$$\dfrac{\begin{matrix} \\ \vdots \\ (\upsilon_2 \smallsetminus \upsilon_1)^{\mathbf{I}} \end{matrix} \qquad \begin{matrix} [\upsilon_2]^{\mathbf{I}} \\ \vdots \\ \upsilon_1^{\mathbf{O}} \ldots \upsilon_1^{\mathbf{O}} \end{matrix}}{} \smallsetminus\text{-E}$$

The deduction rules for subtraction are

$$\smallsetminus\text{-}I\text{:}\quad \dfrac{\epsilon \ \Vdash\ \Upsilon,\ \upsilon_1}{\epsilon \ \Vdash\ \Upsilon,\ \upsilon_2,\ \upsilon_1 \smallsetminus \upsilon_2} \qquad \smallsetminus\text{-}E\text{:}\quad \dfrac{\epsilon \Vdash \Upsilon, \upsilon_1 \smallsetminus \upsilon_2 \qquad \upsilon_1 \ \Vdash\ \Upsilon', \upsilon_2}{\epsilon \ \Vdash\ \Upsilon,\ \Upsilon'}$$

The $\smallsetminus$-introduction rule has following "operational interpretation": if from the conjecture $\epsilon$ the alternative conjectures $\Upsilon$, $\upsilon_1$ follow, then we may we specify our alternative $\upsilon_1$ by taking it as "$\upsilon_1$ *but not* $\upsilon_2$", on one hand, and by considering also $\upsilon_2$ as an alternative, on the other hand.

The $\smallsetminus$-elimination rule can be explained as follows. Suppose we have two arguments, one showing that $\epsilon$ yields the alternatives $\Upsilon$ or else "$\upsilon_1$ *but not* $\upsilon_2$", and another showing that $\upsilon_1$ yields the alternatives $\Upsilon'$ or $\upsilon_2$; then after assuming $\epsilon$ we are left with the alternatives $\Upsilon$ and $\Upsilon'$, but $\upsilon_1 \setminus \upsilon_2$ is no longer a consistent option.

The *dynamics* of our calculus is illustrated by the following reduction of a cut (*maximal formula*) $\upsilon_1 \smallsetminus \upsilon_2$:

$$\dfrac{\dfrac{\epsilon \Vdash \Upsilon, \upsilon_1}{\epsilon \ \Vdash\ \Upsilon, \upsilon_2, \upsilon_1 \smallsetminus \upsilon_2}\smallsetminus\text{-I} \qquad \upsilon_1 \ \Vdash\ \Upsilon', \upsilon_2}{\epsilon \Vdash\ \Upsilon, \Upsilon', \upsilon_2}\smallsetminus\text{-E}$$

reduces to

$$\epsilon \ \Vdash\ \Upsilon,\ \Upsilon', \upsilon_2$$

The rules of inference for disjunction are unprobleматic: they are the exact dual of those for conjunction

$$\dfrac{\begin{array}{c}\vdots\\ \upsilon_i^{\mathbf{O}}\end{array}}{\begin{array}{c}(\upsilon_0 \curlyvee \upsilon_1)^{\mathbf{O}}\\ \vdots\end{array}}\ \curlyvee_i\text{-I} \qquad \text{for } i = 0, 1 \qquad \dfrac{\begin{array}{c}\vdots\\ (\vartheta_0 \cap \vartheta_1)^{\mathbf{I}}\end{array}}{\begin{array}{c}\vartheta_i^{\mathbf{I}}\\ \vdots\end{array}}\ \cap_i\text{-E}$$

and

$$\dfrac{\begin{array}{c}\vdots\\ (\upsilon_0 \curlyvee \upsilon_1)^{\mathbf{I}}\end{array}}{\begin{array}{cc}\upsilon_0^{\mathbf{I}} & \upsilon_1^{\mathbf{I}}\\ \vdots & \vdots\end{array}}\ \curlyvee\text{-E} \qquad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \vartheta_0^{\mathbf{O}} & \vartheta_1^{\mathbf{O}}\end{array}}{\begin{array}{c}(\vartheta_0 \cap \vartheta_1)^{\mathbf{O}}\\ \vdots\end{array}}\ \cap\text{-I}$$

Similarly unproblematic are the deduction rules and the dynamics of normalization for $\curlyvee$.

The rules of inference and deduction for $\mathbf{NJ}^{\supset\cap}$ and $\mathbf{NJ}^{\smallsetminus\curlyvee}$ are listed in the Appendix in Tables 3 and 4. The reduction rules are listed in Table 5 and the commutation rules resulting from the absurdity and validity rules in Table 6.

## §4. Term Assignment for conjectural reasoning.

Definition 2. We are given a countable set of *free variables* (denoted by $x$, $y$, $a$, $b$), a contable set of *globally bound variables* (denoted by $\mathtt{x}$) and an *injective* operation $(\dots)_{(-)}$ which given a variable $x$ and a term $\ell$ returns a bound variable $\mathtt{x}$, written $\mathtt{x}_\ell$.

(i) Terms and lists of terms are defined simulaneously by the following grammar:

$$\begin{array}{rcl} t & := & x \mid \mathtt{x}_\ell \mid \mathtt{false}\ (\ell_1 \dots \ell_n) \mid \mathtt{inl}(\ell) \mid \mathtt{inr}(\ell) \mid \mathtt{casel}\ (\ell) \mid \mathtt{caser}\ (\ell) \mid \\ & & \mathtt{continue\ from}\ (\mathtt{x}_\ell)\ \mathtt{using}(\ell) \mid \mathtt{postpone}\ (x :: \ell)\ \mathtt{with}\ (\mathtt{x}_{\ell'})\ \mathtt{until}(\ell') \\ \ell & := & () \mid t \cdot \ell \end{array}$$

with the usual associative operation of *append*:

$$() * \ell' = \ell' \qquad\qquad (t \cdot \ell) * \ell' = t \cdot (\ell * \ell').$$

If $\ell = (t_1, \dots, t_n)$ is a list and $\sigma$ a permutation on $n$ then we write $\ell_\sigma$ for $(t_{\sigma(1)}, \dots, t_{\sigma(n)})$. Also, if $\overline{\ell} = \ell_1, \dots, \ell_n$ and $\overline{\ell'} = \ell'_1, \dots, \ell'_n$ are *vectors* of lists (of the same length $n$), then $\overline{\ell} * \overline{\ell'} = \ell_1 * \ell'_1, \dots, \ell_n * \ell'_n$.

(ii) *Term expansion:* Let $\mathtt{op}$ ( ) be one of $\mathtt{x}_{(\ )}$, $\mathtt{false}\ (\ell_1 \dots (\ ) \dots \ell_n)$, $\mathtt{inl}(\ )$, $\mathtt{inr}(\ )$, $\mathtt{casel}(\ )$, $\mathtt{caser}(\ )$, $\mathtt{continue\ from}\ (\mathtt{x}_{(\ )})\ \mathtt{using}\ (\ )$ or $\mathtt{postpone}\ (x :: \ell')\ \mathtt{with}\ (\mathtt{x}_{(\ )})\ \mathtt{using}(\ )$.

Then the *expansion* of $\texttt{op}\ (\ell)$ is the list defined inductively thus:

$$\texttt{op}\ () = () \qquad\qquad \texttt{op}\ (t \cdot \ell) = \texttt{op}\ (t) \cdot \texttt{op}\ (\ell)$$

*Remark.* By term expansion, a term consisting of an operator applied to a list of terms can always be turned into a list of terms; thus terms may always be trasformed into an expanded form where operators are applied only to terms, *except for expressions* $(x :: \ell')$ occurring in terms of the form $\texttt{postpone}\ (x :: \ell')\ \texttt{with}\ \texttt{x}_{\ell'}\ \texttt{using}\ (\mathrm{t})$.

Definition 3. The *free variables* $FV(\ell)$ in a list of terms $\ell$ are defined as follows:

$$\begin{array}{rl}
FV(()) = & \emptyset \\
FV(t \cdot \ell) = & FV(t) \cup FV(\ell) \\
FV(x) = & \{x\} \\
FV(\texttt{x}_\ell) = & FV(\ell) \\
FV(\texttt{false}\ (\ell_1 \ldots \ell_n)) = & \bigcup_{i \leq n} FV(\ell_i) \\
FV(\texttt{inl}(\ell)) = FV(\texttt{inr}(\ell)) = & FV(\ell) \\
FV(\texttt{casel}(\ell)) = FV(\texttt{caser}(\ell)) = & FV(\ell) \\
FV(\texttt{continue from}\ (\texttt{x}_\ell)\ \texttt{using}\ (\ell)) = & FV(\ell) \\
FV(\texttt{postpone}\ (x :: \ell)\ \texttt{with}\ (\texttt{x}_{\ell'})\ \texttt{until}\ (\ell') = & FV(\ell') \cup FV(\ell) \setminus \{x\}.
\end{array}$$

Definition 4. Substitution of *lists of terms* within lists of terms is defined from the usual substitution (avoiding capture of free variables) as follows:

$$\begin{array}{rl}
()[\ell'/x] = () & \qquad t \cdot \ell[\ell'/x] = t[\ell'/x] \cdot \ell[\ell'/x] \\
t[()/x] = () & \qquad t[u \cdot \ell/x] = t[u/x] \cdot t[\ell/x]
\end{array}$$

If $\overline{\ell}$ is a vector $(\ell_1, \ldots, \ell_m)$, then $\overline{\ell}[\ell'/x] = (\ell_1[\ell'/x], \ldots, \ell_m[\ell'/x])$.

Definition 5. $\beta$-reduction $\ell \rightsquigarrow_\beta \ell'$ is defined as follows:

$$\begin{array}{rl}
\texttt{casel}\ (\texttt{inl}\ \ell) \rightsquigarrow_\beta\ \ell; & \qquad \texttt{caser}\ (\texttt{inr}\ \ell) \rightsquigarrow_\beta\ \ell; \\
\texttt{casel}\ (\texttt{inr}\ \ell) \rightsquigarrow_\beta (); & \qquad \texttt{caser}\ (\texttt{inl}\ \ell) \rightsquigarrow_\beta (); \\
\texttt{postpone}\ (x :: \ell')\ \texttt{with}\ \texttt{x}_\texttt{c}\ \texttt{until} & \\
(\texttt{continue from}\ (\texttt{y}_\ell)\ \texttt{using}(\ell)) \rightsquigarrow_\beta & \quad \{\texttt{y}_\ell ::= \ell'[\ell/x]\}, \{\texttt{x}_\texttt{c} ::= \ell\}.
\end{array}$$

Here we have written $\texttt{x}_\texttt{c}$ for $\texttt{x}_{\texttt{continue from}\ (\texttt{y}_\ell)\ \texttt{using}(\ell)}$.

**4.1. Typing judgements for the calculus of continuations.** The typing judgements for $\mathbf{NJ}^{\setminus\curlyvee}$ are in the following Table 1.

*Remark.* ($i$) The implicit presence of contraction in the conclusion requires formulas to be labelled with *multisets* of terms, rather than with terms alone. We represent *multisets* by *lists* (*modulo* permutations $\sigma$). Thus Contraction of multisets of terms is implemented by the operation "$*$" (*append*) of lists (*modulo* permutations) and Weakening by the empty list "()".

**Typing judgement for $\mathbf{NJ}^{\smallsetminus\curlyvee}$**

*exchange:*
$$\frac{\epsilon \;\rhd\; \Upsilon, \ell : \upsilon, \ell'\upsilon', \Upsilon'}{\epsilon \;\rhd\; \Upsilon, \ell' : \upsilon', \ell : \upsilon, \Upsilon'}$$

*contraction:*
$$\frac{\Theta \;\rhd\; \Upsilon, \ell : \upsilon, \ell' : \upsilon}{\Theta \;\rhd\; \Upsilon, \ell * \ell' : \upsilon}$$

*weakening:*
$$\frac{\Theta \;\rhd\; \Upsilon}{\Theta \;\rhd\; \Upsilon, () : \upsilon}$$

*assumption* :
$$x : \upsilon \rhd x : \upsilon$$

$$\frac{\epsilon \rhd \Upsilon, \ell : \upsilon \qquad x : \upsilon \rhd \overline{\ell} : \Upsilon'}{\epsilon \rhd \Upsilon, \overline{\ell}[\ell/x] : \Upsilon'}\ \textit{-substitution}$$

$$\frac{\epsilon \rhd \ell : \bigwedge}{\epsilon \rhd \texttt{false}\ (\ell) : \upsilon_1\ \ldots\ \texttt{false}\ (\ell) : \upsilon_n}\ \bigwedge\text{-E}$$

$$\frac{\epsilon \rhd\ \ell : \upsilon_1, \Upsilon}{\epsilon \rhd\ \mathtt{y}_\ell : \upsilon_2,\ \texttt{continue from}\ (\mathtt{y}_\ell)\ \texttt{using}\ (\ell) : \upsilon_1 \smallsetminus \upsilon_2, \Upsilon}\ \smallsetminus\text{-I}$$

$$\frac{\epsilon \;\rhd\; \Upsilon, \ell' : \upsilon_1 \smallsetminus \upsilon_2 \qquad x : \upsilon_1 \rhd \ell : \upsilon_2,\ \overline{\ell} : \Upsilon'}{\epsilon \rhd\ \Upsilon,\ \overline{\ell}[\mathtt{x}_{\ell'}/x] : \Upsilon,\ \texttt{postpone}\ (x :: \ell)\ \texttt{with}\ (\mathtt{x}_{\ell'})\ \texttt{until}\ \ell' : \bullet}\ \smallsetminus\text{-E}$$

$$\frac{\epsilon \rhd \ell : \upsilon_0, \Upsilon}{\epsilon \rhd \texttt{inl}\ (\ell) : \upsilon_0 \curlyvee \upsilon_1, \Upsilon}\ \curlyvee_0\text{-I} \qquad \frac{\epsilon \rhd \ell : \upsilon_1, \Upsilon}{\epsilon \rhd \texttt{inr}\ (\ell) : \upsilon_0 \curlyvee \upsilon_1, \Upsilon}\ \curlyvee_1\text{-I}$$

$$\frac{\epsilon \rhd\ \Upsilon, \ell : \upsilon_0 \curlyvee \upsilon_1}{\epsilon \rhd\ \Upsilon, \texttt{casel}\ (\ell) : \upsilon_0, \texttt{caser}\ (\ell) : \upsilon_1}\ \curlyvee\text{-E}$$

TABLE 1. Typing judgements for $\mathbf{NJ}^{\smallsetminus\curlyvee}$

(*ii*) The operation $(\ldots)_{(-)}$ takes a free variable $y : \upsilon_2$ and a term $\ell : \upsilon_1$, which represents a set of paths (there may be more than one path, in view of contraction) in the proof-graph and generates a bound variable $\mathtt{y}_\ell : \upsilon_2$. Thus the operation $(y)_{(\ )}$ is of type $\upsilon_1 \to \upsilon_2$; since it is injective, the new bound variable is *fresh*, it cannot occur elsewhere in the proof-graph.

(*iii*) The occurrence of $\ell$ as a subterm of

$$\mathbf{t}(\ell) = \texttt{continue from}\ (\mathtt{y}_\ell)\ \texttt{using}\ (\ell) : \upsilon_1 \smallsetminus \upsilon_2$$

allows us to locate the edge $\mathbf{t}(\ell)$ as the *major conclusion* of the inference represented by the $\smallsetminus - I$ node, which has $\ell : \upsilon_1$ as incoming edge, the *premise*. Moreover, the occurrence of the fresh bound variable $\mathtt{y}_\ell$ in $\mathbf{t}(\ell)$ establishes a connection with the edge $\mathtt{y}_\ell : \upsilon_2$ as the *minor conclusion* of the same inference, a conclusion which is essential for the implementation of the rewriting in the $\beta$-reduction. A *bound* variable must be assigned to the edge $\upsilon_2$, as any substitution of a term for $\mathtt{y}$ must occur only within the process of $\beta$-reduction.

(*iv*) Part (iii) gives us terms

$\mathbf{t}() = \mathtt{continue\ from}\ (\mathtt{y}\ )\ \mathtt{using}\ (\ ) : \upsilon_1 \to (\upsilon_1 \setminus \upsilon_2)$ and
$(y)_{(\ )} : \upsilon_1 \to \upsilon_2$.

Hence the term $[\mathbf{t}(), (y)_{(\ )}] : \upsilon_1 \to (\upsilon_1 \setminus \upsilon_2) + \upsilon_2$ is the dual of $app_{\upsilon_2^\perp, \upsilon_2^\perp}$, as required in section 1.

(*v*) Let $\ell = (t_1, \ldots, t_m)$. The expression $x :: \ell$, which could be read as $\lambda x.\ell$, codes the subpaths of the proof-graph from the edge $x$ to the edges $t_i : \upsilon_2$, for all $i \leq m$. Thus the term $\mathtt{postpone}$ binds $x$ in the usual sense. But as a *side effect* of the binding of $x$, *other* paths from $x$ to the edges in $\overline{\ell}$ are affected: indeed no substitution of a term $s$ for $x$ in $\overline{\ell}$ is possible, so long as $x$ remains bounded by the $\mathtt{postpone}$ term: this effect we call a *remote binding*. Thus we need to substitute a bound variable for $x$ in $\overline{\ell}$: the device in (ii) provides us with $\mathtt{x}_{\ell'}$, where $\ell' : \upsilon_1 \setminus \upsilon_2$ represents the paths leading to the *major premise* of the $\setminus$-E inference. Notice also that the $\mathtt{postpone}$ term must access the bound variable $\mathtt{y}_{\ell'}$ for the *remote rewriting* required by the $\beta$-reduction to succeed.

(*vi*) Terms $\mathbf{u}(\ )$ of the form $\mathtt{postpone}\ (y :: \ell)\ \mathtt{with}\ (\mathtt{x}_{(\ )})\ \mathtt{until}(\ )$ are really of type $(\upsilon_1 \setminus \upsilon_2) \to \perp$, hence we should give $\mathbf{u}(\ell')$ the type $\perp$ (i.e., $\bigwedge$). We write instead $\mathbf{u}(\ell') : \bullet$ because these terms are regarded as *control terms*, "global" expressions that cannot occur as subterms of other terms.

(*vii*) The term $\overline{\ell}[\mathtt{x}_{(\ )}/x]$ can be written as $\overline{\ell}_* : (\upsilon_1 \setminus \upsilon_2) \to \Upsilon$, and it is dual to $u^* : \Upsilon^\perp \to (\upsilon_2^\perp \supset \upsilon_2^\perp)$. Hence

$$[\overline{\ell}[\mathtt{x}_{(\ )}/x], \mathbf{u}(\ )] : (\upsilon_1 \setminus \upsilon_2) \to \Upsilon + \perp$$

as required in section 1.

(*viii*) Let us consider now a redex of the form

$$\dfrac{\dfrac{\epsilon \;\rhd\; \Upsilon, \ell : \upsilon_1}{\epsilon \;\rhd\; \Upsilon,\ \mathtt{y}_\ell : \upsilon_2,\ \mathbf{t}(\ell) : \upsilon_1 \setminus \upsilon_2}\setminus\text{-I} \qquad x : \upsilon_1 \;\rhd\; \overline{\ell} : \Upsilon',\ \ell' : \upsilon_2}{\epsilon \;\rhd\; \Upsilon,\ \overline{\ell}[\mathtt{x}_{\mathtt{c}}/x] : \Upsilon',\ \mathtt{y}_\ell : \upsilon_2,\ \mathbf{r} : \bullet}\setminus\text{-E}$$

where
$\mathbf{t}(\ell) = \mathtt{continue\ from}\ \mathtt{y}_\ell\ \mathtt{using}\ \ell$ as in (iii),

$\mathbf{r} = \texttt{postpone}\ (x :: \ell')\ \texttt{with}\ \texttt{x}_\texttt{c}\ \texttt{until}\ (\texttt{continue from}\ \texttt{y}_\ell\ \texttt{using}\ \ell)$ and $\texttt{x}_\texttt{c}$ stands for $\texttt{x}_{\texttt{continue from }(\texttt{y}_\ell)\texttt{ using}(\ell)}$.

With $\beta$-reduction the redex is destroyed, the direct and remote binding are removed and the term $\ell : \upsilon_1$ can be substituted for $x$ both in $\ell'$ and in $\overline{\ell}$:

$$\frac{\epsilon \ \rhd\ \Upsilon, \ell : \upsilon_1 \qquad x : \upsilon_1 \ \rhd\ \overline{\ell} : \Upsilon',\ \ell' : \upsilon_2}{\epsilon \ \rhd\ \Upsilon,\ \overline{\ell}[\ell/x] : \Upsilon',\ \ell'[\ell/x] : \upsilon_2}\ \textit{substitution}$$

Since both $\ell'$ and $\ell$ occur in the redex $\mathbf{r}$, the substitution $\ell'[\ell/x]$ is a transformation of $\mathbf{r}$ and can be immediately executed, but then the command must be *broadcast* to the context to replace the result $\ell'[\ell/x]$ for the term $\texttt{y}_\ell : \upsilon_2$. We express this command by the "control term" $\{\texttt{x}_\ell ::= \ell'[\ell/y]\}$.

Similarly, the substitution $\overline{\ell}[\ell/x]$ is implemented by broadcasting the command to replace $\ell$ for $\texttt{y}_\texttt{c}$ in all the terms $\overline{\ell}[\texttt{x}_\texttt{c}]$. This command is expressed by $\{\texttt{y}_\texttt{c} ::= \ell\}$.

As indicated above, it is possible to broadcast these commands only because the `continue` term has access to $\texttt{y}_\ell$ and the term `postpone` has access to $\texttt{y}_\texttt{c}$.

We shall not try to implement remote substitution here, but we expect the specification of their action may be achieved in an elegant way using familiar techniques from calculi for concurrency.

**4.2. Isomorphism theorem.** To prove the theorem 1, we need to show that (modulo $\alpha$-equivalence) there exists a bijection $(\ )^\perp$ between proof-terms for $\mathbf{NJ}^{\supset\cap}$ and for $\mathbf{NJ}^{\smallsetminus\curlyvee}$ such that

(i) *if* $\overline{x} : \Theta \ \rhd\ t : \vartheta$ *then* $(t)^\perp$ *has the form* $\overline{\ell}$ *and* $x : \vartheta^\perp \ \rhd\ \overline{\ell} : \Theta^\perp$ *and conversely, if* $x : \epsilon \ \rhd\ \overline{\ell} : \Upsilon$ *then* $(\overline{\ell})^\perp$ *has the form* $t$ *and* $\overline{x} : \Upsilon^\perp \ \rhd\ t : \epsilon^\perp$*; moreover,* $(t)^{\perp\perp} = t$ *and* $(\overline{\ell})^{\perp\perp} = \overline{\ell}$.

(ii) *if* $t_0$ *$\beta$-reduces to* $t_1$ *then* $(t_0)^\perp$ *$\beta$-reduces to* $(t_1)^\perp$*; conversely, if* $\overline{\ell}_0$ *$\beta$-reduces to* $\overline{\ell}_1$*, then* $(\overline{\ell}_0)^\perp$ *$\beta$-reduces to* $(\overline{\ell}_1)^\perp$.

**Proof.** We write $\overline{x} : \Upsilon^\perp$ for $x_1 : \upsilon_1^\perp, \ldots x_n : \upsilon_1^\perp$ and $\overline{\texttt{false}}\ (x) : \Upsilon$ for $\texttt{false}_{\upsilon_1}(x) : \upsilon_1, \ldots\ \texttt{false}_{\upsilon_n}(x) : \upsilon_n$, and so on. The duality $(\ )^\perp$ on proof-terms is defined as follows. Setting $x^\perp = x$, the judgement $x : \vartheta \ \rhd\ x : \vartheta$ is mapped to $x^\perp : \vartheta^\perp \ \rhd\ x^\perp : \vartheta^\perp$ and conversely.

$$(1.1)\qquad (x : \bigwedge \ \rhd\ \overline{\texttt{false}}\ (x) : \Upsilon)^{\perp} \ = \ \overline{x} : \Upsilon^{\perp}\ \rhd\ \texttt{true}\ (\overline{x}) : \bigvee$$

$$(1.2)\qquad (\overline{x} : \Theta \rhd \texttt{true}\ (\overline{x}) : \bigvee)^{\perp} \ = \ x : \bigwedge \rhd \overline{\texttt{false}}\ (x) : \Theta^{\perp}$$

$$(2.1)\qquad (z : \vartheta_1 \supset \vartheta_2, y : \vartheta_1 \rhd y : \vartheta_2)^{\perp} \ = \ x : \vartheta_2^{\perp} \rhd \mathtt{y}_x : \vartheta_1^{\perp}, r : \vartheta_2^{\perp} \setminus \vartheta_1^{\perp}$$

$$(2.2)\qquad (x : \upsilon_1 \rhd \mathtt{y}_x : \upsilon_2, r : \upsilon_1 \setminus \upsilon_2)^{\perp} \ = \ y : \upsilon_2^{\perp},\ z : \upsilon_2^{\perp} \supset \upsilon_1^{\perp},\ \rhd\ zy : \upsilon_1^{\perp}$$

where $r =$ `continue from` $\mathtt{y}_x$ `using` $x$.

$$(3.1)\qquad (y : \vartheta_0 \cap \vartheta_1 \rhd \pi_0(y) : \vartheta_0)^{\perp} \ = \ x : \vartheta_0^{\perp} \rhd \texttt{inl}\ x : \vartheta_0^{\perp} \curlyvee \vartheta_1^{\perp}$$

$$(3.2)\qquad (x : \upsilon_0 \rhd \texttt{inl}\ x : \upsilon_0 \curlyvee \upsilon_1)^{\perp} \ = \ y : \upsilon_0^{\perp} \cap \upsilon_1^{\perp} \rhd \pi_0(y) : y : \upsilon_0^{\perp}$$

$$(4.1)\qquad (y : \vartheta_1 \cap \vartheta_1 \rhd \pi_1(y) : \vartheta_1)^{\perp} \ = \ x : \vartheta_1^{\perp} \rhd \texttt{inr}\ x : \vartheta_0^{\perp} \curlyvee \vartheta_1^{\perp}$$

$$(4.2)\qquad (x : \upsilon_1 \rhd \texttt{inr}\ x : \upsilon_0 \curlyvee \upsilon_1)^{\perp} \ = \ y : \upsilon_0^{\perp} \cap \upsilon_1^{\perp} \rhd \pi_1(y) : y : \upsilon_1^{\perp}$$

Now suppose

$$(\overline{x} : \Theta \rhd t_i : \vartheta_i)^{\perp} = y_i : \vartheta_i^{\perp} \rhd \overline{\ell_i} : \Theta^{\perp}$$

for $i = 0$ and 1. We set

$$(5.1)\qquad (\overline{x} : \Theta \rhd < t_0, t_1 >: \vartheta_0 \cap \vartheta_1)^{\perp} \ = \ z : \vartheta_0^{\perp} \curlyvee \vartheta_1^{\perp}\ \rhd\ \mathbf{r}_0 * \mathbf{r}_1 : \Theta^{\perp}$$

where $\mathbf{r}_0 = \overline{\ell_0}[\texttt{casel}\ z/y_0]$ and $\mathbf{r}_1 = \overline{\ell_1}[\texttt{caser}\ z/y_1]$.
Next suppose

$$(y_i : \upsilon_i \rhd \overline{\ell_i} : \Upsilon)^{\perp} = \overline{x} : \Upsilon^{\perp} \rhd t_i : \upsilon_i^{\perp}$$

for $i = 0$ and 1. We set

$$(5.2)\qquad (z : \upsilon_0 \curlyvee \upsilon_1\ \rhd\ \mathbf{r}_0 : \Theta_0^{\perp}, \mathbf{r}_1 : \vartheta_1^{\perp})^{\perp} \ = \ \overline{x} : \Upsilon^{\perp}\ \rhd < t_0, t_1 >: \upsilon_0^{\perp} \cap \upsilon_1^{\perp}$$

where again $\mathbf{r}_0 = \overline{\ell_0}[\texttt{casel}\ z/y_0]$ and $\mathbf{r}_1 = \overline{\ell_1}[\texttt{caser}\ z/y_1]$.

Now suppose

$$(x : \vartheta_1,\ \overline{x} : \Theta\ \ \rhd\ \ t : \vartheta_2)^{\perp} = y : \vartheta_2^{\perp} \rhd \ell_1 : \vartheta_1^{\perp}, \ldots, \ell_m : \vartheta_1^{\perp}, \overline{\ell} : \Theta^{\perp}$$

We set

$$(6.1)\qquad (\overline{x} : \Theta \rhd \lambda x.t : \vartheta_1 \supset \vartheta_2)^{\perp} = z : \vartheta_2^{\perp} \setminus \vartheta_1^{\perp}\ \rhd \overline{\ell}[\mathtt{y}_z/y] : \Theta^{\perp},\ \mathbf{u} : \bullet$$

where $\mathbf{u} =$ `postpone` $(y :: \ell_1 * \ldots * \ell_m)$ `with` $\mathtt{y}_z$ `until` $(z)$.

Finally suppose

$$(x : \upsilon_1 \rhd \ell_1 : \upsilon_2, \ldots, \ell_m : \upsilon_2, \overline{\ell} : \Upsilon)^{\perp} = y : \upsilon_2^{\perp},\ \overline{x} : \Upsilon^{\perp}\ \ \rhd\ \ t : \upsilon_1^{\perp}$$

We set

(6.2) $(z : \upsilon_2 \setminus \upsilon_1 \;\rhd\; \overline{\ell}[\mathsf{y}_z/y] : \Upsilon,\ \mathbf{u} : \bullet)^\perp = \overline{x} : \Upsilon^\perp \;\rhd\; \lambda y.t : \upsilon_2^\perp \supset \upsilon_1^\perp$
where $\mathbf{u} = \texttt{postpone}\ (y :: \ell_1 * \ldots * \ell_m)\ \texttt{with}\ \mathtt{y}_z\ \texttt{until}\ z$.

Next we need to prove the Substitution lemma 1:

*If*

$$(i) \qquad (\Theta_2 \rhd u : \vartheta)^\perp = a : \vartheta^\perp \rhd \overline{\ell}_2 : \Theta_2^\perp$$

*and*

$$(ii) \qquad (x : \vartheta, \Theta_1 \rhd t : \vartheta_0)^\perp = b : \vartheta_0^\perp \rhd \overline{\ell}_1 : \Theta_1^\perp, \ell : \vartheta^\perp$$

*then*

$$(\Theta_2, \Theta_1 \rhd t[u/x] : \vartheta_0)^\perp = b : \vartheta_0^\perp \rhd \overline{\ell}_1 : \Theta_1^\perp, \ell_2[\ell/a] : \Theta_2^\perp$$

*and simmetrically for substitutions in the conjectural part.*

We prove the lemma by induction on $t$. Let us consider the case of $t = \lambda y.s$. Given $(i)$ and

$$(ii) \quad (x : \vartheta, \Theta_1 \rhd \lambda y.s : \vartheta_1 \supset \vartheta_2)^\perp = b : \vartheta_2^\perp \setminus \vartheta_1^\perp \quad \rhd \quad \overline{\ell}_1[\mathsf{c}_b/c] : \Theta_1^\perp, \ell[\mathsf{c}_b/c] : \vartheta^\perp,\ \mathbf{r} : \bullet$$

where $\mathbf{r} = \texttt{postpone}\ (c :: \ell_1)\ \texttt{with}\ \mathsf{c}_b\ \texttt{until}\ b$, we need to show that

$$((\lambda y.s)[u/x])^\perp \;=\; \overline{\ell}_1[\mathsf{c}_b/c], \overline{\ell}_2[\ell[\mathsf{c}_b/c]/a],\ r : \bullet.$$

We may assume that $(ii)$ results by an application of (6.1) and thus that we have

$$(iii) \qquad (x : \vartheta, y : \vartheta_1, \Theta_1 \;\rhd\; s : \vartheta_2)^\perp = c : \vartheta_2^\perp \rhd \overline{\ell}_1 : \Theta_1^\perp, \ell_1 : \vartheta_1^\perp, \ell : \vartheta^\perp$$

The induction hypothesis is that

$$(iii) \quad (y : \vartheta_1, \Theta_1, \Theta_2 \;\rhd\; s[u/x] : \vartheta_2)^\perp \;=\; c : \vartheta_2^\perp \quad\rhd\quad \overline{\ell}_1 : \Theta_1^\perp, \ell_1 : \vartheta_1^\perp, \overline{\ell}_2[\ell/a] : \Theta_2^\perp.$$

By applying (6.1) to $(iii)$ we obtain

$$(\Theta_1, \Theta_2 \rhd \lambda x.s[u/y] : \vartheta_1 \supset \vartheta_2)^\perp = b : \vartheta_2^\perp \setminus \vartheta_1^\perp \;\rhd\; \overline{\ell}_1[\mathsf{c}_b/c] : \Theta_1^\perp, \overline{\ell}_2[\ell/a][\mathsf{c}_b/c] : \Theta_2^\perp, \mathbf{r} : \bullet$$

Since we may assume that $(\lambda x.s)[u/y] = \lambda x.s[u/y]$ and since the variable $c$ does not occur in $\overline{\ell}_2$ , the desired result follows.

Part $(i)$ of Theorem 1 is proved by a straightforward induction on $t$ or $\overline{\ell}$. To prove part $(ii)$ of Theorem 1 there are four cases to check;

we consider only that of a $\setminus$-reduction. Let

$$\overline{\ell} = (\overline{\ell}_1, \mathtt{c}, \overline{\ell}_2[\mathtt{a}/a], \mathtt{s})$$

where
$\mathtt{s} = \mathtt{postpone}\ (a :: \ell_2)\ \mathtt{with\ a\ using}\ (\mathtt{continue\ from}\ (\mathtt{c})\ \mathtt{using}\ (\ell_1))$ (omitting some indices) and suppose

$$\overline{\ell} \quad \rightsquigarrow_\beta \quad \overline{\ell_1}, \mathtt{c}, \overline{\ell}_2[\mathtt{a}/a]\{\mathtt{a} ::= \ell_1\}, \{\mathtt{c} ::= \ell_2[\ell_1/a]\}$$

A typing derivation of $\overline{\ell}$ is obtained as follows: we have a derivation $d_1$ ending with the inference

$$(o) \qquad \frac{\epsilon \ \rhd\ \overline{\ell}_1 : \Upsilon_1, \ell_1 : \upsilon_1}{\epsilon \ \rhd\ \overline{\ell}_1 : \Upsilon_1, \mathtt{c} : \upsilon_2, \mathbf{r} : \upsilon_1 \setminus \upsilon_2}$$

where $\mathbf{r} = \mathtt{continue\ \ from}\ (\mathtt{c})\ \mathtt{using}\ (\ell_1)$ and also a derivation $d_2$ of

$$(i) \qquad a : \upsilon_1 \ \rhd\ \ell_2 : \upsilon_2, \overline{\ell}_2 : \Upsilon_2$$

and we apply the inference $\setminus$-E to the conclusions of $d_1$ and $d_2$, yielding

$$\epsilon \ \rhd\ \overline{\ell}_1 : \Upsilon_1, \mathtt{c} : \upsilon_2, \overline{\ell}_2[\mathtt{a}/a] : \Upsilon_2, \mathtt{s} : \bullet$$

But the same typing of $\overline{\ell}$ may also be obtained by first deriving

$$(ii) \qquad b : \upsilon_1 \setminus \upsilon_2 \ \rhd\ \overline{\ell}_2[\mathtt{a}/a] : \Upsilon_2, \mathtt{s}(b) : \bullet$$

from $(i)$, where $\mathtt{s}(b) = \mathtt{postpone}\ (a :: \ell_2)\ \mathtt{with\ a\ using}\ (b)$ and then substituting $\mathbf{r}$ for $b$ in the terms in $(ii)$ which contain it free, i.e., in the "control term" $\mathtt{s}(b)$. Moreover, we have

$$(iii) \qquad (\epsilon \ \rhd\ \overline{\ell}_1 : \Upsilon_1, \ell_1 : \upsilon_1)^\perp = x : \upsilon_1^\perp, \overline{y} : \Upsilon_1^\perp \rhd u : \epsilon^\perp$$

and

$$(iv) \qquad (a : \upsilon_1 \ \rhd\ \ell_2 : \upsilon_2,\ \overline{\ell}_2 : \Upsilon_2)^\perp = y : \upsilon_2^\perp, \overline{x} : \Upsilon_2^\perp \ \rhd\ t : \upsilon_1^\perp$$

By applying Lemma 1 to $(iii)$ and (2.2) we have

$$(v) \qquad \begin{array}{r} (\epsilon \ \rhd\ \overline{\ell}_1 : \Upsilon_1, \mathtt{c} : \upsilon_2, \mathbf{r} : \upsilon_1 \setminus \upsilon_2)^\perp \ = \\ y : \upsilon_2^\perp, z : \upsilon_2^\perp \supset \upsilon_1^\perp, \overline{y} : \Upsilon_1^\perp \ \rhd u[zy/x] : \epsilon^\perp \end{array}$$

By applying Lemma 1 to $(iv)$ and (6.2) we have

$$(vi) \qquad \begin{array}{r} (b : \upsilon_1 \setminus \upsilon_2 \ \rhd\ \overline{\ell}_2[\mathtt{a}/a] : \Upsilon_2, \mathtt{s}(b) : \bullet)^\perp \ = \\ \overline{x} : \Upsilon_2^\perp \ \rhd\ \lambda y.t : \upsilon_2^\perp \supset \upsilon_1^\perp \end{array}$$

Again by Lemma 1 applied to $(v)$ and $(vi)$ we conclude

$$(vii) \qquad \begin{array}{r} (\epsilon \ \rhd\ \overline{\ell}_1 : \Upsilon_1, \mathtt{c} : \upsilon_2, \overline{\ell}_2[\mathtt{a}/a] : \Upsilon_2, \ \ \mathtt{s} : \bullet = \\ = y : \upsilon_2^\perp, \overline{y} : \Upsilon_1^\perp, \overline{x} : \Upsilon_2^\perp \ \rhd u[(\lambda y.t)y/x] : \epsilon^\perp \end{array}$$

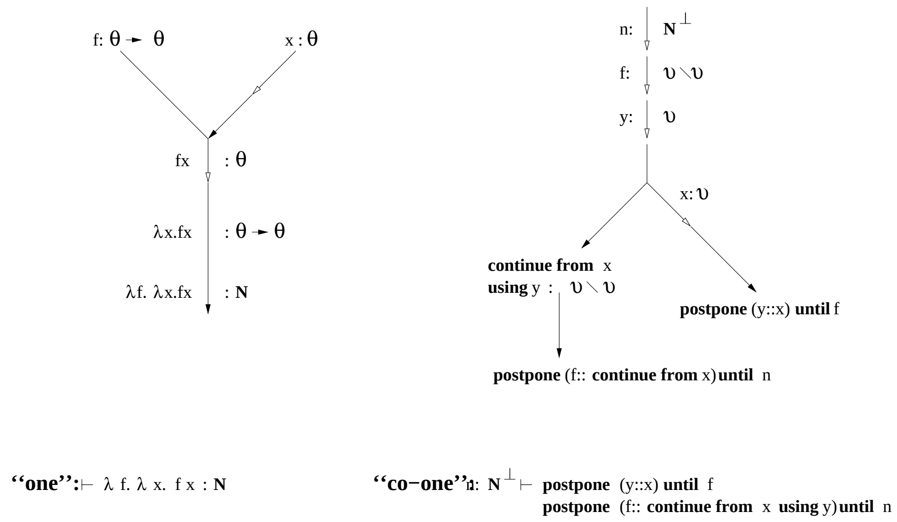



FIGURE 1. Church's one.

Now the right-hand side of $(vii)$ reduces to $u[t/x]$. But also by Lemma 1 applied to $(iii)$ and $(iv)$ we obtain

$$(\epsilon \ \rhd \ \overline{\ell}_1 : \Upsilon_1, \ell_2[\ell_1/a] : \upsilon_2, \overline{\ell}_2[\ell_1/a])^{\perp} = y : \upsilon_2^{\perp}, \overline{y} : \Upsilon_1^{\perp}, \overline{x} : \Upsilon_2^{\perp} \rhd u[t/x] : \epsilon^{\perp}$$

and the argument of the left-hand side is exactly what $\overline{\ell}$ reduces to when the global substitutions are eventually performed. This concludes the proof.

**4.3. Examples.** We give some graphical representation of dual derivations and proof-terms (with some simplification of the notation).

In Fig. 1 we have drawn a refutation

$$n : \mathbf{N}^{\perp} \rhd \ \begin{array}{l} \texttt{postpone}\ (y :: \texttt{x})\ \texttt{until}\ (f) : \bullet \\ \texttt{postpone}\ (f :: \texttt{continue from}\ (\texttt{x})\ \texttt{using}\ (\texttt{y}))\ \texttt{until}\ (n) : \bullet \end{array}$$

which is formally given in $\mathbf{NJ}^{\smallsetminus}$ as follows:

$$\dfrac{n : \mathbf{N}^{\perp} \rhd n : \mathbf{N}^{\perp} \qquad \dfrac{f : \upsilon \smallsetminus \upsilon \rhd f : \upsilon \smallsetminus \upsilon \qquad \dfrac{\dfrac{y : \upsilon \rhd y : \upsilon}{y : \upsilon \rhd \mathtt{x} : \upsilon, \mathbf{s} : \upsilon \smallsetminus \upsilon}}{}}{f : \upsilon \smallsetminus \upsilon \rhd \mathbf{t} : \bullet,\ \mathbf{s}[\mathtt{y}/y] : \upsilon \smallsetminus \upsilon}}{n : \mathbf{N}^{\perp} \rhd \mathbf{t} : \bullet, \mathbf{u} : \bullet}$$

where $\mathbf{s} = \texttt{continue from}\ (\texttt{x})\ \texttt{using}\ (y)$,
$\mathbf{t} = \texttt{postpone}\ (y :: \texttt{x})\ \texttt{until}\ (f)$ and
$\mathbf{u} = \texttt{postpone}\ (f :: \texttt{continue from}\ (\texttt{x})\ \texttt{using}\ (\texttt{y}))\ \texttt{until}\ (n)$.

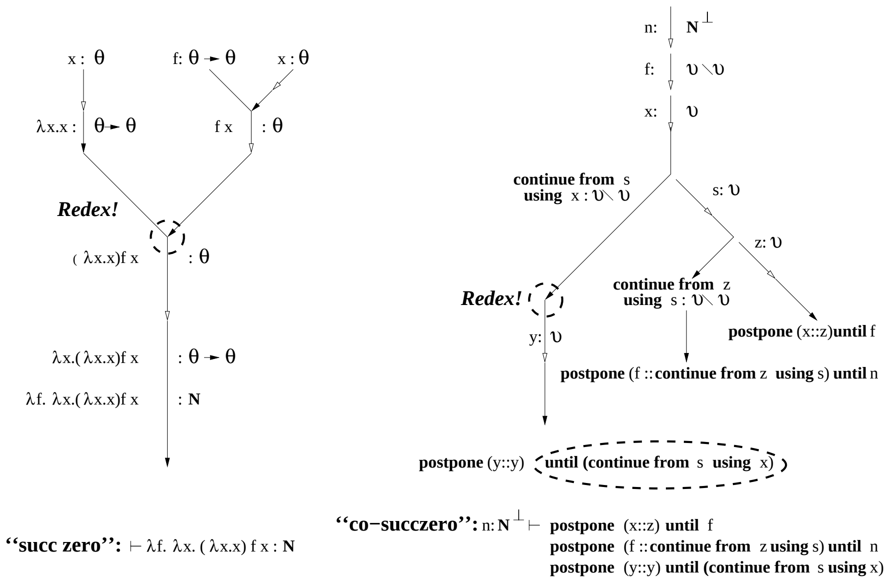



FIGURE 2. "succ zero".

In Fig. 2 we draw a part of the computation that **succ**(**zero**) = **one** and its dual.

We leave it as an exercise to the reader to write the formal proof corresponding to the drawing.

REFERENCES

[1] G. Bellin, M. Hyland, E. Robinson and C. Urban. Categorical Proof Theory of Classical Propositional Calculus, to appear in *Theoretical Computer Science* 2005.

[2] G. Bellin and P. J. Scott. "On the Pi-calculus and linear logic" in: Theoretical Computer Science 135 (1994) pp. 11-65.

[3] G. Bellin. "Chu's Construction: A Proof-theoretic Approach" in Ruy J.G.B. de Queiroz editor, "Logic for Concurrency and Synchronisation", Kluwer Trends in Logic n.18, 2003, pp.93-114.

[4] G. Bellin and C. Biasi. Towards a logic for pragmatics. Assertions and conjectures. In: *Journal of Logic and Computation*, Special Issue with the Proceedings of the Workshop on *Intuitionistic Modal Logic and Application* (IMLA-FLOC 2002), V. de Paiva, R. Goré and M. Mendler eds., Volume **14**, Number 4, 2004, pp. 473-506.

[5] G. Bellin and K. Ranalter. A Kripke-style semantics for the intuitionistic logic of pragmatics **ILP**. In: *Journal of Logic and Computation*, Special Issue with the Proceedings of the Dagstuhl Seminar on *Semantic Foundations of Proof-search*, Schloss Dagstuhl, 1-6 April 2001, Volume **13**, Number 5, 2003, pp. 755-775.

[6] G. Bellin and C. Dalla Pozza. A pragmatic interpretation of substructural logics. In *Reflection on the Foundations of Mathematics (Stanford, CA, 1998)*, Essays in honor of Solomon Feferman, W. Sieg, R. Sommer and C. Talcott eds., Association for Symbolic Logic, Urbana, IL, Lecture Notes in Logic, Volume 15, 2002, pp. 139-163.

[7] G. M. Bierman On intuitionistic linear logic. Technical Report 346 (Ph.D. Thesis), Computer Laboratory, University of Cambridge, August 1994.

[8] C. Biasi. *Verso una logica degli operatori prammatici asserzioni e congetture*, Tesi di Laurea, Facoltà di Scienze, Università di Verona, March 2003.

[9] T. Crolard. *Extension de l'isomorphisme de Curry-Howard au traitement des exceptions (application d'une ètude de la dualité en logique intuitionniste).* Thèse de Doctorat, Université de Paris 7, 1996.

[10] T. Crolard. Substractive logic, in *Theoretical Computer Science* 254:1-2(2001) pp. 151-185.

[11] T. Crolard. A Formulae-as-Types Interpretation of Subtractive Logic. In: *Journal of Logic and Computation*, Special Issue with the Proceedings of the Workshop on *Intuitionistic Modal Logic and Application* (IMLA-FLOC 2002), V. de Paiva, R. Goré and M. Mendler eds., Volume **14**, Number 4, 2004, pp. 529-570

[12] C. Dalla Pozza and C. Garola. A pragmatic interpretation of intuitionistic propositional logic, *Erkenntnis* **43**. 1995, pp.81-109.

[13] C. Dalla Pozza. Una logica prammatica per la concezione "espressiva" delle norme, In: A. Martino, ed. *Logica delle Norme*, Pisa, 1997

[14] R. Goré. Dual Intuitionistic Logic Revisited, In TABLEAUX00: Automated Reasoning with Analytic Tableaux and Related Methods, LNAI 1847:252-267, 2000. Springer.

[15] J-L. Krivine. *Lambda-calcul, types et modèles*, Masson, Paris 1990.

[16] J. Lambek and P. J. Scott. *Introduction to higher order categorical logic*, Cambridge studies in advanced mathematics 7, Cambridge University Press, 1986 (Paperback Edition 1988)

[17] A. S. Murawski and C.-H. L. Ong. Fast verification of MLL proof nets via IMLL. In *ACM Transactions on Computational Logic*, 2004. To appear.

[18] D. Prawitz. *Natural deduction. A proof-theoretic study.* Almquist and Wikksell, Stockholm, 1965.

[19] C. Rauszer. Semi-Boolean algebras and their applications to intuitionistic logic with dual operations, in *Fundamenta Mathematicae*, **83**, 1974, pp. 219-249.

[20] G. Reyes and H. Zolfaghari, Bi-Heyting algebras, Toposes and Modalities, in *Journal of Philosophical Logic*, 25, 1996, pp. 25-43.

[21] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*, Cambridge Tracts in Theoretical Computer Science 43, Cambridge University Press 1996.

FACOLTÀ DI SCIENZE, UNIVERSITÀ DI VERONA
37134 VERONA, ITALY
and
DPT OF COMPUTER SCIENCE, QUEEN MARY UNIVERSITY OF LONDON
LONDON E1 4NS
*E-mail*: bellin@dcs.qmul.ac.uk

## Sequent Calculi $\mathbf{LJ}^{\supset\cap}$ and $\mathbf{LJ}^{\smallsetminus\curlyvee}$

### G3-type $\mathbf{LJ}^{\supset\cap}$

**identity rules**

*logical axiom:*

$$\vartheta, \Theta \Rightarrow \vartheta$$

*cut:*

$$\frac{\Theta \Rightarrow \vartheta \qquad \vartheta, \Theta' \Rightarrow \epsilon}{\Theta, \Theta' \Rightarrow \epsilon}$$

**structural rules**

*exchange:*

$$\frac{\Theta, \vartheta_1, \vartheta_2, \Theta' \Rightarrow \epsilon}{\Theta, \vartheta_2, \vartheta_1, \Theta' \Rightarrow \epsilon}$$

(Contraction and Weakening left implicit in **G3**-type systems)

**logical rules**

*validity axiom:*

$$\Theta \Rightarrow \bigvee$$

*right $\supset$:*

$$\frac{\Theta, \vartheta_1 \Rightarrow \vartheta_2}{\Theta \Rightarrow \vartheta_1 \supset \vartheta_2}$$

*left $\supset$:*

$$\frac{\vartheta_1 \supset \vartheta_2, \Theta \Rightarrow \vartheta_1 \qquad \vartheta_2, \Theta \Rightarrow \epsilon}{\vartheta_1 \supset \vartheta_2,\ \Theta \Rightarrow \epsilon}$$

*right $\cap$:*

$$\frac{\Theta \Rightarrow \vartheta_1 \qquad \Theta \Rightarrow \vartheta_2}{\Theta \Rightarrow \vartheta_1 \cap \vartheta_2}$$

*left $\cap$:*

$$\frac{\vartheta_0, \vartheta_1, \Theta \Rightarrow \epsilon}{\vartheta_0 \cap \vartheta_1, \Theta \Rightarrow \epsilon}$$

### G3-type $\mathbf{LJ}^{\smallsetminus\curlyvee}$

**identity rules**

*logical axiom:*

$$\upsilon \Rightarrow \Upsilon, \upsilon$$

*cut:*

$$\frac{\epsilon \Rightarrow \Upsilon, \upsilon \qquad \upsilon \Rightarrow \Upsilon'}{\epsilon \Rightarrow \Upsilon, \Upsilon'}$$

**structural rules**

*exchange:*

$$\frac{\epsilon \Rightarrow \Upsilon, \upsilon_1, \upsilon_2, \Upsilon'}{\epsilon \Rightarrow \Upsilon, \upsilon_2, \upsilon_1, \Upsilon'}$$

(Contraction and Weakening right implicit in **G3**-type systems)

**logical rules**

*absurdity axiom:*

$$\bigwedge \Rightarrow \Upsilon$$

*right $\smallsetminus$:*

$$\frac{\epsilon \Rightarrow \Upsilon, \upsilon_1 \qquad \upsilon_2 \Rightarrow \Upsilon, \upsilon_1 \smallsetminus \upsilon_2}{\epsilon \Rightarrow \Upsilon, \upsilon_1 \smallsetminus \upsilon_2}$$

*left $\smallsetminus$:*

$$\frac{\upsilon_1 \Rightarrow \Upsilon, \upsilon_2}{\upsilon_1 \smallsetminus \upsilon_2 \Rightarrow \Upsilon}$$

*right $\curlyvee$:*

$$\frac{\epsilon \Rightarrow \Upsilon, \upsilon_0,\ \upsilon_1}{\epsilon \Rightarrow \Upsilon, \upsilon_0 \curlyvee \upsilon_1}$$

*left $\curlyvee$:*

$$\frac{\upsilon_1 \Rightarrow \Upsilon \qquad \upsilon_2 \Rightarrow \Upsilon}{\upsilon_1 \curlyvee \upsilon_2 \Rightarrow \Upsilon}$$

TABLE 2. The sequent calculi $\mathbf{LJ}^{\supset\cap}$ and $\mathbf{LJ}^{\smallsetminus\curlyvee}$

## Natural Deduction $\mathbf{NJ}^{\supset\cap}$ and $\mathbf{NJ}^{\smallsetminus\curlyvee}$ - Rules of Inference

### ASSERTIVE RULES - $\mathbf{NJ}^{\supset\cap}$

$$\frac{\begin{matrix}\vdots & & \vdots\\ \vartheta_1 & \ldots & \vartheta_m\end{matrix}}{\begin{matrix}\bigvee\\ \vdots\end{matrix}}\ \bigvee\text{-I} \qquad \frac{\begin{matrix}[\vartheta_1]\\ \vdots\\ \vartheta_2\end{matrix}}{\begin{matrix}\vartheta_1 \supset \vartheta_2\\ \vdots\end{matrix}}\ \supset\text{-I} \qquad \frac{\begin{matrix}\vdots & \vdots\\ \vartheta_1 \supset \vartheta_2 & \vartheta_1\end{matrix}}{\begin{matrix}\vartheta_2\\ \vdots\end{matrix}}\ \supset\text{-E}$$

$$\frac{\begin{matrix}\vdots & \vdots\\ \vartheta_0 & \vartheta_1\end{matrix}}{\begin{matrix}\vartheta_0 \cap \vartheta_1\\ \vdots\end{matrix}}\ \cap\text{-I} \qquad \frac{\begin{matrix}\vdots\\ \vartheta_0 \cap \vartheta_1\end{matrix}}{\begin{matrix}\vartheta_0\\ \vdots\end{matrix}}\ \cap_0\text{-E} \qquad \frac{\begin{matrix}\vdots\\ \vartheta_0 \cap \vartheta_1\end{matrix}}{\begin{matrix}\vartheta_1\\ \vdots\end{matrix}}\ \cap_1\text{-E}$$

### CONJECTURAL RULES- $\mathbf{NJ}^{\smallsetminus\curlyvee}$

$$\frac{\begin{matrix}\vdots\\ \bigwedge\end{matrix}}{\begin{matrix}\upsilon_1 & \ldots & \upsilon_n\\ \vdots & & \vdots\end{matrix}}\ \bigwedge\text{-E} \qquad \frac{\begin{matrix}\vdots\\ \upsilon_2\end{matrix}}{\begin{matrix}\upsilon_1 & \upsilon_2 \smallsetminus \upsilon_1\\ \vdots & \vdots\end{matrix}}\ \smallsetminus\text{-I} \qquad \frac{\begin{matrix}\vdots\\ \upsilon_2 \smallsetminus \upsilon_1\end{matrix}}{\begin{matrix}\upsilon_2\\ \vdots\\ [\upsilon_1]\end{matrix}}\ \smallsetminus\text{-E}$$

$$\frac{\begin{matrix}\vdots\\ \upsilon_0\end{matrix}}{\begin{matrix}\upsilon_0 \curlyvee \upsilon_1\\ \vdots\end{matrix}}\ \curlyvee_0\text{-I} \qquad \frac{\begin{matrix}\vdots\\ \upsilon_1\end{matrix}}{\begin{matrix}\upsilon_0 \curlyvee \upsilon_1\\ \vdots\end{matrix}}\ \curlyvee_1\text{-I} \qquad \frac{\begin{matrix}\vdots\\ \upsilon_0 \curlyvee \upsilon_1\end{matrix}}{\begin{matrix}\upsilon_0 & \upsilon_1\\ \vdots & \vdots\end{matrix}}\ \curlyvee\text{-E}$$

TABLE 3. $\mathbf{NJ}^{\supset\cap}$ and $\mathbf{NJ}^{\smallsetminus\curlyvee}$ - Rules of Inference

## Natural Deduction $\mathbf{NJ}^{\supset\cap}$ and $\mathbf{NJ}^{\smallsetminus\curlyvee}$

**$\mathbf{NJ}^{\supset\cap}$ - structural rules**

*exchange:* $$\frac{\Theta, \vartheta, \vartheta', \Theta' \vdash \epsilon}{\Theta, \vartheta', \vartheta, \Theta' \vdash \epsilon}$$

*contraction:* $$\frac{\vartheta, \vartheta, \Theta \vdash \epsilon}{\vartheta, \Theta \vdash \epsilon}$$

*weakening:* $$\frac{\Theta \vdash \epsilon}{\vartheta, \Theta \vdash \epsilon}$$

**identity rules**

*assumption:* $$\vartheta \vdash \vartheta$$

*substitution:* $$\frac{\Theta \vdash \vartheta \qquad \vartheta, \Theta' \vdash \epsilon}{\Theta, \Theta' \vdash \epsilon}$$

**logical rules**

*validity axiom:* $$\frac{}{\Theta \vdash \bigvee}$$

*$\supset$-I:* $$\frac{\Theta, \vartheta_1 \vdash \vartheta_2}{\Theta \vdash \vartheta_1 \supset \vartheta_2}$$

*$\supset$-E:* $$\frac{\Theta_1 \vdash \vartheta_1 \supset \vartheta_2 \qquad \Theta_2 \vdash \vartheta_1}{\Theta_1, \Theta_2 \vdash \vartheta_2}$$

*$\cap$-I:* $$\frac{\Theta \vdash \vartheta_1 \qquad \Theta \vdash \vartheta_2}{\Theta \vdash \vartheta_1 \cap \vartheta_2}$$

*$\cap_0$-E:* $$\frac{\Theta \vdash \vartheta_0 \cap \vartheta_1}{\Theta \vdash \vartheta_0}$$

*$\cap_1$-E:* $$\frac{\Theta \vdash \vartheta_0 \cap \vartheta_1}{\Theta \vdash \vartheta_1}$$

**$\mathbf{NJ}^{\smallsetminus\curlyvee}$ - structural rules**

*exchange:* $$\frac{\epsilon \vdash \Upsilon, \upsilon, \upsilon', \Upsilon'}{\epsilon \vdash \Upsilon, \upsilon', \upsilon, \Upsilon'}$$

*contraction:* $$\frac{\epsilon \vdash \Upsilon, \upsilon, \upsilon}{\epsilon \vdash \Upsilon, \upsilon}$$

*weakening:* $$\frac{\epsilon \vdash \Upsilon}{\epsilon \vdash \Upsilon, \upsilon}$$

**identity rules**

*assumption:* $$\upsilon \vdash \upsilon$$

*substitution:* $$\frac{\epsilon \vdash \Upsilon, \upsilon \qquad \upsilon \vdash \Upsilon'}{\epsilon \vdash \Upsilon, \Upsilon'}$$

**structuralrules**

*absurdity axiom:* $$\frac{}{\bigwedge \vdash \Upsilon}$$

*$\smallsetminus$-I:* $$\frac{\epsilon \vdash \Upsilon, \upsilon_1}{\epsilon \vdash \Upsilon, \upsilon_2, \upsilon_1 \smallsetminus \upsilon_2}$$

*$\smallsetminus$-E:* $$\frac{\epsilon \vdash \Upsilon, \upsilon_1 \smallsetminus \upsilon_2 \qquad \upsilon_1 \vdash \upsilon_2, \Upsilon'}{\epsilon \vdash \Upsilon, \Upsilon'}$$

*$\curlyvee_0$-I:* $$\frac{\epsilon \vdash \Upsilon, \upsilon_0}{\epsilon \vdash \Upsilon, \upsilon_0 \curlyvee \upsilon_1}$$

*$\curlyvee_1$-I:* $$\frac{\epsilon \vdash \Upsilon, \upsilon_1}{\epsilon \vdash \Upsilon, \upsilon_0 \curlyvee \upsilon_1}$$

*$\curlyvee$-E:* $$\frac{\epsilon \vdash \Upsilon', \upsilon_0 \curlyvee \upsilon_1 \qquad \upsilon_0 \vdash \Upsilon \qquad \upsilon_1 \vdash \Upsilon}{\epsilon \vdash \Upsilon, \Upsilon'}$$

TABLE 4. $\mathbf{NJ}^{\supset\cap}$ and $\mathbf{NJ}^{\smallsetminus\curlyvee}$ - Rules of Deduction

## Natural Deduction $\mathbf{NJ}^{\supset\cap}$ and $\mathbf{NJ}^{\smallsetminus\curlyvee}$ - Reduction Rules

**$\cap$-REDUCTION**

$$\dfrac{\dfrac{\begin{matrix}\vdots & \vdots\\ \vartheta_0 & \vartheta_1\end{matrix}}{\vartheta_0 \cap \vartheta_1}\,\cap\text{-I}}{\begin{matrix}\vartheta_i\\ \vdots\end{matrix}}\,\cap_i\text{-E} \qquad \text{reduces to} \qquad \begin{matrix}\vdots\\ \vartheta_i\\ \vdots\end{matrix}$$

**$\supset$-REDUCTION**

$$\dfrac{\dfrac{\begin{matrix}[\vartheta_1]\\ \vdots\\ \vartheta_2\end{matrix}}{\vartheta_1 \supset \vartheta_2}\,\supset\text{-I} \qquad \begin{matrix}\vdots\\ \vartheta_1\end{matrix}}{\begin{matrix}\vartheta_2\\ \vdots\end{matrix}}\,\supset\text{-E} \qquad \text{reduces to} \qquad \begin{matrix}\vdots\\ [\vartheta_1]\\ \vdots\\ \vartheta_2\\ \vdots\end{matrix}$$

**$\curlyvee$-REDUCTION**

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \upsilon_i\end{matrix}}{\upsilon_0 \curlyvee \upsilon_1}\,\curlyvee_i\text{-I}}{\begin{matrix}\upsilon_0 & \upsilon_1\\ \vdots & \vdots\end{matrix}}\,\curlyvee\text{-E} \qquad \text{reduces to} \qquad \begin{matrix}\vdots\\ \upsilon_i\\ \vdots\end{matrix}$$

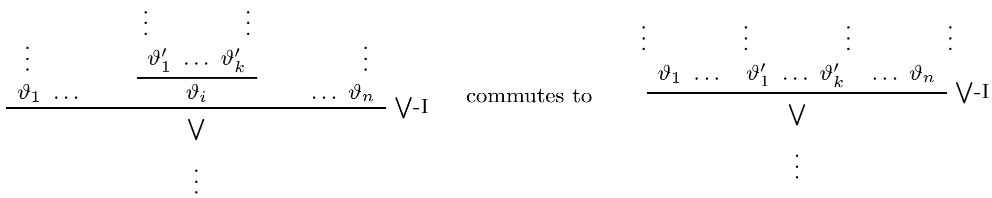
**$\smallsetminus$-REDUCTION**

$$\smallsetminus\text{-I}\;\dfrac{\begin{matrix}\vdots\\ \upsilon_2\end{matrix}}{\begin{matrix}\upsilon_1 & \dfrac{\upsilon_2 \smallsetminus \upsilon_1}{\begin{matrix}\upsilon_2\\ \vdots \quad \vdots\\ [\upsilon_1] \quad \vdots\end{matrix}}\,\smallsetminus\text{-E}\\ \vdots & \end{matrix}} \qquad \text{reduces to} \qquad \begin{matrix}\vdots\\ \upsilon_2\\ \vdots \quad \vdots\\ [\upsilon_1] \quad \vdots\\ \vdots\end{matrix}$$

TABLE 5. $\mathbf{NJ}^{\supset\cap}$ and $\mathbf{NJ}^{\smallsetminus\curlyvee}$ - Reduction Rules

## Natural Deduction $\mathbf{NJ}^{\supset\cap}$ and $\mathbf{NJ}^{\smallsetminus\curlyvee}$ - Commutation rules

**$\bigvee$-COMMUTATIONS**




**$\bigwedge$-COMMUTATIONS**

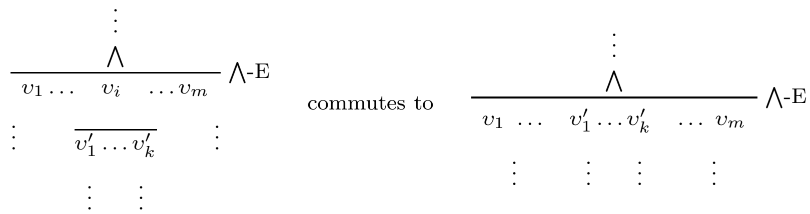



TABLE 6. $\mathbf{NJ}^{\supset\cap}$ and $\mathbf{NJ}^{\smallsetminus\curlyvee}$ - Commutation Rules