Introduction to Part 0
Example C. Any set can be viewed as a category with a single object.

Example D. Many objects of interest in mathematics constitute categories.

Example E. The category of sets is the category whose objects are sets and whose arrows are functions. The composition of arrows in a category is denoted as $\circ$. For any two arrows $f: A \to B$ and $g: B \to C$, their composition is an arrow $g \circ f: A \to C$.

Example F. The category of groups has groups as objects and group homomorphisms as arrows. The composition of group homomorphisms is defined as function composition.

Definition 1. A category consists of a collection of two sorts, objects and arrows, with three properties: for any two objects $A$ and $B$, there is at least one arrow $f: A \to B$, composition of arrows is associative, and there is an identity arrow for each object.

In this section, we present what our reader is expected to know about categories. We begin with a rather informal definition:

Introduction to category theory
2. Show that for any concrete category $C$, there is a functor $U : C \to Set$

1. Prove Proposition 1.4 and 1.5

**Examples**

$C \times C \\ (g \times h) : \to C \times C$ ($f \times g) \equiv (f \times g) : \to (C \times C)$

**Proposition 1.4** For any category $C$, and $f, g : C \to C$

A category $C$ with objects and arrows.

In particular, a category $C$ is monoidal if there are two categories $I$ isomorphic.

**Definition 1.5** An arrow $f : A \to B$ in a monoidal category $C$ is monoidal.

A monoidal arrow is defined as

$\gamma : \gamma (A) \to \gamma (B)$

where $\gamma (A)$ and $\gamma (B)$ are the objects of the category $C$.

**Example C:** From any category $C$ (object $A$ and $B$) one forms a new category $C \times C$.

**Example D:** From any category $C$ (object $A$ and $B$) one forms a new category $C \times C$. There are two ways of forming new categories from old.

Once we define functors, we can define a category $C$ to be monoidal if there is a monoidal arrow $\gamma : A \to B$ such that $\gamma (A) = A$ and $\gamma (B) = B$.

In particular, the identity functor $I : C \to C$ is monoidal.

1. Prove Proposition 1.4 and 1.5

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In particular, the identity functor $I : C \to C$ is monoidal.
Proposition 2.2. When objects are sets, small graphs and 1-sets are.

To apply the usefulness of natural transformations, the reader should,...

Example 2.3. Given categories $A$ and $B$, the functor category $B^{A}$ has as...

Definition 2.4. For any categories $A$ and $B$, and any object of $B$, issue such that...

Proposition 1.6. We record these more basic isomorphisms in the spirit of...

Definition 1.7. A natural transformation is a pair $(T, 	heta)$, where $T$ is a...

Introduction to category theory
In one-to-one correspondence with \( F \)

**Proposition 2.7.** If \( \mathcal{C} \) is a small and \( \mathcal{D} \) is a set, then \( \text{Nat}(\mathcal{C}, \mathcal{D}) \) is a set.

where \( \text{Nat}(\mathcal{C}, \mathcal{D}) \) is the category of natural transformations of \( F \) and \( G \) is a functor from \( \mathcal{C} \) to \( \mathcal{D} \).

**Proof.** We claim that \( \text{Nat}(\mathcal{C}, \mathcal{D}) \) is a set. Each \( \eta : F \Rightarrow G \) is a function from \( \mathcal{C} \) to \( \text{mor} \mathcal{D} \times \text{mor} \mathcal{C} \), and the axioms of a functor show that \( \text{Nat}(\mathcal{C}, \mathcal{D}) \) is a set.

The axioms in \( \mathcal{C} \) are natural transformations. We therefore

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**Proposition 2.8.** If \( \mathcal{C} \) is a small category and \( \mathcal{D} \) is a set, then \( \text{Nat}(\mathcal{C}, \mathcal{D}) \) is a set.

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Before going into the examples, let us give another formulation of what will
calls us the two definitions.

One says that $L$ is right ordered if $P$ on $L$ is left adjoint to $J$ and one

$$1 \circ J = (L f) \circ (f) \quad 1 \circ J = (L f) \circ (f)$$

and $\delta$ and $\Delta$ are natural transformations such that

$$\delta \circ J = (L f) \circ (f) \quad \Delta \circ J = (L f) \circ (f)$$

**Definition:** A $\delta$ and $\Delta$ are natural transformations between categories $P$ and $Q$ given by a

$$\delta : \eta \circ J_1 = (L f) \circ (f) \quad \Delta : \eta \circ J_1 = (L f) \circ (f)$$

under the assumption that $\eta$.

Suppose $\eta : \delta \circ J = (L f) \circ (f)$, then $\delta$ and $\Delta$ are natural transformations.

**Example:** Take $\eta : \delta \circ J = (L f) \circ (f)$ the set of points of a plane $X$ in the set of all-

subject of $x$.

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subject of $x$.\)
**Definition 3.2.** A solution to the universal mapping problem for a functor $U : \mathscr{B} \to \mathscr{A}$ is given by the following data: for each object $A$ of $\mathscr{A}$ an object $F(A)$ of $\mathscr{B}$ and an arrow $\eta(A) : A \to UF(A)$ such that, for each object $B$ of $\mathscr{B}$ and each arrow $f : A \to U(B)$ in $\mathscr{A}$, there exists a unique arrow $f^* : F(A) \to B$ in $\mathscr{B}$ such that $U(f^*)\eta(A) = f$.

**Example U1.** Let $\mathscr{B}$ be the category of monoids, $\mathscr{A}$ the category of sets, $U : \mathscr{B} \to \mathscr{A}$ the forgetful (= underlying) functor, $F(A)$ the free monoid generated by the set $A$ and $\eta(A)$ the obvious mapping of $A$ into the underlying set of the monoid $F(A)$.

**Definition 3.2.** Of special interest is the case of Definition 3.2 in which $\mathscr{B}$ is a full subcategory of $\mathscr{A}$ and $U : \mathscr{B} \to \mathscr{A}$ is the inclusion. Then $\eta(A) : A \to F(A)$ may be called the best approximation of $A$ by an object of $\mathscr{B}$ in the sense that, for each arrow $f : A \to B$ with $B$ in $\mathscr{B}$, there is a unique arrow $f^* : F(A) \to B$ such that $f^*\eta(A) = f$. One then says that $\mathscr{B}$ is a full reflective subcategory of $\mathscr{A}$ with reflector $F$ and reflection $\eta$.

**Example U2.** Let $\mathscr{A}$ be the category of Abelian groups, $\mathscr{B}$ the full subcategory of torsion free Abelian groups and $F(A) = A/T(A)$, where $T(A)$ is the torsion subgroup of $A$.

**Proposition 3.3.** Given two categories $\mathscr{A}$ and $\mathscr{B}$, there is a one-to-one correspondence between adjointnesses $(F, U, \eta, \epsilon)$ and solutions $(F, \eta, *)$ of the universal mapping problem for $U : \mathscr{B} \to \mathscr{A}$.

**Proof.** If $(F, U, \eta, \epsilon)$ is given, put $f^* = \epsilon(B)F(\epsilon(f))$. Conversely, if $U$ and $(F, \eta, *)$ are given, for each $f : A \to A'$ put $F(f) = (\eta(A) f)^*$ and check that this makes $F$ a functor and $\eta$ a natural transformation; moreover define $\epsilon(B) = (1_{\text{UB}})^*$. It follows from symmetry considerations that an adjointness is also equivalent to a `co-universal mapping problem', obtained by dualizing $A \to A'$ into $A' \to A$.

**Proposition 3.4.** An adjointness $(F, U, \eta, \epsilon)$ between locally small categories $\mathscr{A}$ and $\mathscr{B}$ gives rise to and is determined by a natural isomorphism $\text{Hom}_\mathscr{A}(F(-), -) \cong \text{Hom}_\mathscr{B}(-, U(-))$ between functors $\mathscr{A}^{op} \times \mathscr{B} \to \text{Sets}$.

We leave the proof of this to the reader.

Even if $\mathscr{A}$ is not locally small, there is a natural bijection between arrows $F(A) \to B$ in $\mathscr{B}$ and arrows $A \to UB$ in $\mathscr{A}$. Logicians may think of such a bijection as comprising two rules of inference; and this point of view has been quite influential in the development of categorical logic. An analogous situation in the propositional calculus would be the bijection between proofs of the entailments $C \land A \to B$ and $A \land C \to B$ (see Exercise 4 below). Inasmuch as implication is a more sophisticated notion than conjunction, the adjointness here explains the emergence of one concept from another. This point of view, due to Lawvere, may be summarized by yet another slogan, illustrations of which will be found throughout this book (see, for instance, Exercise 6 below).

**Slogan IV.** Many important concepts in mathematics arise as adjoints, right or left, to previously known functors.

We summarize two important properties of adjoint functors, which will be useful later.

**Proposition 3.5.** (i) Adjoint functors determine each other uniquely up to natural isomorphisms.

(ii) If $(U, F)$ and $(U', F')$ are pairs of adjoint functors, as in the diagram

\[
\begin{array}{ccc}
\mathscr{B} & \overset{F}{\to} & \mathscr{A} \\
\mathscr{A} & \overset{U}{\leftarrow} & \mathscr{B} \\
\end{array}
\]

then $(U', F')$ is also an adjoint pair.

**Exercise**

1. If $(F, G)$ is a Galois correspondence between posets $\mathscr{A}$ and $\mathscr{B}$, show that $F$ preserves suprema and $G$ preserves infima. If $\mathscr{A}$ has and $F$ preserves suprema, show that its right adjoint $G : \mathscr{B} \to \mathscr{A}$ can be calculated by the
Proposition 4.1. An equivalence of categories is an isomorphism of categories.

Proof. Let $F : C \rightarrow D$ be an equivalence of categories. Then, $F$ is a full and faithful functor $F : C \rightarrow D$. Since $F$ is faithful, we have $\ker F = 1$. Moreover, $\ker F = 1$ implies that $F$ is an isomorphism. Therefore, $F$ is an equivalence of categories.

Therefore, by Proposition 4.1, we have an equivalence of categories $\mathcal{C} \leftrightarrow \mathcal{D}$. Since $\mathcal{C} \leftrightarrow \mathcal{D}$, there is an isomorphism of categories $\mathcal{C} \leftrightarrow \mathcal{D}$. Hence, $\mathcal{C} \leftrightarrow \mathcal{D}$.

We will now extend the notion of equivalence.

4 Equivalence of categories

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Theorem 4.2. Given an equivalence of categories $\mathcal{C} \leftrightarrow \mathcal{D}$, there is an isomorphism of categories $\mathcal{C} \leftrightarrow \mathcal{D}$.

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Elements

Shorn A. Many advantages and disadvantages in mathematics arise directly from the definitions of objects and processes that are at the heart of any logical system. For example, in category theory, the category of groups, denoted by Grp, is the category of groups and group homomorphisms. The objects in this category are groups, and the morphisms are group homomorphisms.

Grp is a category in the sense that it has objects (groups) and arrows (group homomorphisms). The composition of arrows is associative, and there are identity arrows for every group. The objects and morphisms can be thought of as the building blocks of the category.

Limits in categories

Some objects in Grp are isomorphic, but their morphisms may differ. For example, the groups (Z/2Z, +) and (Z/4Z, +) are isomorphic, but there is no group homomorphism between them. This is because the only group homomorphism from (Z/4Z, +) to (Z/2Z, +) is the zero map, which is not injective.

However, there is a natural isomorphism between (Z/2Z, +) and (Z/4Z, +). This isomorphism can be constructed by considering the projections from Z/4Z to Z/2Z and Z/4Z to Z/2Z. These projections are both group homomorphisms, and they have the property that the composition of the projections is the identity map on Z/4Z.

In category theory, a limit is a way of combining objects and morphisms to form a new object that is universal with respect to a certain property. Limits can be thought of as the categorical generalization of the concept of a limit in analysis.

The limit of a diagram in a category is a way of combining objects and morphisms to form a new object that is universal with respect to a certain property. Limits can be thought of as the categorical generalization of the concept of a limit in analysis.
If it clearly seen that the equalizing object $C$ is unique up to isomorphism

\[ \forall g. \exists ! f. f \circ g = \text{eq} \]

then the \( f \) is the unique \( g \) such that $f \circ g = \text{eq}$. If it is not clear, then the \( g \) is not unique and is not equalizing.

\[ \forall g. \exists ! f. f \circ g = \text{eq} \]

For the sake of simplicity, we will use the notation $f$ to denote the unique equalizing object.

\[ \forall g. \exists ! f. f \circ g = \text{eq} \]

We can define the equalizer of two parallel arrows as the kernel of the morphism that takes the first arrow to the second.

\[ \forall g. \exists ! f. f \circ g = \text{eq} \]

The equalizer is the initial object in the category of all functors $\mathcal{C} \to \mathcal{D}$.

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where $f$ is a unique morphism such that $f \circ g = \text{eq}$. If it is not clear, then the \( g \) is not unique and is not equalizing.

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Proposition 5.6. If a category has binary products and equalizers, pullbacks may be constructed as follows: A pullback of a diagram in a category with binary products and equalizers. By the pullback $x \rightarrow y \times z$, $x \rightarrow y$, $z \rightarrow x$ is constructed from products and equalizers in category $C$. In other special categories, equalizers are given by the pullback of a diagram with a binary product $f \rightarrow g$. Pullbacks are defined as the pullback of a diagram in a category with binary products, a pullback in category $C$ is obtained by constructing a pullback in a category with binary products.

Commutes

Proof

commutes in categories

Limits in categories

In other words, a pullback of a diagram

Definition 5.5. A pullback of a diagram in category $C$ is given by a diagram $P$.

Proposition 5.4. Pullbacks for all pairs of arrows $f, g$ are

Here can be obtained in front of and 6 units

(5.4)

Done. By

It seems to express the influence of $\gamma (a, a')$ continuously. So suppose

\( \gamma = (a, a') \) and so we conjecture

\( \eta (a, a') = (a, a') \) \hfill (5.4)

As it happens that $f = \theta$, this becomes equal to

\( (\gamma, \gamma, \gamma)(\theta, \theta, \theta) = (\gamma, \gamma, \gamma)(\theta, \theta, \theta) \) \hfill (5.4)

Then

\( (\gamma, \gamma, \gamma)(\theta, \theta, \theta) = (\gamma, \gamma, \gamma)(\theta, \theta, \theta) \) \hfill (5.4)

Assuming $\theta = \eta$, the following diagram

From the two special cases we can define $(f', f')$ satisfying $(f', f') \rightarrow (\gamma, \gamma, \gamma)(\theta, \theta, \theta)$

\( f' = (\gamma, \gamma, \gamma)(\theta, \theta, \theta) \) \hfill (5.4)

Hence we conjecture an arrow $(f', f') \equiv (f, f)$ satisfying as

Introduction to category theory
Many functions occurring in nature possess limits (up to isomorphism).

**Definition 4.1.** Let $f : A \to B$ be a function and let $g : B \to C$. Then $g \circ f : A \to C$ is said to be the composition of $g$ and $f$. The composition $g \circ f$ is also denoted by $f \cdot g$.

**Theorem 4.2.** Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$ be two sets. Then $|X \times Y| = |X| \cdot |Y|$.

**Proof.** Let $f : X \to Y$ be a function. Then $f$ assigns a unique element of $Y$ to each element of $X$. Thus, for each $x \in X$, there exists a unique $y \in Y$ such that $f(x) = y$. This correspondence defines a function $g : Y \to X$ given by $g(y) = x$ if and only if $f(x) = y$. Therefore, $f \cdot g : Y \to X$ is defined and is a bijection. Thus, $|X \times Y| = |X| \cdot |Y|$.

**Proposition 5.1.** Given two categories $C$ and $D$, a functor $F : C \to D$ is a special type of function that preserves the structure of the categories. It is a mapping from objects to objects and from arrows to arrows in a way that respects composition and identity.

**Proposition 5.2.** In an object $X$ of the category $C$, then $X \times X$ denotes a set of pairs $\{(x, y) : x, y \in X\}$, called the product of $X$ and $X$. The product is denoted by $X \times X$.

**Definition 5.3.** Let $F : C \to D$ be a functor. Then $F$ is said to be a faithful functor if for all objects $A, B \in C$, the arrow $f : A \to B$ in $C$ is isomorphic to the arrow $F(f) : F(A) \to F(B)$ in $D$.

**Proposition 5.4.** Every category $C$ has a terminal object $1$ such that for every object $X \in C$, there exists a unique morphism $i_X : X \to 1$.

**Proof.** Let $X \in C$. Define $i_X : X \to 1$ by $i_X(x) = x' \in 1$, where $x'$ is any element of $1$. This defines a morphism $i_X : X \to 1$. To show that $i_X$ is unique, suppose $j_X : X \to 1$ is another morphism with $j_X(x) = x'' \in 1$. Since $1$ is a terminal object, there exists a unique morphism $k : 1 \to 1$ such that $k(i_X(x)) = j_X(x)$. Therefore, $j_X(x) = k(i_X(x)) = i_X(x)$, showing that $i_X$ is unique.
1. For all limits can be constructed from products and equalizers.

Exercises

3. Suppose we have two arrows $g, h$ and let $k$ be an arrow in the full sub-

Now, for any object $c$ of $g, h$.

Suppose we have two arrows $g, h$ and let $k$ be an arrow in the full sub-

For any arrow in the full sub-

If $a$ is a small category, then $\mathcal{C}$ is a small category. Show that there is an arrow $f: a \to c$ in $\mathcal{C}$.

If $a$ is a small category, then $\mathcal{C}$ is a small category. Show that there is an arrow $f: a \to c$ in $\mathcal{C}$.

Since the proof is complete.
The equations are illustrated by the following commutative diagrams:

\[
\begin{align*}
\text{Definition 6.} & \quad \text{Define a triple } (\mathcal{L}, \phi, \psi) \text{ on a category } \mathcal{C} \text{ as follows:} \\
& \quad \text{Let } \mathcal{L} \text{ be a functor on a category } \mathcal{C}, \phi : \mathcal{L} \to \mathcal{C}, \psi : \mathcal{C} \to \mathcal{L}.
\end{align*}
\]

For example, let us prove one of the unit laws.

**Proposition:** Let \( \mathcal{L} \) be a functor with adjunctions, and let \( \phi, \psi \) be the adjoints to the functor \( \mathcal{L} \).

\[
\text{Let } \phi : \mathcal{L} \to \mathcal{C} \text{ and } \psi : \mathcal{C} \to \mathcal{L}, \text{ then there is an isomorphism } \psi \circ \mathcal{L} \cong \mathcal{C} \frac{\psi \circ \mathcal{L}}{\mathcal{C}} \frac{\mathcal{L}}{\mathcal{L}} \phi.
\]

The reader will recall how natural transformations are composed (see).
The first component of \( \Phi \) is \( \Phi_1 \). Moreover, \( \Phi_1 \subseteq \Gamma \) and the \( \Gamma \)-factors of \( \Phi \) are not necessarily the same as the \( \Gamma \)-factors of \( \Phi \) itself.

Then \( \Phi \) is the \( \Phi \)-factors of \( \Phi \) and the \( \Gamma \)-factors of \( \Phi \).

\[ (\theta) \cup = \gamma_1 \] for any \( \gamma_1 \) of \( \Theta \).

The power set \( \mathcal{P}(\Phi) \) of the \( \Phi \)-factors of \( \Phi \) is the \( \Phi \)-factors of \( \Phi \).

\[ \Phi = \gamma_1 \] for any \( \gamma_1 \) of \( \Theta \).
The desired property of a functor $X$ is simply that the unique function with $\mathcal{(X)}(\mathcal{A}) \rightarrow \mathcal{(X)}(\mathcal{B})$ for any object $\mathcal{A}$ of $\mathcal{A}$, $\mathcal{(X)}(\mathcal{A}) \rightarrow \mathcal{(X)}(\mathcal{B})$. This property holds if and only if $\mathcal{(X)}(\mathcal{A})$ and $\mathcal{(X)}(\mathcal{B})$ are isomorphic as objects of $\mathcal{B}$.

To prove this, suppose $\mathcal{(X)}(\mathcal{A}) \rightarrow \mathcal{(X)}(\mathcal{B})$ is a map of $\mathcal{B}$. For any object $\mathcal{X}$ of $\mathcal{X}$, we have

$$\mathcal{(X)}(\mathcal{A}) \mathcal{(X)}(\mathcal{A}) \rightarrow \mathcal{(X)}(\mathcal{B})$$

We claim that $\mathcal{(X)}(\mathcal{A})$ is an object of $\mathcal{B}$. Indeed, for any object $\mathcal{X}$ of $\mathcal{X}$, we have

$$\mathcal{(X)}(\mathcal{A}) \mathcal{(X)}(\mathcal{A}) \rightarrow \mathcal{(X)}(\mathcal{B})$$

By definition, $\mathcal{(X)}(\mathcal{A})$ is an object of $\mathcal{B}$. Therefore, $\mathcal{(X)}(\mathcal{A})$ is an object of $\mathcal{B}$.

It follows that $\mathcal{(X)}(\mathcal{A}) \rightarrow \mathcal{(X)}(\mathcal{B})$ is a map of $\mathcal{B}$. Therefore, $\mathcal{(X)}(\mathcal{A})$ is an object of $\mathcal{B}$.

Hence, the desired property holds.

### Example (continued)

When the Kripke category of the power set

$$\mathcal{P}(\mathcal{X})$$

To the order to check the associativity of composition in

$$\mathcal{P}(\mathcal{X})$$

since the identity arrow $\mathcal{1}$ in $\mathcal{P}(\mathcal{X})$. We have

$$f(\mathcal{1}) = f(\mathcal{1}) \mathcal{1} = f(\mathcal{1}) \mathcal{1} \mathcal{1} = f(\mathcal{1})$$

and

$$f = f(\mathcal{1})$$

In particular,

$$f(\mathcal{1}) \mathcal{1} = f(\mathcal{1}) \mathcal{1}$$

We define the operation $\mathcal{1}$ on objects as follows. The operation $\mathcal{1}$ on objects is the identity map $\mathcal{1}$. Hence, the category $\mathcal{1}$ is a category.

The category of resolutions of a type also has an initial object.

**Proposition.** The category of resolutions of a type also has an initial object.

This completes the proof.
Examples of cartesian closed categories

1. Categories of sets

2. Categories of pointed sets

3. Categories of functions

4. Categories of functors

A cartesian closed category is a category with finite products (hence a cartesian category), exponentiation, and terminal object. This means that there is a way to "multiply" objects and that there is a "map" operation that associates to each object a function from that object to itself.

\[ (\forall X)(\exists Y)(\forall f : \text{hom}(X,Y))(\exists g : \text{hom}(Y,Z))(\forall h : \text{hom}(Y,X))(\exists ! \gamma : \text{hom}(Z,Y))(\forall x : X)((\gamma \circ g) \circ f = h) \]

This implies that for any object \( X \) and any morphism \( f : X \to Y \) there is a unique morphism \( \gamma : Y \to X \) such that \( \gamma \circ g = f \) for any morphism \( g : Y \to X \).

Moreover, the concept of cartesian closed category is useful in the study of computational structures and programming languages, as it allows for the formation of function spaces as objects in the category.
A \cap (A - X) = A \\
A \cup (A - X) = A \\
A \cap (A - X) \subseteq A - X \\
A \cup (A - X) \supseteq A - X \\

**Example 7.3:** A Heyting algebra is the lattice of open subsets of a topological space $X$, with the following structure:

$(\cap, \cup, \wedge, \vee, 0, 1)$

where $\wedge$ is the meet (greatest lower bound) and $\vee$ is the join (least upper bound).

The category of Heyting algebras is cartesian closed.

**Example 7.4:** Although the category of topological spaces and continuous maps is cartesian closed, the category of Heyting algebras is not.

**Example 7.5:** The category of small categories is cartesian closed.

**Examples of cartesian closed categories**

1. The category of sets.
2. The category of small categories.
3. The category of topological spaces.
4. The category of Heyting algebras.

**Remark:** The Cartesian product of categories is defined as the category whose objects are pairs of objects from the original categories, and whose morphisms are pairs of morphisms from the original categories.

**Example 7.6:** More generally, for any small category $C$, the functor which assigns to each set $X$ the set of all small categories over $X$, i.e., the functor $\text{Cat}(\cdot, X)$, where $\text{Cat}$ is the category of all small categories, is a fibration.

**Example 7.7:** The Cartesian product of sets is cartesian closed. Hence $A \times B$ is the set of all functions $A \to B$, i.e., the natural numbers.
Cartesian closed categories

and λ-calculus

Exercises

1. Carry out the detailed proof in any of the above examples.

2. Show that Heyting semantists may be derived equivalently.

Kuratowski (Chapter 7)

Introduction to category theory

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The proposition $A \rightarrow B$ is in some sense equivalent to the proof of its use. However, functional completeness of propositional calculus presents a deductive system of logic that is an extension of $\text{Propositional Calculus}$. The introduction to Part I.
Historical perspective on Part I

Historical perspective on Part I

Historical perspective on Part I

Historical perspective on Part I

Historical perspective on Part I
The successor function and the usual operations on natural numbers are

\[
(x + 1)^y = x \cdot x^y + 1 \quad (x + 1)^0 = 0
\]

so in general, one has

\[
f \cdot f = f, f \cdot Z = f, Z \cdot f = f, \text{etc.}
\]

One extends \( f \) the process of raising to every function \( f \) of the form \( f(x, y) = (x, y). \)

\[
((x, y), f)^y = y \cdot f
\]

When this process is applied to rational numbers one gets the usual operation called \( \phi = (\phi(x, y)). \)

\[
\phi(x, y) = (x \cdot x^y + 1, y \cdot x^y + 1)
\]

As pointed out in the book by Curry and Feys, these three types of

closure are:

1. Closure in the usual way.
2. Closure in the way prescribed by Whitehead and Russell.
3. Closure in the way prescribed by the lambda calculus.

The lambda calculus was introduced by Schönfinkel (1914) in a paper

\[
\lambda f f = \lambda f (f \cdot f)
\]

Closed connectives and lambda-calculus

Historical perspective

Historical background on the development of the lambda calculus

F. P. Ramsey, "The Foundations of Mathematics" (1925)

A. N. Whitehead and B. Russell, "Principia Mathematica" (1910-1913)

R. B. Braithwaite, "Philosophical Studies" (1944)

W. V. Quine, "Mathematical Logic" (1951)

H. B. Curry, "Combinatory Logic" (1951)

S. C. Kleene, "Introduction to Metamathematics" (1952)

G. H. Hardy, "A Mathematician's Apology" (1940)

R. L. Goodstein, "The Foundations of Mathematics" (1947)

A. A. Fraenkel, "Set Theory and the Foundations of Mathematics" (1948)

J. R. Newman, "The World of Mathematics" (1956)

G. Frege, "The Foundations of Arithmetic" (1884)

P. Bernays, "Axiomatic Set Theory" (1958)

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Propositional calculus as a deductive system

For further historical comments, the reader is referred to the end of Part I.

Propositional calculus as a deductive system
A positive propositional calculus is a conjunction calculus

\[
\begin{align*}
\text{where}\quad & A \iff \overline{A} \\
\text{Introduce Rule of Introduction}\quad & \text{are also the following two derived rules of} \\
\text{Converse Rule of Introduction}\quad & \text{for future reference, we also note the following two derived rules of} \\
\text{We note that from \( \overline{A} \), with the help of \( \overline{A} \), one may derive} \\
& \text{understand the general} \\
& \text{So, the following new and rule of inference} \\
& \text{specify the following new and rule of inference.} \\
& \text{As a result,} A \iff A \\
& \text{We also}
\end{align*}
\]

**Propositional Calculus as a Deductive System**

Thus, \( A \iff A \) and \( \overline{A} \) are formulas.

\[
\frac{A \land B \land \ldots \land F}{A} \quad \frac{A \land B \land \ldots \land F}{B} \quad \frac{A \land B \land \ldots \land F}{C}
\]

This may now be simplified.

\[
\begin{align*}
\text{If we choose a combination of fields, we obtain derived rules of} \\
& \text{as follows:}
\end{align*}
\]

\[
\begin{align*}
& A \iff A \\
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& \text{Converse Rule of Introduction}\quad & \text{for future reference, we also note the following two derived rules of} \\
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& \text{So, the following new and rule of inference.} \\
& \text{As a result,} A \iff A \\
& \text{We also}
\end{align*}
\]

The presentation of this problem in no form, where inference is superfluous:

\[
\begin{align*}
& A \iff A \\
& \text{Introduce Rule of Introduction}\quad & \text{are also the following two derived rules of} \\
& \text{Converse Rule of Introduction}\quad & \text{for future reference, we also note the following two derived rules of} \\
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& \text{understand the general} \\
& \text{So, the following new and rule of inference.} \\
& \text{As a result,} A \iff A \\
& \text{We also}
\end{align*}
\]

**Cartesian closed categories and calculus**
The deduction theorem asserts

$$\text{If } A \rightarrow C, \text{ then } A \vdash C \rightarrow B.$$  

Devised by Gerhard Gentzen

The usual deduction theorem asserts

$$\text{If } A \rightarrow C, \text{ then } A \vdash C \rightarrow B.$$  

Devised by Gerhard Gentzen

1. Deduction theorem

2. For the appropriate deduction systems, the following derived rules are

\[
\begin{align*}
\text{\textit{Simplification}} & : \quad C \vdash C \\
\text{\textit{Conjunction Introduction}} & : \quad A, B \vdash A \land B \\
\text{\textit{Conjunction Elimination}} & : \quad A \land B \vdash A \\ & : \quad A \land B \vdash B
\end{align*}
\]

The usual deduction theorem asserts

$$\text{If } A \rightarrow C, \text{ then } A \vdash C \rightarrow B.$$  

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\end{align*}
\]
A cartesian category is a category equipped with additional structures.

A cartesian closed category is a cartesian category with additional structures.

\[ \text{terms and functions by the distributive laws:} \]
\[ \text{and, continued above, follow:}\]
\[ \text{We shall also write:} \]
\[ \text{We show the following continuity subsuperscripts:}\]
\[ \text{Since a cartesian closed category is a cartesian category,} \]
\[ \text{we shall also consider the distributive law:} \]
\[ \text{is a product of objects of and with projections } \]
\[ \text{and } \text{is an object in the cartesian closed category.} \]
\[ \text{For all } \text{and } \text{we shall write:} \]

\[ \text{A cartesian closed category is a cartesian category equipped with additional structures.} \]

\[ \text{Thus, any deduction system may obtain a category by imposing} \]
\[ \text{the following conditions:} \]

\[ \text{A category is a deduction system in which the following conditions hold between processes:} \]

\[ \text{Example:} \]

\[ \text{In a cartesian closed category,} \]
\[ \text{we can compose two functions.} \]

\[ \text{The reader is advised that we do not distinguish notionally between} \]
\[ \text{theorems and theorems not yet proven.} \]

\[ \text{Remark:} \]

\[ \text{In the presence of a} \text{,} \]
\[ \text{the above notion was introduced on the length of the proof} \phi \text{.} \]

\[ \text{where} \phi \text{is the product of associativity} \]

\[ \text{A cartesian category is a cartesian category equipped with additional structures.} \]

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\[ \text{the above notion was introduced on the length of the proof} \phi \text{.} \]

\[ \text{where} \phi \text{is the product of associativity} \]
Theorem 5.1. Let $f: X \to Y$ be a function. Then, $f$ is a bijection if and only if there exist functions $g: Y \to X$ and $h: X \to Y$ such that $h \circ f = 1_X$ and $f \circ g = 1_Y$.

Proof. Assume $f$ is a bijection. Let $y \in Y$. Then there exists a unique $x \in X$ such that $f(x) = y$. Define $g: Y \to X$ by $g(y) = x$. Then $f \circ g(y) = f(x) = y$, so $f \circ g = 1_Y$. Similarly, for any $x \in X$, there exists a unique $y \in Y$ such that $f(x) = y$. Let $h: X \to Y$ be defined by $h(x) = y$. Then $h \circ f(x) = f(x)$ for all $x \in X$, so $h \circ f = 1_X$.

Conversely, assume that $f$ is a bijection. Let $y \in Y$. Then there exists a unique $x \in X$ such that $f(x) = y$. Let $g: Y \to X$ be defined by $g(y) = x$. Then for all $y \in Y$, $f \circ g(y) = f(x) = y$. Hence $f \circ g = 1_Y$. Similarly, for any $x \in X$, let $h: X \to Y$ be defined by $h(x) = y$. Then for all $x \in X$, $h \circ f(x) = h(x) = y$. Hence $h \circ f = 1_X$.
In an appropriate universe, we can introduce here one more axiom. Those may have to be referred as sets in the theory without those axioms, but in the intermediate $H_1$, they are referred as sets.

The above universal property means that $\sigma$ is a functor $\mathcal{C} \rightarrow \mathcal{D}$ such that for all objects $A \in \mathcal{D}$ and for all morphisms $f : A \rightarrow B$ in $\mathcal{D}$, $\sigma B = \emptyset$.

For all objects $A \in \mathcal{D}$, if $\sigma B = \emptyset$ then $f \circ \sigma = \sigma$.

Let $\mathcal{C}$ be a category closed under products. Then, the natural projection $\pi : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a functor $\mathcal{C} \rightarrow \mathcal{C}$ such that for all objects $A \in \mathcal{C}$ and for all morphisms $f : A \rightarrow B$ in $\mathcal{C}$, $\pi B = \emptyset$.

For all objects $A \in \mathcal{C}$, if $\pi B = \emptyset$ then $f \circ \pi = \pi$.