A pragmatic logic of hypotheses

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Summary. In this paper we consider Dalla Pozza and Garola *pragmatic interpretation of intuitionistic logic* [13] focussing on the role of illocutionary forces and justification conditions and we extend their approach by interpreting intuitionistic and co-intuitionistic logic through an "intended interpretation" in the style of a game-theoretic semantics and a notion of duality between *assertions* and *hypotheses*. The compatibility of these constructions from the viewpoint of an intuitionistic philosophy is shown by checking that the construction can be performed in an intuitionistic metatheory.

1 Introduction

This paper is about *intended interpretations* of *co-intuitionistic* (also known as *anti-intuitionistic* or *dual-lintuitionistic*) logic in the context of a variant of Cecylia Rauzer's *bi-intuitionistic* logic [25, 26]. The quest for the *intended interpretation* of a formal system often arises when several mathematical structures have been proposed to characterise an informal, perhaps vague notion and furthermore more unfamiliar and vaguer extensions arise by analogy or by opposition: here philosophical analysis may be invoked to assess which formal systems indeed belong to *logic* in the sense that they capture actual forms of human reasoning rather than to applied mathematics.

This is indeed the case of co-intuitionism and bi-intuitionism and of their proof-theory: researchers in this area sometimes extend intuitionistic or classical logic with the connective of subtraction, as in Tristan Crolard ([11, 12]) and develop co-intuitionistic proof theory (somehow anticipated by the notes in appendix to Prawitz [24]) using the sequent calculus, as in work by Czemark [14] and Urbas [29], the display calculus by Goré [19] or natural deduction by Uustalu [30]. Here he obvious symmetry with the connectives of intuitionistic logic is exploited; for instance, Luca Tranchini [28] works out in detail the idea of turning Prawitz deduction trees upside down (also suggested by one of the present authors in [4, 2, 5]).

But bi-intuitionism is obviously an offspring of intuitionism. Bouwer's ideas have been developed in the huge corpus of mathematical intuitionism, but also in philosophical intuitionism. Our mathematical references are mainly the Brouwer-Heyting-Kolmogorov interpretation of intuitionistic and the Extended Curry-Howard correspondence between the typed λ-calculus, intuitionistic Natural Deduction and Cartesian Closed Categories, in the interpretation of William Lawvere. Our philosophical focus is mainly in Michael Dummett and Dag Prawitz's theory of *meaning as use* and *justificationist* philosophy. How are these ideas extended from intuitionism to co-intuitionism and bi-intuitionsm? One of the present authors has developed a computational interpretation and a categorical semantics for co-intuitionistic linear logic [5, 6] and some variants of Kripke semantics for co-intuitionstic logic [4, 2]. Here we ask some philosophical questions, namely, what is the "intended interpretation" of co-intuitionism and bi-intuitionsm, if any? and are they acceptable from the viewpoint of philosophical intuitionism? We try first an "intended interpretation", loosely related to game theoretic semantics.

1.1 A Mathematical Prelude

A Heyting algebra is a (distributive) lattice $\mathcal{C}$ where the operation of Heyting implication $B \to A$ is defined as the right adjoint to meet ($\wedge$); a *co-Heyting algebra* is a lattice such that its opposite $\mathcal{C}^{op}$ (reversing the order) is a Heyting algebra. Here *subtraction* $A \smallsetminus B$ (the dual of *implication*, read as "A but not B") is defined as the left adjoint of disjunction:

$$\frac{C \wedge B \ \leq \ A}{C \ \leq \ B \to A} \qquad\qquad \frac{A \ \leq \ B \vee C}{A \smallsetminus B \ \leq \ C}$$

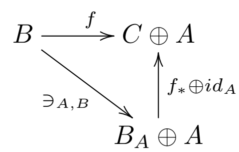
A *bi-Heyting* algebra is a lattice that has both the structure of Heyting and of a co-Heyting algebra. Bi-intuitionistic logic, modelled by bi-Heyting algebras, has also a Kripke semantics introduced by Cecylia Rauszer [25, 26]; the first treatment in category theory is by Makkai, Reyes and Zolfaghari [21, 27]. Let *strong* and *weak* negation be defined as $\sim A =_{def} A \to \bot$ and $\frown A =_{def} \bigvee \smallsetminus A$, where $\bot$ and $\bigvee$ are logical constant for falsity and truth, respectively. As $A \vee \sim A$ is not valid in intuitionism so $A \wedge \frown A$ is not contradictory in co-intuitionism, a feature of paraconsistent logics.

It was soon discovered that *first-order* bi-intuitionistic logic is the logic of constant domains, i.e., an intermediate system between classical and intuitionistic logic [17]. Moreover, every topological or categorical model of bi-intuitionistic logic is isomorphic to a partial order (see [11]): this rules out a *rich proof theory* with categorical models of Rauszer logic analogue to Cartesian Closed Categories for intuitionism. A solution suggested in [4, 2] is to "keep the dual parts separate" and connected by "mixed operators".

But a problem arises already in *co-intuitionism*:

Proposition. [11] *In the category* **Set** *the co-exponent* B_A *of two sets* A *and* B *is defined if and only if* $A = \emptyset$ *or* $B = \emptyset$.

Proof: In **Set** coproducts are *disjoint unions*. The co-exponent of A and B is an object B_A together with an arrow $\ni_{A,B}\colon B \to B_A \oplus A$ such that for any arrow $f : B \to C \oplus B$ there exists a unique $f_* : B_A \to C$ making the following diagram commute:



If $A \neq \emptyset \neq B$ then the functions f and $\ni_{A,B}$ for every $b \in B$ must *choose a side*, left or right, of the coproduct in their target and moreover $f_\star \sqcup 1_A$ leaves the side unchanged. Hence, if we take a nonempty set C and f with the property that for some b different sides are chosen by f and $\ni_{A,B}$, then the diagram does not commute.

The solution advocated in [6] is to construct a categorical model of *linear co-intuitionistic logic*, where *disjunction* is J-Y. Girard's *par*, and to use Girard's storage operator *whynot?* (**?**) and apply the dual of Girard's translation of intuitionistic logic into linear logic:

$$\begin{aligned}(p)^\circ &= p\\ (C \curlyvee D)^\circ &= ?(C^\circ \oplus D^\circ) = ?(C^\circ)\wp ?(D^\circ)\\ (C \smallsetminus D)^\circ &= C^\circ \smallsetminus (?D^\circ)\\ (E \vdash C_1, \ldots, C_n)^\circ &= ?(E^\circ) \vdash ?(C_1^\circ), \ldots, ?(C_n^\circ))\end{aligned}$$

In what follows we make no use of these mathematical results but do retain that *commas* in the co-intuitionistic consequence relation

$$E \vdash C_1 \ldots, C_n$$

must be understood as Girard's *par* and that *disjunction* is understood *multiplicatively*.

2 Pragmatic interpretations of intuitionism and co-intuitionism.

The task of this paper is to explore *intended interpretations* of bi-intuitionism where intuitionistic and co-intuitionistic formulas and connectives receive distinct interpretations. If, according to a suggestion by M. Dummett intuitionism is the logic of *assertions* and of their justifications, then we regard co-intuitionism as the logic of the justification of *hypotheses*, in so far as the notion of a hypothesis can be seen as dual to that of an assertion.

We develop our interpretation by expanding and reinterpreting Dalla Pozza and Garola's *pragmatic interpretation of intuitionistic logic* [13]. Its main feature is to take *elementary expressions* of the form $\vdash p$, where Frege's symbol "$\vdash$" represents an (impersonal) *illocutionary force of assertion* and p is a proposition. The grammar of Dalla Pozza and Garola's language $\mathcal{L}^P$ is as follows:

$$A, B \quad := \quad \vdash p \mid \bigvee \mid A \supset B \mid A \cap B \mid A \cup B \tag{1}$$

and (strong) negation "$\sim$" is defined as $\sim A = A \supset \mathbf{u}$. Here $\bigvee$ is an assertions which is always justified and $\mathbf{u}$ is always unjustified.

Then the justification of intuitionistic formulas is given precisely by Brouwer-Heyting-Kolmogorov's interpretation of intuitionistic connectives: the justification of $\vdash p$ is given by *conclusive evidence* for p (e.g., a proof of the mathematical proposition p) and the justification of an *implication* $A \supset B$ is a method that transforms a justification of A into a justification of B. Moreover a justification of a conjunction $A \cap B$ is a pair $\langle j, k\rangle$ where j is a justification of A and k a justification of B; a justification of a disjunction $A_0 \cup A_1$ is a pair $\langle j, 0\rangle$ where j is a justification of A_0 or $\langle k, 1\rangle$ where k is a justification of A_1.

To be sure, from an intuitionistic viewpoint the proposition p must be such that conclusive evidence for it can be effectively given: i.e., the (informal) proof justifying $\vdash p$ must be intuitionistic. If this is granted, then the expressions of $\mathcal{L}^P$ are *types of justification methods*; in a *propositions as types* framework they are *intuitionistic propositions.*

Having introduced the consideration of illocutionary forces in the elementary expression of logical languages, we can then ask in which sense intuitionistic types are *assertive expressions*: do molecular expressions inherit illocutionary force from their elementary components? is an illocutionary assertive force implicit in the way of presenting their justification? This is an interesting question: Dalla Pozza and Garola do not give an explicit answer and we may leave it open here.

However in the justification of an elementary expression $\vdash p$ a classically-minded logician may be satisfied with a classical proof: for instance let p be $q \vee \neg q$ where q is intuitionistically undecidable. More generally, if we do not develop such a pragmatic interpretation in an *intuitionistic metatheory*, then what we obtain is a constructive interpretation of intutionism in a classical framework, which is certainly unacceptable by an intuitionistic philosopher. This appears the spirit of Dalla Pozza and Garola's pragmatics: it is a two-layers formal system where the propositions p occurring in an elementary expression $\vdash p$ are interpreted according to classical semantics; broadly speaking, their goal is to show how classical logic can be reconciled with justificationist theories of meaning.

Gödel, McKinsey, Tarski and Kripke's modal **S4** interpretation are naturally considered as a *reflection* of the *pragmatic layer* of the logic for pragmatics into the *semantic layer*, where the image $\Box A'$ of a pragmatic expression A is indeed a proposition of classical modal logic **S4**, and the necessity operator of **S4** is read as an operator of "abstract knowability". Briefly put, the modal meaning of pragmatic assertions is provided by the following translation of pragmatic connectives:

$$\begin{aligned}(\vdash p)^M &= \Box p;\\ (A \supset B)^M &= \Box(A^M \to B^M);\\ (A \cap B)^M &= A^M \wedge B^M;\\ (A \cup B)^M &= A^M \vee B^M;\\ (\bigvee)^M \ = \ \mathbf{t}; &= (\mathbf{u})^M \ = \ \mathbf{f}.\end{aligned} \tag{2}$$

Here "$\to$", "$\wedge$", "$\vee$" are the classical connectives, **t** and **f** the truth values. It appears that in Carlo Dalla Pozza view the sign of assertion can only be applied to *classical propositions*[1]. If this is the case, then claiming that intuitionistic molecular expressions have assertive illocutionary force amounts to accepting the "invariance principle"

$$A \ \approx \ \vdash(A^M)$$

for all expressions A. Manifestly, an interpretation of intuitionism that validates such an identification is intuitionistically unacceptable.

Since our goal here is to find *intuitionistically acceptable* interpretations of are co-intuitionism and bi-intuitionsm, we shall seek interpretations where types of justifications are *pragmatically understood* intuitionistic or co-intuitionistic propositions and will develop our discussion in an intuitionistic metatheory.

We shall present an interpretation of (our versions of) co-intuitionistic logic with the flavour of the *game-theoretic* interpretation of linear logic and of Nelson's work on *constructive falsity*. Namely, we define *evidence for* and *evidence against* co-intuitionistic justification types, exploiting the duality between subtraction and implication but using the semantics of Girard's *par* for co-intuitionistic disjunction. The advantage of this interpretation is that it springs from a natural generalization of the Brouwer-Heyting-Kolmogorov interpretation and it is not tied to a particular formal presentation of the logic.

3 Co-Intuitionistic Logic as a logic of hypotheses

A clear example of how a change of epistemic attitudes, particularly as expressed in the elementary formulas, drastically affects the resulting logic is

[1] Personal communication.

given by considering the illocutionary force of *hypothesis* as basic. Obviously when hypothetical force is given also to molecular formulas the meaning of the connectives changes. Indeed it is even possible to have all sorts of *mixed connectives* operating on assertive and hypothetical sentences and building assertive or hypothetical connectives: this has been done in [4] and completeness of the resulting logic with respect to the classical **S4** translation has been checked. But the notion of a *duality* (informally understood) between assertions and hypotheses allows us to focus on a core fragment of the logic of hypotheses regarded as a pragmatic interpretation of *co-intuitionistic logic.*

The *minimal fragment of a co-intuitionistic logic of hypothesis* is built from elementary hypothetical expressions $\mathcal{H}\, p$ and a constant $\bigwedge$ for a hypothesis which is always unjustified, using the connectives *subtraction* $C \smallsetminus D$ (*"possibly C but not D"*), *hypothetical disjunction* $C \curlyvee D$ and *hypothetical conjunction* $C \curlywedge D$.

$$C, D \quad := \quad \mathcal{H}\, p \mid \bigwedge \mid C \smallsetminus D \mid C \curlyvee D \mid C \curlywedge D \tag{3}$$

and (weak) negation "$\frown$" (a *doubt*) is defined as $\frown C = (\mathbf{j} \smallsetminus C)$. Here "$\bigwedge$" is an always unjustified hypothesis and "$\mathbf{j}$" an always justified one. In this paper we shall not discuss hypothetical conjunction at all.

It is not obvious what an extension of the Brouwer-Heyting-Kolmogorov (BHK) interpretation to co-intuitionistic logic is. We certainly have constructive justifications for the *refutation* of hypothetical expressions, thus defining a logic of hypotheses to a *refutation calculus*: in particular, a refutation of $C \smallsetminus D$ is a method transforming every refutation of D into a refutation of C, i.e., conclusive evidence for $(D^{\perp} \supset C^{\perp})$[2],where we write $C^{\perp}, D^{\perp}$ for the assertion that are dual to C, D.

3.1 Evidence, Negative evidence and the duality between assertions and hypotheses.

But what constitutes a *justification* for a *hypothesis* ($\mathcal{H}\, p$) and how does it differ from a justification of an *assertion* ($\vdash p$)? In the familiar BHK interpretation of intuitionistic logic evidence for a mathematical statement p is a proof of it; in the case of non-mathematical assertive statements, we speak of *conclusive evidence* for p. What constitutes *conclusive evidence* for p depends on the context and scientific discipline.

Consider for example, the *theory of argumentation* in legal reasoning. Here five *proof-standards* have been identified from an analysis of legal practice: *scintilla of evidence*, *preponderance of evidence*, *clear and convincing evidence*, *beyond reasonable doubt* and *dialectical validity*, in a linear order of strength [18, 10]. Can such distinctions be taken up in our approach in some way?

It seems that a *scintilla of evidence* suffices to justify $\mathcal{H}\, p$, making the hypothesis that p, and that *dialectical evidence* ought to coincide with assertability

[2] See Angelelli [1].

$\vdash p$, which in our framework is *conclusive evidence*; to these, we may add the case of *no evidence at all.* The other proof-standards are defined through probabilities; this goes beyond our purely logical approach here.[3].

If we assume the notion of "negative evidence", or *evidence against* the truth of a proposition, as basic, in addition to "positive evidence", or *evidence for*, then another logical relation is evident between *scintilla of evidence* and *conclusive evidence*, in addition to the order of strength: *we cannot have at the same time* conclusive evidence for *and a* scintilla of evidence against *the truth of a proposition.* On this basis we can attempt an interpretation of intuitionistic and co-intuitionistic connectives which is reminiscent of *game semantics* and also of Nelson's treatment of *constructive falsity.*

3.2 A game-theoretic semantics?

For elementary formulas we have the following.

Definition 1. *Let p be a proposition.* Evidence for $\vdash p$ *is conclusive evidence of the truth of p;* evidence against $\vdash p$ *is a scintilla of evidence that p may be false.* Evidence for $\mathcal{H}\, p$ *is a scintilla of evidence that p may be true;* evidence against $\mathcal{H}\, p$ *is conclusive evidence of the falsity of p.*

Remark 1. Definition 1 seems unproblematic in a *classical* metatheory, where the terms "proposition", "truth" and "falsity" have a well-established use, and "conclusive evidence" and "scintilla of evidence" may be understood in a *modal epistemic* sense as the grounds for $\Box p$ and $\Diamond p$, respectively. But suppose "proposition", "truth" and "falsity" are understood intuitionistically: bivalence is not presupposed for intuitionistic propositions. Then on one hand it seems that *conclusive evidence of the falsity of p* must be regarded as grounds for ordinary *intuitionistic negation*, namely, the fact that assuming the truth of p would lead to a contradiction. On the other hand, the notion of a *scintilla of evidence* of the falsity of p seems to be a new concept, perhaps corresponding to the notion of *doubt.*

For co-intuitionistic expressions we have the following interpretation.

Definition 2. (game-theoretic semantics for co-intuitionistic logic)

1. *Evidence for* $\mathcal{H}\, p$ *is a scintilla of evidence that p may be true;*
   - *evidence against* $\mathcal{H}\, p$ *is conclusive evidence that p is not true.*

[3] Incidentally, legal reasoning teaches us an interesting methodological point. Even the strongest standard of evidence, which may be needed in a criminal court to justify the assertion p that a defendant is guilty, does not imply that p is true: indeed successive evidence may justify reopening the case. A theory of assertions suitable for representing common sense reasoning must accept this "pragmatic" feature of assertion and relinquish the axiom $\Box p \rightarrow p$ in its semantic projection.

2. *Evidence* for a subtraction $C \setminus D$ *is given by evidence* for C *together with evidence* against D*;*
   \- *evidence* against $C \setminus D$ *is a method that transforms evidence* for C *into evidence* for D *and also evidence* against D *into evidence* against C*.*
3. *Evidence* for a disjunction $C \curlyvee D$ *is a method that transforms evidence* against C *into evidence* for D *and also evidence* against D *into evidence* for C*;*
   \- *evidence* against $C \curlyvee D$ *is evidence* against both C *and* D*.*
4. *Evidence* for a conjunction $C \curlywedge D$ *is evidence* for C *and also* for D*; evidence* against $C \curlywedge D$ *is conclusive evidence* against C *or* against D*.*

We write $c : H^+$ and $c' : H^-$ to denote evidence c *for* H and evidence c' *against* H. The basic consequence relation $C \vdash D$, which for the time being we may consider as being *reflexive* and *transitive* only, is interpreted by a pair of functions $\langle f_1, f_2 \rangle$ between types of justification methods, where $f_1 : C^+ \to D^+$ sends evidence *for* C to evidence *for* D, and $f_2 : D^- \to C^-$ sends evidence *against* D to evidence *against* C.

With this definition we can prove the following

Proposition 1. *Let C, D and E be hypothetical expressions. Then the basic adjunction*

$$\frac{C \vdash D \curlyvee E}{\overline{C \setminus D \vdash E}} \tag{4}$$

is valid in the game-theoretic semantics.

Proof. (1) Let f_1, f_2 be methods, where $f_1 : C^+ \to (D \curlyvee E)^+$ and $f_2 : (D \curlyvee E)^- \to C^-$. We define methods $g_1 : (C \setminus D)^+ \to E^+$ and $g_2 : E^- \to (C \setminus D)^-$. We know that if $c : C^+$ then $f_1(c) : (D \curlyvee E)^+$ is a map k with $k : D^- \to E^+$ and $k : E^- \to D^+$. Moreover we have $f_2(\langle d, e \rangle) : C^-$ if $d : D^-$ and $e : E^-$.

(i) Now evidence *for* $C \setminus D$ is a pair $\langle c, d \rangle$ where $c : C^+$ and $d : D^-$. Hence we may define $g_1(c, d) = (f_1(c))(d)$ where $f_1(c) = k : D^- \to E^+$ as above and $k(d) : E^+$ is evidence for E as required.

(ii) Moreover let $e : E^-$. We need to define $g_2(e) : (C \setminus D)^-$ as a map $m : C^+ \to D^+$ and $m' : D^- \to C^-$. But if $c : C^+$ then $f_1(c)$ is a map $k : E^- \to D^+$ as above, so we let $g_2(e)(c) = f_1(c)(e) : D^+$. Also if $d : D^-$ then $f_2(\langle d, e \rangle) : C^-$, so we define $g_2(e)(d) = f_2(\langle d, e \rangle) : C^-$ as required.

(2) Now given methods $g_1 : (C \setminus D)^+ \to E^+$ and $g_2 : E^- \to (C \setminus D)^-$ we define $f_1 : C^+ \to (D \curlyvee E)^+$ and $f_2 : (D \curlyvee E)^- \to C^-$.

(iii) Given $c : C^+$ and $d : D^-$ we may let $(f_1(c))(d) = g_1(\langle c, d \rangle) : E^+$ and $(f_1(c))(e) = (g_2(e))(c) : D^+$, since $g_2(e)$ maps C^+ to D^+.

(iv) Finally, given $d : D^-$ and $e : E^-$ we let $f_2(\langle d, e \rangle) = (g_2(e))(d) : C^-$ since $g_2(e)$ maps D^- to C^-. **qed**

Remark 2. Suppose in clause (3) of definition 2 we let *evidence for* $C \curlyvee D$ be either a pair $(c, 0)$ where c is is a scintilla of evidence *for* C or a pair $(d, 1)$ where d is a scintilla of evidence *for* D and *evidence against* $C \curlyvee D$ as a pair $\langle c, d\rangle$ where c and d are conclusive evidence against C, D respectively. This corresponds to an *additive* interpretation of disjunction. We claim that with an additive interpretation the above lemma cannot be proved, at least not in an intuitionistic metatheory.
The argument goes through in case $(1)(i)$ since now $f_1(c) = (d, 0)$ or $(e, 1)$ with $d : D^+$ or $e : E^+$, but we have $d : D^-$ and d is *conclusive evidence against* D; therefore $f_1(c)$ can only be $(e, 1)$ as required. Also in case $(1)(ii)$ given $e : E^-$ and $c : C^+$, we have $f_1(c) : (D \curlyvee E)^+$, but since e is *conclusive evidence against* E, we have $f_1(c) = (d, 0)$ for some $d : D^+$. The argument goes through also in case $(2)(iii)$, since we can take $f_1(c) = (g_1(\langle c, d\rangle), 1) : (D \curlyvee E)^+$ and also in case $(2)(iv)$ without changes. But consider case $(2)(i)$: given $c : C^+$, if there is some $d : D^+$ then we can set $f_1(c) = (d, 0)$ and if $d : D^-$ then we take $f_1(c) = (e, 1)$ where $e = g_1(\langle c, d\rangle)$ but it may still be possible that there is no evidence whatsoever for or against D. However, if we take positive or negative evidence for D to be represented in *classical* **S4** as $w \Vdash \Diamond D^M$ or $w \Vdash \Box\neg D^M$ in some possible world w belonging to a Kripke model $\mathcal{M} = (W, \leq, \Vdash)$ then the absence of any evidence for D in w entails the existence of negative evidence for D.

The above remark shows that we need to take hypothetical disjunction as multiplicative *par* in order to have our game-theoretic semantics for co-intuitionism: indeed at present, our treatment deals only with the linear fragment of co-intuitionism. Of course in a full-fledged treatment one would also define the game-theoretic interpretation of the exponentials and use it prove the equivalence

$$?C \wp ?D \equiv ?(C \oplus D)$$

We shall not do this here.

4 Bi-intuitionism.

We may obtain "intended interpretation" of our version of *bi-intuitionistic logic* simply by exploiting the notion of a *duality* between assertions and hypotheses. We start from a duality between *elementary formulas*, i.e., the assertion that p is true and the hypothesis that *not* p. Given an infinite sequence of propositional variables and of their negations[4] $p_0, \neg p + 0, p_1, \neg p_1, \ldots$, we

[4] No use is made here of the logical properties of the classical connective of negation, except for its involutory character: $\neg\neg p \equiv p$; in particular, no interaction is considered of classical negation with pragmatic operators and connectives. More abstractly, we are given an infinite sequence of propositional atoms with an involution without fixed point on them.

define the duality between the assertive and hypothetical expressions in the obvious way:

$$
\begin{gathered}
({\vdash}p)^{\perp} = \mathcal{H}\neg p \quad ({\vdash}\neg p)^{\perp} = \mathcal{H}\, p \quad (\mathcal{H}\, p)^{\perp} = {\vdash}\neg p \quad (\mathcal{H}\neg p)^{\perp} = {\vdash}p; \\
(\bigvee)^{\perp} = \bigwedge = (\bigwedge)^{\perp} = \bigvee; \\
\begin{array}{ll}
(A \supset B)^{\perp} = B^{\perp} \smallsetminus A^{\perp} & (C \smallsetminus D)^{\perp} = D^{\perp} \supset C^{\perp}; \\
(A \cap B)^{\perp} = A^{\perp} \curlyvee B^{\perp} & (C \curlyvee D)^{\perp} = C^{\perp} \cap D^{\perp}; \\
(A \cup B)^{\perp} = A^{\perp} \curlywedge B^{\perp} & (C \curlywedge D)^{\perp} \cup C^{\perp} \cap D^{\perp};
\end{array}
\end{gathered}
\tag{5}
$$

The "game-theoretic interpretation" of intuitionistic expressions, is the exact dual of definition 2.

Definition 3. (game-theoretic semantics for intuitionistic logic)

1. *Evidence for* ${\vdash}p$ *is conclusive evidence that* p *may be true;*
   \- *evidence against* $\mathcal{H}\, p$ *is a scintilla of evidence that* p *is not true.*
2. *Evidence* for $A \supset B$ *is a method that transforms evidence* for A *into evidence* for B *and also evidence* against B *into evidence* against A.
   \- *evidence* against an implication $A \supset B$ *is given by evidence* for A *together with evidence* against B*;*
3. *Evidence* for *a conjunction* $A \cap B$ *is evidence* for A *and* for B.
   \- *evidence* against a conjunction $A \cap B$ *is a method that transforms evidence* for A *into evidence* against B *and also evidence* for B *into evidence* against A*;*
4. *Evidence* for a disjunction $A \cup B$ *is evidence* for A *or* for B*;*
   evidence against $A \cap B$ *is evidence*against S *and* against B.

Notice that the interpretation of assertive conjunction corresponds to that of *times* ($\otimes$) in linear logic.

The proof of the following proposition is completely analogous to that of the proposition 4.

Proposition 2. *Let* A, B *and* C *be hypothetical expressions. Then the basic adjunction*

$$
\frac{A \cap B \vdash C}{A \vdash B \supset C} \tag{6}
$$

is valid in the game-theoretic semantics.

Remark 3. (i)Notice also that the "game-theoretic interpretation" justifies regarding the duality $(\ )^{\perp}$ not only as a meta-theoretic property, but also as a (pair of) connectives representing the duality within the language,

(ii) The correspondence given by the duality between the intuitionistic and co-intuitionistic sides of the bi-intuitionistic system sketched here is very tight.

Once the full system is developed beyond the linear fragment it may be interesting to ask whether a different, less symmetric view of the relations between the inuitionistic and co-intuitionistic sies may reveal more interesting properties: indeed the categorical motivations requiring us to develop co-intuitionistic logic as an extension of its linear part, as sketched in our "mathematical prelude" do not apply to intuitionism.

5 Conclusions

In this paper we have taken Dalla Pozza and Garola's pragmatic interpretation of intuitionistic logic and its extension to co-intuitionism and bi-intuitionism as a framework for logical analysis from the viewpoint of an intuitionistic philosophy. This means that we made sure that such investigations can be performed within an intuitionistic meta-theory and thus, for instance, that any reference to Kripke semantics for classical **S4** is not taken as the foundation of the concepts to be investigated. We have given an intended interpretation" of co-intuitionistic logic as a logic of hypotheses and of their justifications, in the style of a game-theoretic semantics for the (linear fragment of) co-intuitionistic logic. Here *evidence against* a hypothesis is taken as as conclusive evidence for its dual assertion, but *evidence for* a hypothesis is the notion of a *scintilla of evidence*, related to the *doubt*.

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