Disambiguating bi-intuitionism.

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0. Plan of the talk.

1. Bi-intuitionistic logic revisited: **mathematical collapse** of its topological and categorical models and **philosophical implausibility** of its modal-tense interpretations.

2. No categorical model of co-intuitionism in Set: translation of **co-IL** into linear logic and categorical model of **linear co-IL** in monoidal left-closed categories with extra structure.

3. Bi-intuitionistic logic and the logic for pragmatics: an *intended model* for **'polarized' bi-intuitionism.** A game-like semantics of justifications for **co-IL** and intuitionistic acceptability of **'polarized' bi-IL**.

4. No collapse of topological and categorical models of **'polarized' bi-IL**. Bi-Intuitionistic modalities as intuitionistically acceptable polarity-cahnging modalities. A *logic of expectations* as a pragmatic interpretation of the double negation rule.

1.1. A "rich" proof-theory.

• For **Intuitionistic Logic IL** we have the **Extended Curry-Howard** correspondence:

• Natural Deduction **NJ** (+ sequent calculus **LJ**)
- extended to higher order logic;

• Simply Typed λ-**calculus**
- extended to Dependent Types and system **F**;

• Cartesian Closed Categories **CCC**s
- subsuming Heyting algebras, topological models etc.

A "rich" proof-theory for bi-Intuitionism?

1.2. Co- and Bi-Heyting algebras

• A *Heyting algebra* is a (distributive) lattice with an operation $\to$, the **right adjoint** to the **meet** $\wedge$;
• A *co-Heyting algebra* is a lattice with an operation - (*subtraction*) which is the **left adjoint** to the **join** $\vee$.
Thus we have

in **HA**	in **co-HA**
$\dfrac{c \wedge a \ \leq \ b}{c \ \leq \ a \to b}$	$\dfrac{b \ \leq \ a \vee c}{b - a \ \leq \ c}$

• A *bi-Heyting algebra* is a lattice

$$\mathcal{C} = (C, \wedge, \vee, \to, -, \bot, \top)$$

with both Heyting and co-Heyting structures.

Strong $\neg$ and weak $\sim$ negations:
• $\neg a = a \to \bot$ (the largest c such that $c \wedge a = \bot$);
• $\sim a = \top - a$ (the smallest c such that $c \vee a = \top$).

1.3. Negations and modalities.

Immediate properties of negations:

- $a \leq \neg\neg a \qquad \sim\sim a \leq a$;
- $\neg a \ \leq \ \sim a$. *Proof:* $\dfrac{\top \leq a \vee \sim a}{\neg a \leq \neg a \wedge (a \vee \sim a) = \bot \vee (\neg a \wedge \sim a).}$

Hence

$$\neg \sim a \ \leq \ \sim\sim a \ \leq \ a \ \leq \ \neg\neg a \ \leq \ \sim \neg a$$

Define

$$\begin{aligned} \Box_0 a &= a & \Diamond_0 a &= a; \\ \Box_{n+1} a &= \neg \sim \Box_n a & \Diamond_{n+1} a &= \sim \neg \Diamond_n a \\ \boxdot a &= \bigwedge_n \Box_n a & \diamond a &= \bigvee_n \Diamond_n a. \end{aligned}$$

Proposition.
$\boxdot a$ is the largest **complemented** *x such that $x \leq a$*
$\diamond a$ is the smallest **complemented** *x such that $a \leq x$.*

Proof. Indeed $\boxdot a \leq \neg \sim \boxdot a$ implies $\boxdot a \wedge \sim \boxdot a \leq 0$ hence $\sim \boxdot a = \neg \boxdot a$. Proceed dually for $\diamond a$.
Thus $\neg\neg \boxdot a = \boxdot a$ *and* $\diamond a = \sim\sim \diamond a$.

- Reyes and Zolfaghari, Bi-Heyting algebras, toposes and modalities, *J.Phil.Log* **25**, 1996:
  - *"a new approach to the modal operators of necessity and possibility".*
- *Are* $\boxdot$ *and* $\diamond$ **intuitionistic** *modalities?*

1.4. Co-Heyting Boundaries.

Lawvere 1991 advocates co-Heyting algebras for representing the notion of a **boundary**.

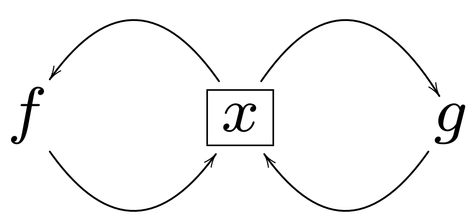
Let $\mathbf{S}$ be the set of all subgraphs of a graph $G = (V, E)$.

For $Y, Z \in \mathbf{S}$ define

- $Y \wedge Z =$ the intersection of Y, Z;
- $X \vee Y =$ the union of X, Y;
- $\neg X =$ the largest subgraph Z such that $X \wedge Z = \emptyset$;
- $\sim X =$ the smallest subgraph Z s.t. $X \vee Z = G$.

$X \wedge \sim X$ is the *boundary* of X.

In the following graph G let $Y = \{x, f\}, Z = \{x, g\}$:



- *Subgraphs* of G: $\{G, Y, Z, \{x\}, \emptyset\}$; $\sim Y = Z$, $\sim Z = Y$.
- $Y \wedge Z = \{x\}$, the **boundary** of Y and of Z.
- *Dual De Morgan law:*

$\sim (Y \vee Z) = \emptyset \neq \{x\} = \sim Y \wedge \sim Z$

$\sim (Y \wedge Z) = \sim \{x\} = G = \sim Y \vee \sim Z.$

Bibliographic note: F. W. Lawvere, Intrinsic co-Heyting boundaries and the Leibniz rule in certain toposes. In *Category Theory (Como 1990)*, Springer L.N.Math 1488, 1991, pp. 279-297.
P. Pagliani. Intrinsic co-Heyting boundaries and information incompleteness in Rough Set Analysis. In: *RSCTC 1998*. Springer L.N.C.S., 1424, 2009 pp. 123-130.
No advances on this topic in this paper.

1.5. Bi-Intuitionism and co-Intuitionism.

• *Bi-Intuitionistic Logic* (**bi-IL**) (also Heyting-Brouwer) is the logic on the following language

• Atoms $p_0, p_1, \ldots$

$A, B \ := \ p \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \to B \mid A - B$

with bi-Heyting algebras as algebraic models.

• *Co-Intuitionistic Logic* (**co-IL**), (aka *dual intuitionistic*), the fragment of **bi-IL** on the language

$$A, B \ := \ p \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A - B$$

with co-Heyting algebras as algebraic models.

• C. Rauszer. Semi-Boolean algebras and their applications to intuitionistic logic with dual operations, in *Fundamenta Mathematicae*, **83**, 1974, pp. 219-249.
• C. Rauszer. Applications of Kripke Models to Heyting-Brouwer Logic, in *Studia Logica* **36**, 1977, pp. 61-71.

Also: Goré 2000, Crolard 2001, 2004, Shramko 2005, Wansing 2008, Pagliani 2009, Pinto and Uustalu 2010, Tranchini 2012.

1.6. Kripke Models for bi-IL

- $\mathcal{M} = (W, \leq, \mathcal{V})$ where

- $(W, \leq)$ a preordered frame

- $\mathcal{V} : \mathbf{Atoms} \to \wp(W)$ **monotone**.

Monotonicity: *if* $w \leq w'$ *and* $w \in \mathcal{V}(p)$ *then* $w' \in \mathcal{V}(p)$

- *Forcing conditions:* (Rauszer 1977)

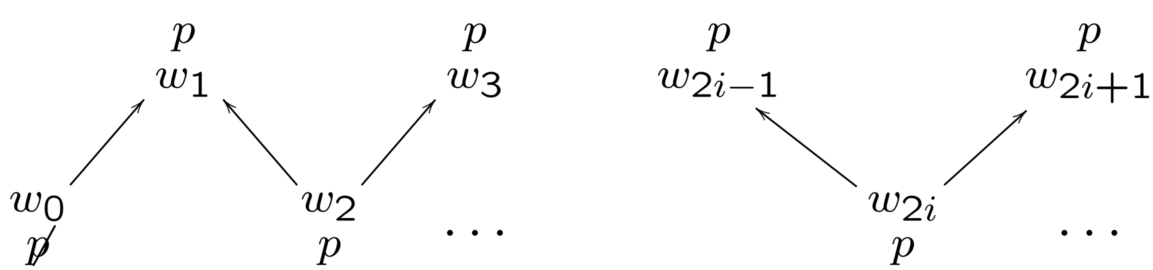
- $w \Vdash p$ iff $w \in \mathcal{V}(p)$;

- $w \Vdash A \to B$ iff $\forall w' \geq x$ if $w' \Vdash A$ then $w' \Vdash B$;

- $w \Vdash B - A$ iff $\exists w' \leq x$ $w' \Vdash B$ and $w' \nVdash A$;

- $w \Vdash A \wedge B$ iff $w \Vdash A$ and $w \Vdash B$ etc.

To show that in $\neg\sim A \ \lneqq \ A \ \lneqq \ \sim\neg A$ the order may be strict, consider the infinite Kripke model:



$w_m \Vdash \neg \sim p$	iff $\forall v \geq w_m . \forall u \leq v . u \Vdash p$	iff $\mathbf{m > 2}$
$w_m \Vdash \neg \sim \Box_n p$	iff	iff $\mathbf{m > 2n + 2}$

Similarly

$w_m \Vdash \Diamond_n \sim p$ iff $\exists v \leq w_m . \exists u \geq v . u \Vdash \sim p$ iff $\mathbf{m \leq 2n + 1}$.

1.7. Topological Models of bi-IL.

- *A bi-topological space* $(X, \mathcal{O})$ is given by

    A set X and a collection $\mathcal{O} \subseteq \wp(X)$
    - $\mathcal{O}$ contains X, $\emptyset$ and
    - is closed under arbitrary unions
    - *and arbitrary intersections*.

A bi-topological space is a Boolean algebra if all $S \in \mathcal{O}$ are *clopen.* There exist bi-topological spaces that aren't Boolean algebras.

Models of **bi-IL** in bi-topological $(X, \mathcal{O})$:

Let $[\![p_i]\!] \in \mathcal{O}$, $\quad [\![\top]\!] = X,\ [\![\bot]\!] = \emptyset$;
$[\![A \wedge B]\!] = [\![A]\!] \cap [\![B]\!]$, $\quad [\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!]$.
$[\![A \to B]\!] = int([\![A]\!]^C \cup [\![B]\!])$, $\quad ([\![A - B]\!] = ext([\![A]\!] \cap [\![B]\!]^C)$

Lemma. *A topological space* $(X, \mathcal{O})$ *is bi-topological iff* $\mathcal{O}$ *is the set of all final (initial) sections of some preorder.*

Thus non-trivial topological models of **bi-IL** exist but **"collapse to preorders"**.

1.8. Extending Gödel, McKinsey and Tarski S4 interpretation.

Pinto and Uustalu 2010:
"It is also a basic observation that the Gödel translation of **IL** *into the modal logic* **S4** *extends to a translation of* **bi-IL** *into the future-past tense logic* **KtT4***. As the semantics of* **KtT4** *does not enforce monotonicity of interpretations, atoms must be translated as future necessities or past possibilities (these are always monotone)":*

$$p^M = \Box p \qquad \text{or} \qquad p^M = \blacklozenge p$$

Also we have $(\ )^M$: **bi-IL** $\to \mathbf{KtT4}$

$$(A \to B)^M = \Box(A^M \to B^M) \quad (B - A)^M = \blacklozenge(B^M \wedge \neg A)$$
$$(A \wedge B)^M = A^M \wedge B^M \quad (A \vee B)^M = A^M \vee B^M$$

• But how can atoms have an **ambiguous** epistemic interpretation between *necessarily in the future* and *possibly in the past*?

Problem 1: *Linguistic ambiguity* of **KtT4** modal interpretations.

Bibliographical Note:

L. Pinto and T. Uustalu. Relating sequent calculi for Bi-intuitionistic Propositional Logic, van Bakel, Berardi and Berger eds. *Proceedings Third International Workshop on Classical Logic and Computation.* EPTCS 47, 2010. pp.57-72.

1.9. Collapse of bi-IL models.

Proposition (Gabbay 1972) *First order* **bi-IL** *is the logic of constant domains* (an *intermediate* logic between intuitionistic and classical).

Theorem (Crolard 2001) *Every categorical model of* **bi-IL** *is isomorphic to a partial order*.
Proof: Joyal's argument showing that bi-cartesian closed categories are degenerate applies here.

Problem 2: *No 'rich proof theory'* for **bi-IL**!

T. Crolard. Subtractive logic, in *Theoretical Computer Science* **254**,1-2, 2001, pp. 151-185.

2.1. (Philosophical) Comments to 1.

• Problem 1 is conceptually 'fatal' for the **KtT4** interpretation: it is untenable, because of the ambiguous translation of *atomic formulas*.

• Philosophically, we need an *intended interpretation* of bi-intuitionistic logic. What determines the meaning of an atomic formula in **bi-IL**? Is the meaning of atomic formulas the same in *intuitionistic* and *co-intuitionistic* logic?

Proposed solution to 1: (i) *Separate*

• **classical logic** as logic of **proposition** and **truth** from

• **bi-intuitionism** as logic of **judgements** and their **justifications**,

Dalla Pozza and Garola 1995, Bellin and Dalla Pozza 2002, following Dummett.

(ii) *Disambiguate the interpretation of* **bi-IL**:

• **intuitionism** as logic of **assertions**.

• **co-intuitionism** as logic of **hypotheses**

Bellin 2004, 2012, 2013, B.et al 2012a, 2012b, 2013.

2.2. (Mathematical) Comments to 2.

• Problem 2 is mathematical: there must be more structure in bi-intuitionistic logic for it to have a *rich proof theory*.
What additional structure? This depends on the desired applications.
However the 'linguistic disambiguation' of bi-intuitionism (problem 1) motivates the following solution.

Proposed solution to 2: *'Polarize'* **bi-IL**.
Keep the dual Heyting and co-Heyting structure separate, related by negations implementing the duality

$$(\ )^{\perp} : \mathbf{IL} \longrightarrow \textbf{co-IL} \qquad (\ )^{\perp}\textbf{co-IL} \longrightarrow \mathbf{IL}$$

Bellin 2004, 2012, 2013?, B.et al 2012a?, 2012b, 2013?.

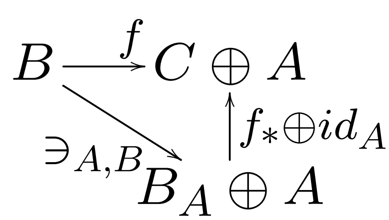
Bibliographical Note:

- C. Dalla Pozza and C. Garola 1995. A pragmatic interpretation of intuitionistic propositional logic, *Erkenntnis* **43**. 1995, pp.81-109.
- B. and C. Dalla Pozza 2002. A pragmatic interpretation of substructural logics. In *S. Feferman Festschrift*, ASL LN in Logic, **15**, 2002, pp. 139-163.
- B. and C. Biasi 2004. Towards a logic for pragmatics. Assertions and conjectures. In: *Journal of Logic and Computation*, **14**, 4, 2004, pp. 473-506.
- B. 2013. Assertions, hypotheses, conjectures: Rough-sets semantics and proof-theory,*Advances in Natural Deduction*, 2013.
- B., M. Carrara and D. Chiffi 2012a?. A pragmatic framework for intuitionistic modalities: Classical Logic and Lax logic, subm. *JLC*, 2012.
- B. and A. Menti 2012b. On the π-calculus and co-intuitionistic logic. Notes on logic for concurrency and λP systems, accepted *Fundam. Informaticae*
- B. 2012? Categorical Proof Theory of Co-Intuitionistic Linear Logic, *LOMECS*, 2012.
- B., M. Carrara and D. Chiffi 2013?. A pragmatic logic of hypotheses, *Logic and Logical Philosophy*.

2.3. Categorical models of co-IL.

• Disjunction is modelled by co-products and subtraction by co-exponents. In **Set** co-products are *disjoint unions*, but in **Set** nontrivial co-exponents don't exist!

Proposition. (Crolard 2001) *The co-exponent B_A of two sets A and B is defined iff $A = \emptyset$ or $B = \emptyset$.*
Proof: The co-exponent of A and B is an object B_A together with an arrow $\ni_{A,B}$: $B \to B_A \oplus A$ such that for any arrow $f : B \to C \oplus B$ there exists a unique $f_* : B_A \to C$ making the following diagram commute:



If $A \neq \emptyset \neq B$ then the functions f and $\ni_{A,B}$ for every $b \in B$ must *choose a side*, left or right, of the coproduct in their target and moreover $f_\star \sqcup 1_A$ leaves the side unchanged. Hence, if we take a nonempty set C and f with the property that for some b different sides are chosen by f and $\ni_{A,B}$, then the diagram does not commute.

Problem 3. *No model of* **co-IL** *in* **Set**.

2.4. A solution to Problem 3.

• Problem 3 shows that *co-intuitionistic disjunction* ($\curlyvee$) cannot be the exact dual of *intuitionistic* ($\cup$):

$$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_0 \cup A_1} \cup_i \text{ I} \quad i = 0,1 \qquad\qquad \frac{E \vdash \Delta, C_0, C_1}{E \vdash \Delta, C_0 \curlyvee C_1} \curlyvee \text{ I}$$

• *Intuitionistic Linear Logic* **ILL** can be modelled by *monoidal categories*!

BBHdP 1993: P.N.Benton, G.M.Bierman, J.M.E.Hyland and V.C.V.dePaiva. A term calculus for Intuitionistic Linear Logic. In: *Typed Lambda Calculi and Applications*, L.N.C.S., **664**, 1993, pp.75-90.

• Intuitionistic logic **IL** is translated into **ILL** (Girard 1986)

Proposed way out: (i) Define *co-Intuitionistic Linear Logic* **co-ILL**;

(ii) represent **co-IL** into **co-ILL** by the dual of Girard's translation.

(iii) Define categorical models of **co-ILL**, by *dualizing the construction in* BBHdP 1993.

2.5. Translation co-IL $\rightarrow$ linear co-IL.

We sketch the solution in Bellin 2012? with no detail.

Main logical features are:

• Both **co-IL** and **co-ILL** have a consequence relation with *single assumption* and (a list of) conclusions

$$E \vdash C_1, \ldots, C_n$$

co-IL has unrestricted *weakening* and *contraction* right; **co-ILL** does not.

• In the categorical construction we assign *lists of terms in context* thus:

$$x : E \ \triangleright \ t_1 : C_1, \ldots, t_n : C_n.$$

• The fragment of **co-IL** on the language with $(\perp, \curlyvee, \setminus)$ is mapped to the fragment of **co-ILL** with $(\perp, \wp, \setminus, ?)$ where '**?**' is Girard's *exponential whynot?*:

$$\begin{aligned}
(p)^\circ &= \mathsf{p} \\
(\perp)^\circ &= \perp \\
(C \curlyvee D)^\circ &= ?(C^\circ \oplus D^\circ) \\
&= ?(C^\circ)\wp?(D^\circ) \\
(C \setminus D)^\circ &= C^\circ \setminus (?D^\circ) \\
(E \vdash C_1, \ldots, C_n)^\circ &= ?(E^\circ) \vdash ?(C_1^\circ), \ldots, ?(C_n^\circ))
\end{aligned}$$

2.6. A sequent calculus for co-IL.

Identity:

axiom

$$A \Rightarrow A$$

$$\frac{H \Rightarrow \Gamma, C \qquad C \Rightarrow \Delta}{H \Rightarrow \Gamma, \Delta}\ \textit{cut}$$

Structural rules:

$$\frac{H \Rightarrow \Gamma, C, D, \Delta}{H \Rightarrow \Gamma, D, C, \Delta}\ \textit{exch}$$

$$\frac{H \Rightarrow \Gamma}{H \Rightarrow \Gamma, C}\ \textit{weak} \qquad \frac{H \Rightarrow \Gamma, C, C,}{H \Rightarrow \Gamma, C}\ \textit{contr}$$

Logical rules:

unjustifiability:

$$\bot \Rightarrow \Delta$$

$$\frac{H \Rightarrow \Gamma, C \qquad D \Rightarrow \Delta}{H \Rightarrow \Gamma, C \setminus D, \Delta}\ \setminus \mathsf{R} \qquad \frac{C \Rightarrow D, \Delta}{C \setminus D \Rightarrow \Delta}\ \setminus \mathsf{L}$$

$$\frac{H \Rightarrow \Gamma, C_0, C_1}{H \Rightarrow C_0 \curlyvee C_1}\ \curlyvee \mathsf{R} \qquad \frac{C_0 \Rightarrow \Gamma \qquad C_1 \Rightarrow \Delta}{C_0 \curlyvee C_1 \Rightarrow \Gamma, \Delta}\ \curlyvee \mathsf{L}$$

2.6.1. A sequent calculus for linear co-IL.

Identity:

axiom

$$A \Rightarrow A$$

$$\frac{H \Rightarrow \Gamma, C \qquad C \Rightarrow \Delta}{H \Rightarrow \Gamma, \Delta}\ \textit{cut}$$

Structural: Exchange and **Exponential rules:**

$$\frac{C \Rightarrow ?\Gamma}{?C \Rightarrow ?\Gamma}\ \textbf{?}\ \mathsf{L} \qquad \frac{H \Rightarrow \Gamma, C}{H \Rightarrow \Gamma, ?C}\ \textit{der}$$

$$\frac{H \Rightarrow \Gamma}{H \Rightarrow \Gamma, ?C}\ \textit{weak} \qquad \frac{H \Rightarrow \Gamma, ?C, ?C,}{H \Rightarrow \Gamma, ?C}\ \textit{contr}$$

Logical rules:

unjustifiability:

$$\bot \Rightarrow \Delta$$

$$\frac{H \Rightarrow \Gamma, C \qquad D \Rightarrow \Delta}{H \Rightarrow \Gamma, C \smallsetminus D, \Delta}\ \smallsetminus \mathsf{R} \qquad \frac{C \Rightarrow D, \Delta}{C \smallsetminus D \Rightarrow \Delta}\ \smallsetminus \mathsf{L}$$

$$\frac{H \Rightarrow \Gamma, C_0, C_1}{H \Rightarrow C_0 \wp C_1}\ \wp\ \mathsf{R} \qquad \frac{C_0 \Rightarrow \Gamma \qquad C_1 \Rightarrow \Delta}{C_0\ \wp\ C_1 \Rightarrow \Gamma, \Delta}\ \wp\ \mathsf{L}$$

In a sequent-style natural deduction system in place of *left* rules we have *elimination* rules of the form

$$\frac{H \Rightarrow \Gamma, C \smallsetminus D \qquad C \Rightarrow D, \Delta}{H \Rightarrow \Gamma, \Delta}\ \smallsetminus\ E$$

$$\frac{E \Rightarrow \Gamma, ?C \qquad C \Rightarrow ?\Delta}{E \Rightarrow \Gamma, ?\Delta}\ \textbf{?}\ \mathsf{E}$$

2.7. Natural deduction (sequent-style).

Read $E \vdash C_1, \ldots, C_n$ as
- for all $i \leq n$, C_i is compatible with E,
- witness a *"thread of evidence"* $E \mapsto C_i$.

"Thread of evidence": informal notion, related to DR-graphs in a proof net, Sam Buss' logical flow graph, with adjustments for *weakening*.

Rules for subtraction:

$$\smallsetminus\textit{-intro}\,\frac{H \vdash \Gamma, C \quad D \vdash \Theta}{H \vdash \Gamma, C \smallsetminus D, \Theta}$$ **"connect threads"**

"set aside " $$\smallsetminus\textit{-elim}\,\frac{H \vdash \Delta, C \smallsetminus D \quad C \vdash D, \Upsilon}{H \vdash \blacktriangle, \Delta, \Upsilon}$$

"Set aside": evidence threads $C \mapsto D$ are incompatible with threads $H \mapsto C \smallsetminus D$. *Store all of them away* (in some location $\blacktriangle$)!

2.7.1. Inversion principle for subtraction.

In a derivation of the form

$$\frac{\displaystyle \frac{H \vdash \Gamma, \mathbf{C} \quad \mathbf{D} \vdash \Theta}{H \vdash \Gamma, \Theta, \mathbf{C} \setminus \mathbf{D}}\ \setminus\text{-}intro \qquad \mathbf{C} \vdash \mathbf{D}, \Upsilon}{H \vdash \Gamma, \Theta, \blacktriangle, \Upsilon}\ \setminus\text{-}elim$$

The formula $\mathbf{C} \setminus \mathbf{D}$ is *maximal* (*a cut*).

Can remove the pair *intro/elim*:

$$\frac{\displaystyle \frac{H \vdash \Gamma, \mathbf{C} \quad \mathbf{C} \vdash \mathbf{D}, \Upsilon}{H \vdash \Gamma, D, \Upsilon}\ subst. \qquad \mathbf{D} \vdash \Theta}{H \vdash \Gamma, \Theta, \Upsilon}\ subst. \cdot$$

Here we use the *"stored away threads* $C \mapsto D$.

Substitution also "connects threads".

2.8. Term assignment to subtraction.

• a set $\{x_1, \ldots, x_i \ldots\}$ of free variables, exactly one for each sequent;
• a set $\{\mathtt{x}_1, \ldots, \mathtt{x}_i \ldots\}$ of unary functions; - $\mathtt{x}(M)$ means *"variable x is bound, depending on M"*;
• $\mathtt{mkc}(M, \mathtt{y})$: *"from M* `make a coroutine` *starting with* y (y becomes bound, rewritten $\mathtt{y}(M)$ everywhere);
- (*threads reaching M are extended to threads from y*);
- the term $\mathtt{postp}(y \mapsto N, M)$ stores the threads $y \mapsto N$ and is set aside in an *untyped location* (and y becomes bound, rewritten as $\mathtt{y}(M)$ everywhere).
• κ, ζ are sequences of terms.

subtraction introduction

$$\frac{x : D \triangleright \kappa : \Gamma, M : A \qquad y : B \triangleright \zeta : \Delta}{x : D \triangleright \kappa : \Gamma, \zeta[y := \mathtt{y}(M)] : \Delta, \mathtt{mkc}(M, \mathtt{y}) : A \setminus B} \setminus \mathrm{I}$$

subtraction elimination

$$\frac{x : D \triangleright \overline{M} : \Gamma, M : A \setminus B \qquad y : A \triangleright \overline{N} : \Delta, N : B}{x : D \triangleright \overline{M} : \Gamma, \overline{N}[y := Y(M)], \mathtt{postp}(y \mapsto N, M)} \setminus \mathrm{E}$$

• There are β and η equations formalizing the normalization procedure.

• A *dual* calculus to the λ-calculus.

2.9. A categorical model of linear co-IL.

Definition. *A* left-closed symmetric monoidal category *(SMC)* $(\mathbb{C}, \bullet, 1, \alpha, \lambda, \rho, \gamma)$*, is a category* $\mathbb{C}$ *equipped with*

- *a bifunctor* $\bullet : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *with a neutral element* 1*,*
- *natural isomorphisms* α, λ, ρ *and* γ *(satisfying the usual diagrams for associativity, left and right identity and commutativity)*
- and where $\bullet$ has a **left adjoint** $\setminus$ (subtraction).

Theorem 1. *Left-closed symmetric monoidal categories model* multiplicative co-ILL.

To prove it, define *typed terms in context* of the form $x : E \lhd \kappa : \Gamma$, where κ is a list of terms, for the logical rules and a suitable set $\mathcal{A}$ of *equations in context* for them and showing $\mathcal{A}$ is satisfied in any model over $\mathcal{C}$.

- Next define the *syntactic category* as the category $\mathcal{C}$ which has the formulas of **multiplicative co-ILL** as objects and typed terms as morphisms and set

$x : E \rhd \kappa : \Gamma \;\; = \;\; y : E \rhd \zeta : \Gamma$ iff $\kappa = \zeta[y := x]$ is derivable from the equations in context $\mathcal{A}$. It follows

Theorem 2 *The syntactic category is a symmetric monoidal left-closed category.*

The categorical completeness theorem follows.

2.9.1. Categorical model of co-ILL (cont.)

Dualize Benton, Bierman, Hyland and De Paiva 1993 to get the extra structure to model Girard's *whynot?*.

Definition. *A* dual linear category $\mathbb{C}$ *consists of*
- *A symmetric monoidal left-closed category with*
- *a symmetric co-monoidal monad* $(?, \eta, \mu, n_{-,-}, n_{\perp})$ *such that*

(i) - *each free ?-algebra* $(?A, \mu_A)$ *carries naturally the structure of a commutative* $\wp$*-monoid;*
(ii) - *whenever* $f : (?A, \mu_A) \to (?B, \mu_B)$ *is a morphism of free algebras, then it is also a monoid morphism.*

Note: The term assigned to the rules of *storage* literally 'store' the terms $\overline{N}$ in a separate area; terms for *dereliction* and *contraction* build *lists of terms*.

$$\frac{v : E \rhd \kappa : \Gamma, M : ?C \qquad x : C \rhd \overline{Q} \mid \overline{N} : ?\Delta}{v : E \rhd \kappa : \Gamma, \overline{Q}[x := \mathtt{x}(M)], \mathtt{store}(\overline{N}, \overline{\mathrm{y}}, \mathtt{x}, M) \mid \overline{\mathrm{y}}(\mathtt{x}(M)) : ?\Delta}$$

dereliction

$$\frac{x : E \rhd \kappa : \Gamma, M : C}{x : E \rhd \kappa : \Gamma, [\, M \,] : ?C}$$

weakening

$$\frac{x : E \rhd \kappa : \Gamma}{x : E \rhd \kappa : \Gamma, \mathtt{connect\ to}(R) : ?C}$$

where $R \in \kappa$.

contraction

$$\frac{x : E \vdash \kappa : \Gamma, M : ?C, N : ?C}{x : E \vdash \kappa : \Gamma, [\, M, N \,] : ?C}$$

3.1. Semantics + Pragmatics of p-bi-IL

Classically, *propositions* are **true** or **false** (Frege).

Claim: Intuitionistically, sentences are **types of illocutionary acts**.

• *Illocutionary acts* are events that can be **justified** or **unjustified**, i.e., have a **justification value**.
- Also in a given social context they are *felicitous* or *infelicitous* and have *perlocutionary effects* (Austin).
Examples: making *assertions, hypotheses, questions, answers, commands, promises*, etc.
• Illocutionary acts must have a *propositional content*. But the *propositional content* of an assertion A does not suffice to determine the meaning and the *justification value* of A.
• Illocutionary acts can be *impersonal*, e.g., the statement of a theorem can be seen as an impersonal assertion, and a statute or law as an impersonal obligation.

3.2. Logic for pragmatics.

Formalizing *types* of illocutionary acts:

• **Elementary assertions:** $\vdash p$

- Dalla Pozza and Garola 1995.

• **Elementary hypotheses:** $\mathcal{H}p$.

- Bellin 2004, 2012, 2013?, B.et al 2012a?, 2012b, 2013?.

- Here '$\vdash$', '$\mathcal{H}$' are signs of *illocutionary force*

- p is the *propositional content.*

Question: *Under which conditions are such acts intuitionistically meaningful?*

Further 'illocutionary act candidates'':

• **Elementary conjecture:** $\mathcal{C}p$

- i.e., the hypothesis that in some circumstances it may be assertable that p.

• **Elementary expectation:** $\mathcal{E}p$

- i.e., the assertion that in all circumstances it may be possible to make the hypothesis that p.

- *Need to investigate these judgements and their intuitionistic status.*

3.3. 'Polarized' bi-intuitionism.

Language $\mathcal{L}^{AHEC}$ of polarized **bi-IL** (p**bi-IL**):

$$(\textbf{As})\ A, B := \quad \vdash p \mid \mathcal{E}p \mid \top \mid A \supset B \mid A \cap B \mid A \cup B \mid \neg X$$

$$(\textbf{Hy})\ C, D := \quad \mathcal{H}p \mid \mathcal{C}p \mid \bot \mid C \setminus D \mid C \curlyvee D \mid C \curlywedge D \mid \approx X$$

$$X := \quad A \mid C \mid$$

with $\neg X =_{df} X \supset \bot$: *certainly not* X

and $\approx X =_{df} \top \setminus X$: *perhaps not* X.

As = the type of *assertive expressions*.

- $\vdash p$: *it is assertable that* p;
- $\mathcal{E}p$: *it is to be expected that* p.

Hy = the type of *hypothetical expressions*.

- $\mathcal{H}p$: *the hypothesis that* p *can be made*;
- $\mathcal{C}p$: *the conjecture that* p *can be made*.

Two negations (intuitionistic and co-intuitionistic):

$\neg$: $\mathbf{As} \to \mathbf{As}$, $\approx$: $\mathbf{Hy} \to \mathbf{Hy}$.

Dualities:

$\neg$: $\mathbf{Hy} \to \mathbf{As}$, $\approx$: $\mathbf{As} \to \mathbf{Hy}$,

with the axiom

$(\star)$ $\neg\approx A \equiv A$ and $\approx\neg C \equiv C$.

Note. In Bellin 2004, 2012, 2013?, B.et al 2012a?, 2012b, 2013? we used
'$\sim$' instead of '⫬' (*strong negation*) and
'$\frown$' instead of '≈' (*strong negation*),
confusing notation in discussing bi-Heyting algebras.

3.4. Dummett's justificationism.

Can the language $\mathcal{L}^{AHEC}$ represent intuitionistic reasoning *in an intuitionistic metatheory*?

Dummett: Intuitionism is the **logic of assertions** and of their *justifications*.

• Some assertions about the past, the future, Laplace's determinism, some applications of the classical continuum to physics, etc. are in principle unjustifiable.
- In this case Dummett holds that not only these *assertions* are unjustified, but also their *propositional content* ought to be regarded as **meaningless**.

• Dummett refuses to apply a *correspondence theory of truth* to abstract mathematical constructions.

• He gives a different ontological status to *objects of perception* and to thoughts (*Thought and Reality*).
- The justification of an empirical sentence relies on interaction with nature.
- The justification of a mathematical statement depends on a mental construction.

Claim: *If* p *is intuitionistically meaningful, so is* $\vdash p$.

Note: See e.g.,
- M. Dummett 1991 *The Logical Basis of Metaphysics* Harward University Press, 1991.
- M.Dummett 2006 *Thought and Reality* Oxford UP, 2006.

3.4.1. Prawitz: proofs and justifications.

(Digression from personal notes, CLMPS Nancy, 2011.)
The conceptual problem: how and why a proof succeeds in giving knowledge.

• A proof justifies the last assertion by giving **conclusive grounds** for that assertion.

• Why an inference succeeds in justifying the conclusion given the justification of the premisses?

- *Inference acts* operate on grounds for the premises.

• What constitutes a *justification of an assertion*?

- *Direct, canonical* means to justify an assertion
(e.g., by an introduction rule in Natural Deduction);
- *Indirect, non-canonical* means
(e.g., by an elimination rule in Natural Deduction);
- *Indirect* means must be reduced to canonical ones.
(*principle of harmony* between intro and elim rules).

• **Prawitz:** To know the **meaning** of a sentence A is to know what forms a canonical ground for A has and what conditions the parts of A satisfy.

Note. The grounds of composite sentences ultimately depend on the grounds for *elementary expressions*, which vary according to the **illocutionary force** (*elementary assertions* versus *elementary hypotheses*).

3.5. Is co-IL strongly paraconsistent?

Add *hypothetical conjunction* $\curlywedge$, with sequent rules

$$\frac{H \Rightarrow \Delta, C_0 \qquad H \Rightarrow \Delta, C_1}{H \;\Rightarrow\; \Delta, C_0 \curlywedge C_1} \curlywedge \text{R} \qquad\qquad \frac{C_i \Rightarrow \Gamma}{C_0 \curlywedge C_1 \Rightarrow \Gamma} \curlywedge_i \text{L}$$

for $i = 0$ or 1

Question: (R. Ertola) *Is* **co-IL** strongly paraconsistent *in the sense that there is a class of formulas* Γ *such that from* $C \curlywedge \sim C$ *we cannot derive some formulas in* Γ*?*

Possible solution. Define *co-Harrop formulas* thus:

$$\begin{array}{rrl|l|l|l|l|l|} (\textbf{Hy}) & C, D := & \mathcal{H}p & \bot & C \smallsetminus D & C \curlyvee D & \approx C & C \curlywedge D \\ (\textbf{Har}) & H, K := & \mathcal{H}p & \bot & H \smallsetminus D & H \curlyvee K & \approx C & \end{array}$$

- **Co-Harrop** formulas have the *conjunction property*:

- *if* $\Gamma \subset \mathbf{Har}$ *then* $H \curlywedge K \vdash \Gamma$ *implies* $H \vdash \Gamma$ *or* $K \vdash \Gamma$.

Proof: From the *disjunction property* for intuitionistic Harrop formulas, by duality.

- Is **co-IL** with conjunction $\curlyvee$ strongly paraconsistent w.r.t. co-Harrop formulas?

3.6. What is co-IL about?

Shramko 2005: **co-IL** is about *sentences that have not yet been refuted*.
It is the logic of scientific research according to *Popper's refutationism*.

Y. Shramko. Dual Intuitionistic Logic and a Variety of Negations: The Logic of Scientific Research, *Studia Logica* 80, 2005, pp. 347-367.

$$\dfrac{\dfrac{E \vdash C_1, \ldots C_{n-1}, C_n \quad C_n \vdash}{E \vdash C_1, \ldots C_{n-1}}\,cut \quad C_{n-1} \vdash}{\vdots \atop E \vdash C_1}$$

• Intuitionistic logic is *expansive*: the more you search, the more theorems you find.

• Co-Intuitionistic logic is *recessive*: the more you search for refutations, the less laws you are left with. [cfr. the classes Σ^0_1 and Π^0_1 (Girard *The Blind Spot*).]

• *Is* **co-IL** *only a logic of refutations?*

• Better: it a logic of *what is compatible with* the sentences that have not yet been refuted.

- We look for *positive grounds* for inferring unrefuted statements.

3.7. Extending the BHK interpretation.

For *assertive types* follow the **Brouwer-Heyting-Kolmogorov-[Kreisel]** interpretation:

- $\vdash p$ is justified by *conclusive evidence* that p is true;
- $\top$ is always justified and $\bot$ is never justified;
- $A \supset B$ is justified by a *method* transforming a justification of A into a justification of B
- $A \cap B$ is justified by evidence for A together with evidence for B
- $A \cup B$ is justified by evidence for A or by evidence for B.

Claim: *If elementary formulas are intuitionistically meaningful, so are all* assertive types.

But how to extend the **BHK** interpretation to hypothetical types?
From legal argomentation theory, borrow the notion of **scintilla of evidence** [Gordon and Walton 2009].

- $\mathcal{H}p$ is justified by a *scintilla of evidence* that p is true;
- $C \smallsetminus D$ is justified by a *scintilla of evidence* that there is a justification of C and no justification of D; etc.

NO: start with **co-ILL** where $\curlyvee$ *is replaced by* par !

3.8. A game-like semantics for co-ILL.

Define simultaneously *evidence pro* and *con*.

elementary:

evidence pro $\mathcal{H}p$: a *scintilla of evidence* that p is true;

evidence con $\mathcal{H}p$: *conclusive evidence* that p is false;

subtraction:

evidence pro $C \smallsetminus D$: a *scintilla of evidence* that there is *evidence con* C and *evidence con* D;

evidence con $C \smallsetminus D$: a method transforming *evidence pro* C into *evidence pro* D and *evidence con* D into *evidence con* C;

disjunction:

evidence pro $C \wp D$: a method transforming *evidence con* C into *evidence pro* D and *evidence con* D into *evidence pro* C;

evidence con $C \wp D$: *evidence con* C together with *evidence con* D.

[From the game-semantics for linear logic and Nelson 1949.]

Claim: *The game interpretation of* **co-ILL** *is intuitionistically meaningful. Try to extend this to* **p-bi-IL**.

4.1. 'Polarized' bi-Heyting Algebras.

- A bi-Heyting algebra $\mathcal{C} = (C, \wedge, \vee, \to, -, \top, \bot,)$ is **polarized** if it has substructures A and H such that

- A is the sub-*Heyting* algebra of $\mathcal{C}$ generated by $\{a_1, \ldots\}$;

- H is sub-*co-Heyting* algebra of $\mathcal{C}$ generated by $\{c_1, \ldots\}$;

- there is a bijection p of generators $a_i \mapsto c_i$ with $a_i \leq c_i$;

- the negations of $\mathcal{C}$ yield a *duality*, namely,

(1) $\sim (a \wedge b) = \sim a \vee \sim b$, $\quad \neg(c \vee d) = \neg c \wedge \neg d$;

(2) $\sim (a \vee b) = \sim a \wedge \sim b$, $\quad \neg(c \wedge d) = \neg c \vee \neg d$;

(3) $\sim (a \to b) = \sim b - \sim a$, $\quad \neg(c - d) = \neg d \to \neg c$

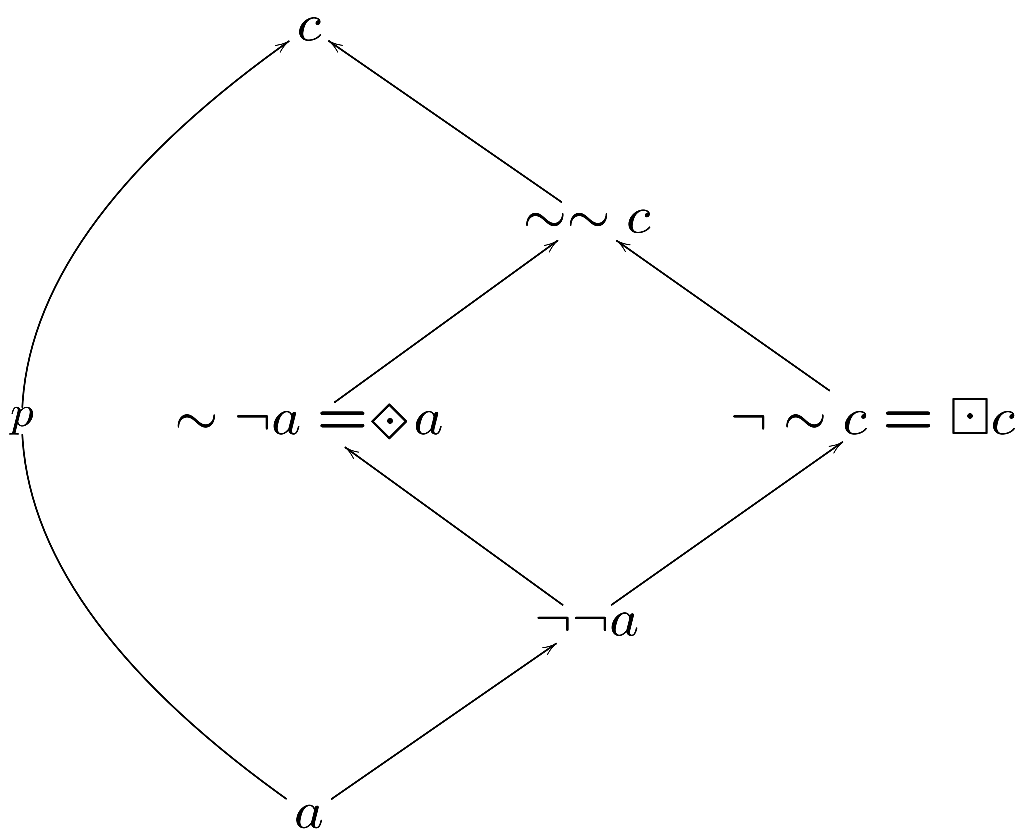
for all $a, b \in A$ and $c, d \in H$, and

($\star$) $\quad \neg \sim a = a \quad$ and $\quad \sim \neg c = c$.

From ($\star$) it follows that

$$\neg \sim c = \boxdot c = \neg \sim \boxdot c \quad \text{and} \quad \sim \neg a = \diamond a = \sim \neg \diamond a.$$

4.1.1. Polarized bi-Heyting algebra (cont.)



• The sets $\mathcal{E}xp = \{\boxdot a_i, \ldots\}$ and $\mathcal{C}onj = \{\odot c_i \ldots\}$ generate Boolean algebras, that *aren't sub-lattices of* $\mathcal{C}$ (Johnstone 1983, prop.1.13)

- $\mathcal{E}xp$ has joins $\boxdot(A \vee B)$ and $\mathcal{C}onj$ has meets $\odot(C \wedge D)$.

4.2. Classical Logic, Intuitionistic Modalities.

Claim 1: In **polarized bi-IL** $\boxdot =_{df} \neg\!\approx$ and $\diamond =_{df} \approx\!\neg$ *are intuitionistic acceptable polarity-changing modalities.*

Let $\mathcal{L}^E$ be the language

$$\begin{array}{rl} \textbf{Exp}\ E, F := & \mathcal{E}p \mid \top \mid E \supset F \mid E \cap F \mid E \cup F \mid \neg E \\ \textbf{Hy-at} := & \mathcal{H}p \mid \approx \mathcal{H}p \text{ with the axioms } \mathcal{E}p \equiv \boxdot\, \mathcal{H}p. \end{array}$$

Let us call the fragment of **polarized bi-IL** on the language $\mathcal{L}^E$ **logic of expectations**.

Claim 2: The *logic of expectations* is an *intuitionistically acceptable intermediate logic* where $\neg\neg E \equiv E$ but the law of excluded middle does not hold.

Fact: A Natural Deduction system for the *logic of expectations* is a typing system for Parigot's $\lambda\mu$-calculus.

4.3. Translation in classical in S4.

Language $\mathcal{L}^{\Box}$ *of* **classical S4**.

$A, B \;:=\; p \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \to B \mid \Box A$

Define $\neg A =_{df} A \to \bot$ and $\Diamond A =_{df} \neg \Box \neg A$.
From now on, '$\neg, \wedge, \vee, \to$*' are reserved for* **classical connectives.**

$$(\top)^M =_{df} \top \qquad (\bot)^M =_{df} \bot$$

$$(\vdash \mathbf{p})^{\mathrm{M}} =_{\mathrm{df}} \Box \mathbf{p} \qquad (\mathcal{H}\mathbf{p})^{\mathrm{M}} =_{\mathrm{df}} \Diamond \mathbf{p}$$

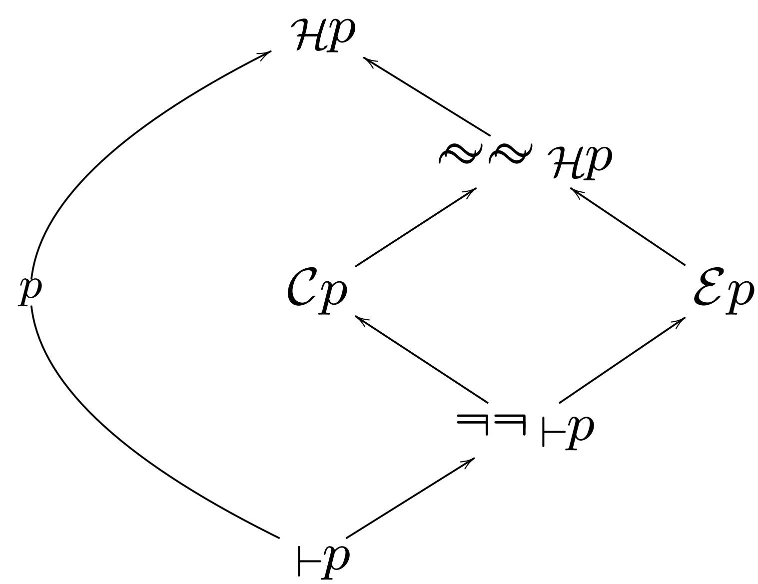
$$(\mathbf{A} \supset \mathbf{B})^{\mathrm{M}} =_{\mathrm{df}} \Box(\mathbf{A}^{\mathrm{M}} \to \mathbf{B}^{\mathrm{M}}) \qquad (\mathbf{C} \smallsetminus \mathbf{D})^{\mathrm{M}} =_{\mathrm{df}} \Diamond(\mathbf{C}^{\mathrm{M}} \wedge \neg \mathbf{D}^{\mathrm{M}})$$

$$(A_1 \cap A_2)^M =_{df} A_1^M \wedge A_2^M \qquad (C_1 \curlyvee C_2)^M =_{df} C_1^M \vee C_2^M$$

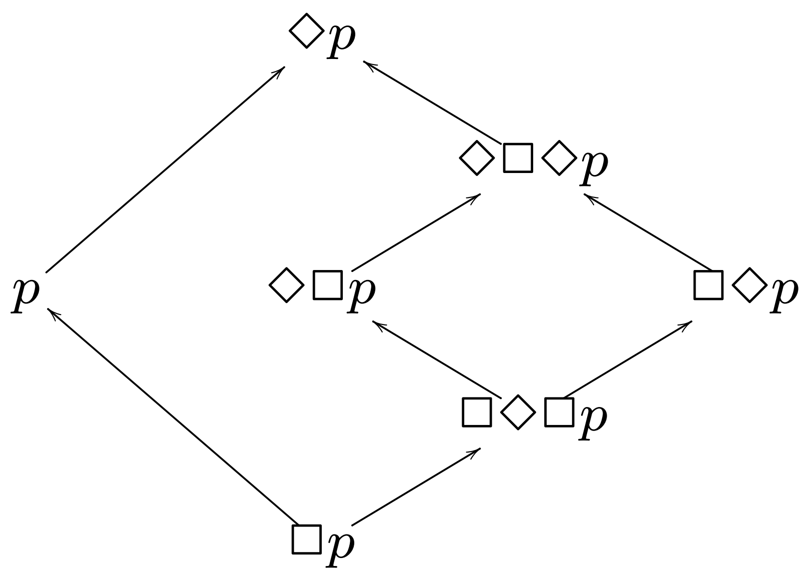
$$(A_1 \cup A_2)^M =_{df} A_1^M \vee A_2^M \qquad (C_1 \curlywedge C_2)^M =_{df} C_1^M \wedge C_2^M$$

$$(\urcorner\, \mathbf{A})^{\mathrm{M}} = \Box \neg \mathbf{A}^{\mathrm{M}} \qquad (\approx \mathbf{C})^{\mathrm{M}} = \Diamond \neg \mathbf{C}^{\mathrm{M}}$$

$$(\urcorner\, C)^M = \neg C^M \qquad (\approx A)^M = \neg A^M$$



The modalities of polarized bi-IL



The modalities of S4

4.4. Features of polarized bi-IL

• **Polarized bi-IL** has models in (ordinary) topological spaces.
- Assertive formulas become *open sets* and
- hypothetical formulas *closed sets*.

• A sequent calculus for **polarized bi-IL** where sequents are of the form

$$\Theta \; ; \; \Rightarrow \; A \; ; \; \Upsilon$$

$$\Theta \; ; \; C \; \Rightarrow \; ; \; \Upsilon$$

• Θ a sequence of *assertive* $A_1, \ldots, A_m$;
• Υ a sequence of *hypothetical* $C_1, \ldots, C_n$.
(see rules in table below).

Theorem. *The sequent calculus for* **p-bi-IL** *is sound and complete for the Kripke semantics induced by the modal translation.*

identity rules

logical axiom:

$$\Theta \;;\; C \;\Rightarrow\; ;\; C, \Upsilon$$

logical axiom:

$$A, \Theta \;;\; \Rightarrow\; A \;;\; \Upsilon$$

cut$_A$:

$$\frac{\Theta \;;\; \Rightarrow\; A \;; \Upsilon \qquad A, \Theta' \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon'}{\Theta, \Theta' \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \Upsilon'}$$

cut$_H$:

$$\frac{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;; \Upsilon, C \qquad \Theta' \;;\; C \;\Rightarrow\; ;\; \Upsilon'}{\Theta, \Theta' \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \Upsilon'}$$

logical rules for implication, subtraction

right $\supset$*:*

$$\frac{\Theta, A \;;\; \Rightarrow\; B \;;\; \Upsilon}{\Theta \;;\; \Rightarrow\; A \supset B \;;\; \Upsilon}$$

left $\smallsetminus$*:*

$$\frac{\Theta \;;\; C \;\Rightarrow\; ;\; \Upsilon, D}{\Theta \;;\; C \smallsetminus D \;\Rightarrow\; ;\; \Upsilon}$$

left $\supset$*:*

$$\frac{A \supset B, \Theta \;;\; \Rightarrow\; A \;;\; \Upsilon \qquad B, \Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon}{A \supset B,\; \Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon}$$

right $\smallsetminus$*:*

$$\frac{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, C \qquad \Theta \;;\; D \;\Rightarrow\; ;\; \Upsilon, C \smallsetminus D}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, C \smallsetminus D}$$

Rules for dualities:

right $\approx$*:*

$$\frac{A, \Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \approx A}$$

left $\approx$*:*

$$\frac{\Theta \;;\; \Rightarrow\; A \;;\; \Upsilon}{\Theta \;;\; \approx A \;\Rightarrow\; ;\; \Upsilon}$$

right $\urcorner$*:*

$$\frac{\Theta \;;\; C \;\Rightarrow\; ;\; \Upsilon}{\Theta \;;\; \Rightarrow\; \urcorner C \;;\; \Upsilon}$$

left $\urcorner$*:*

$$\frac{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, C}{\urcorner C, \Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon}$$

5. Conclusions.

(1) We have reconsidered C. Rauszer's bi-Heyting algebras [1974], and G. Reyes and H.Zolfaghari's treatment of modalities [1996] in them.

(2) We have shown that the usual *tense-epistemic* **KtT4** of **bi-IL** is untenable because of an ambiguous interpretation of atomic sentences.

(3) We have reviewed results by T. Crolard [2001] showing that **bi-IL** has only degenerate topological and categorical models.

(4) T. Crolard's result that even **co-IL** does not have a model in **Set** gave motivations for *linearizing* **co-IL**. We provide a categorical model of **linear co-IL** in *monoidal left-closed categories with extra structure* by dualizing Benton, Bierman, dePaiva and Hyland's 1993 model of **ILL**.

(5) A philosophical analysis of bi-intuitionistic logic as a logic of assertions and hypotheses, extending Dalla Pozza and Garola's *logic for pragmatics* framework [1995] motivates the introduction of **'polarized' bi-IL**, in which topological models are no longer degenerate and the modal translation is again in **S4**.

(6) A ‘rich’ proof-theory for **polarized bi-IL** is now possible by combining the dual categorical models of **IL** (*cartesian closed categories*) and the model of **co-ILL** in *monoidal categories*.

Note. Another promising way to obtain categorical models of **polarized bi-IL** is to modify the categorical construction of *mixed linear and non-linear logic* in [Benton 1995]. We have not done (6) here.

(7) We have extended the BHK interpretation of **IL** to **polarized bi-IL** obtaining a *“game-like semantics”* which we claim to be intuitionistically acceptable.

(8) We have shown that in the framework of **polarized bi-IL** Reyes and Zolfaghari’s modalities become *intuitionistically acceptable polarity-changing modalities* and allow us to define a *logic of expectations* satisfying the **double negation rule**, but not the **law of excluded middle**.

APPENDIX. I.1.Crolard's computational bi-IL

Note. Crolard (2001, 2004) studies *subtractive logic* as an extension of classical logic: rules for subtraction are added to a Gentzen system for classical logic.

- He defines a calculus for *constructive* **bi-IL** by restricting *permissible logical dependencies* in the classical proof-system.
- The analysis of dependencies is reminiscent of Hyland and De Paiva proof-system for **FILL** (*intuitionistic linear logic with par*).
- Crolard's approach is relevant to the analysis of the *call-by-name* and *call-by-value* strategies of computation (Curien 2002).

A.I.2. Computational Interpretations.

The $\lambda\mu$-calculus.

variables:	$x_0, x_1, \ldots$		*names:*	$\alpha_0, \alpha_1, \ldots$
terms:	M, N	$:=$	$x \mid \lambda x.M \mid MN \mid \mu\alpha.Q$	
commands:	Q	$:=$	$[\alpha]M$	(α *abstraction*)

Substitutions:

ordinary: $M[N/x]$ (*capture avoiding*);
renaming: $Q[\alpha/\beta]$;
structural: $T[\alpha \Leftarrow L]$: $[\alpha]N$ replaced by $[\alpha]NL$ in T.

Reductions:

(β)	$(\lambda x.M)N$	$\rightsquigarrow$	$M[N/x]$;
(μ)	$(\mu\beta.Q)N$	$\rightsquigarrow$	$\mu\beta.Q[\beta \Leftarrow N]$;
(*ren*)	$[\alpha]\mu\beta.Q$	$\rightsquigarrow$	$Q[\alpha/\beta]$;
$(\mu\eta)$	$\mu\alpha.[\alpha]M$	$\rightsquigarrow$	M.

Typed $\lambda\mu$-calculus and classical logic.

- *Types:* $A, B := p \mid \bot \mid A \supset B$
- *Sequents:* $\Gamma \vdash t : A \mid \Delta$ where

$\Gamma = x_1 : C_1, \ldots, x_m : C_m$ and $\Delta = \alpha_1 : D_1, \ldots, \alpha_n : D_n$.

To the *Simply Typed λ-calculus* add **naming rules**:

$$\frac{\Gamma \vdash t : A \mid \alpha : A, \Delta}{\Gamma \vdash [\alpha]t : \bot \mid \alpha : A, \Delta}[\alpha] \qquad \frac{\Gamma \vdash t : \bot \mid \alpha : A, \Delta}{\Gamma \vdash \mu\alpha.t : A \mid \Delta}\mu$$

Type system: *classical logic* (of $\supset$) (Parigot 1992).

Categorical models: *control categories* (Selinger 2001).

A.I.3. Crolard's calculus of coroutines.

$$\frac{\Gamma \vdash t : A \mid \Delta}{\Gamma \vdash \texttt{make-coroutine}(t, \beta) : A \setminus B \mid \beta : B, \Delta} \setminus I$$

$$\frac{\Gamma \vdash t : A \setminus B \mid \Delta \qquad \Gamma,\ x : A \vdash u : B \mid \Delta}{\Gamma \vdash \texttt{resume}\ t\ \texttt{with}\ x \mapsto u : C \mid \Delta} \setminus E$$

A *redex* of the form

$$\frac{\dfrac{\Gamma \vdash t : A \mid \Delta}{\Gamma \vdash \texttt{mk-cor}(t, \beta) : A \setminus B \mid \beta : B, \Delta} \setminus\text{-I} \qquad \Gamma, x : A \vdash u : B \mid \Delta}{\Gamma \vdash \texttt{resume}\,(\texttt{mk-cor}(t, \beta))\,\texttt{with}\,x \mapsto u\ : C \mid \beta : B, \Delta} \setminus\text{-E}$$

reduces to

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash t : A \mid \Delta \qquad \Gamma, x : A \vdash u : B \mid \Delta}{\Gamma \vdash u[t/x] : B \mid \Delta} \textit{substitution}}{\Gamma \vdash [\beta]u[t/x] : \bot \mid \beta : B, \gamma : C, \Delta'} [\beta]}{\Gamma \vdash \mu\gamma.[\beta]u[t/x] : C \mid \beta : B, \Delta'} \mu$$

[In the $\setminus$-E there is an implicit *weakening*: the type of `resume` could be $\bot$.]

- Crolard defines a class of *safe coroutines* typable in his system of constructive **bi-IL**.

APPENDIX 2. Bi-IL Rough-sets semantics.

• Nelson 1949, Constructive falsity. To characterize a logic constructively, need to characterize not only *provability* but also *refutability*.
- idea related to *game semantics* (see also Bellin Chu's construction. A proof-theoretic apporach 2003).
- for **bi-IL** need interpretations where the refutations of A do not coincide trivially with proofs of $A^{\perp}$.

A.II.1. Rough equivalence.

Definition. *Indiscernibility space* (U, E), U finite set, E equivalence relation.

$\mathbf{AS}(U) =$ the atomic Boolean algebras having the set of equivalence classes U/E as atoms

• $(U, \mathbf{AS}(U))$ is a topological space (the *Approximation Space* of (U, E));

$\mathcal{I}, \mathcal{C} : \wp(U) \to \mathbf{AS}(U)$ the induced interior operator and a closure operators.

X *is roughly equal to* Y iff $\mathcal{I}(X) = \mathcal{I}(Y)$ and $\mathcal{C}(X) = \mathcal{C}(Y)$.

• Any subset $G \subseteq U$ is a representative of $(\mathcal{I}(G), \mathcal{C}(G))$.

• Use the *disjoint representation*

$$(\mathcal{I}(G), -\mathcal{C}(G))$$

using the *complement of the closure* of G.

A.II.2. Pagliani's bi-IL semantics.

Pagliani 2009:

[1] $1 = (U, \emptyset)$, $0 = (\emptyset, U)$;

[2] $(X^+, X^-) \wedge (Y^+, Y^-) = (X^+ \cap Y^+, X^- \cup Y^-)$ (*conjunction*);

[3] $(X^+, X^-) \vee (Y^+, Y^-) = (X^+ \cup Y^+, X^- \cap Y^-)$ (*disjunction*);

[4] $(X^+, X^-) \rightarrow (Y^+, Y^-) = (-X^+ \cup Y^+, X^+ \cap Y^-)$ (*Nelson's implication*)

[5] $\llcorner\!\!-\ (X^+, X^-) = (-X^+, X^+)$ (*weak negation* or *supplement*);

[6] $(X^+, X^-)^\perp = (X^-, X^+)$ (*orthogonality*);

[7] $(X^+, X^-) \Rightarrow (Y^+, Y^-) = ((-X^+ \cup Y^+) \cap (-Y^- \cup X^-), -X^- \cap Y^-)$ (*Heyting's implication*);

[8] $\neg\ (X^+, X^-) = (X^+, X^-) \Rightarrow (\emptyset, U) = (X^-, -X^-)$ (*intuitionistic negation*);

[9] $(X^+, X^-) \smallsetminus (Y^+, Y^-) = (X^+ \cap -Y^+, (-X^+ \cup Y^+) \cap (-Y^- \cup X^-)$ (*co-intuitionistic subtraction*).

A.II.3. Problem: completeness + polarization.

Problem 1. Need to start with *infinite sets* to obtain a complete semantics for intuitionistic logic.

Problem 2. To represent *polarized* **bi-IL** need to keep the representations of **IL** and **co-IL** separate:
idea: represent *assertive* A as (A_o^+, A_c^-), A_o^+ *open*, A_c^- *closed* and
hypothetical C as (C_c^+, C_o^-), C_c^+ *closed*, C_o^- *open*.

A.II.4. Desiderata.

[1] $\curlyvee^R = (U, \emptyset)$ and $\curlywedge^M = (\emptyset, U)$ (*clopen, clopen*);

[2] $(A \cap B)^R = (A_o^+, A_c^-) \wedge (B_o^+, B_c^-) = (A_o^+ \cap B_o^+, A_c^- \cup B_c^-)$;

[3] $(C \curlyvee D)^R = (C_c^+, C_o^-) \vee (D_c^+, D_O^-) = (C_c^+ \cup D_c^+, C_o^- \cap D_o^-)$;

[4] $(A_o^+, A_c^-) \to (B_o^+, B_c^-) = (\mathcal{I}(-A_o^+ \cup B_o^+), \mathcal{C}(A_o^+ \cap B_c^-))$

[5] $(\approx C)^R = \llcorner(C_c^+, C_o^-) = (\mathcal{C}(-C_c^+), \mathcal{I}(C_c^+))$ and $(\approx A)^R = \llcorner(A_o^+, A_c^-) = (-A_o^+, A_o^+)$;

[6] $(A_o^+, A_c^-)^\perp = (A_c^-, A_o^+)$ and $(C_c^+, C_o^-)^\perp = (C_o^-, C_c^+)^*$;

[7] $(A \supset B)^R = (A_o^+, A_c^-) \Rightarrow (B_o^+, B_c^-) =$
$= (\mathcal{I}(-A_o^+ \cup B_o^+) \cap \mathcal{I}(-B_c^- \cup A_c^-), \mathcal{C}(-A_c^- \cap B_c^-))$;

[8] $(\neg A)^R = \neg(A_o^+, A_c^-) = (\mathcal{I}(A_c^-), \mathcal{C}(-A_c^-))$ and $(\neg C)^R = \neg(C_c^+, C_o^-) = (C_o^-, -C_o^-)$;

[9] $(C \setminus D)^R = (C_c^+, C_o^-) \setminus (D_c^+, D_c^-) =$
$= (\mathcal{C}(C_c^+ \cap -D_c^+), \mathcal{I}(-C_c^+ \cup D_c^-) \cap \mathcal{I}(-D_o^- \cup C_o^-))$.

*There is no specific connective for orthogonality in $\mathcal{L}^{AH}$.

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