Homework.

Matematical Logic - Gianluigi Bellin

October 25, 2013

1(i). **Write derivations in the calculus of sequents** of the following expressions, by applying the procedure *semantic tableaux*

1. $A \to (B \to C), B \Rightarrow A \to C$;
2. $A \to B, B \to C \Rightarrow A \to C$;
3. $(A \to B) \to A \Rightarrow A$.

Consider a **Hilbert system** with axioms

Ax 1 $A \to (B \to A)$;

Ax 2 $(A \to (B \to C)) \to ((A \to B) \to (A \to C)$;

Ax 3 $\neg\neg A \to A$

and with Modus Ponens as only rule of inference:

$$\frac{A \to B \qquad A}{B}\ \mathbf{MP} \quad .$$

Find derivations of

H1 $A \to (B \to C), B \vdash A \to C$;

H2 $A \to B, B \to C \vdash A \to C$

H3 $(A \to B) \to A \vdash A$

by applying the procedure in the proof of equivalence between Gentzen's systems and Hilbert systems.
Hints: In (H1) build a derivation $A \to (B \to C), B, A \vdash C$ and then apply the Deduction Theorem. You do not need to use Axiom 3. In (H3) you do need to use Axiom 3.

Remark. A *Natural Deduction* calculus is proof-system without axioms, with *introduction* and *elimination* inference rules for each connective and with rules that specify which sets of assumptions are *open* and which are *discharged* at each stage of the derivation. Proofs have the form of *trees with annotations* concerning the discharge of assumptions. Here *proof trees* are either an *open assumption* A or result from derivations d, d_1 and d_2 by the inferences

$$\dfrac{\begin{matrix}(1)\\ [A]\\ d\\ B\end{matrix}}{A \to B}\ (1)\ \to\text{-intro}$$

$[A]$ is a set of leaves labelled with A.

$$\dfrac{\begin{matrix}d_1 & & d_2\\ A \to B & & A\end{matrix}}{B} \to \text{elim} \qquad\qquad \dfrac{\begin{matrix}\text{d}\\ \neg\neg A\end{matrix}}{A.}\ \neg\neg\ \text{rule}$$

with the following *deduction rules*:
(1) if $\Gamma, A \vdash B$ then $\Gamma \vdash A \to B$;
(2) if $\Gamma \vdash A \to B$ and $\Delta \vdash A$ then $\Gamma, Delta \vdash B$.
(3) if $\Gamma \vdash \neg\neg A$ then $\Gamma \vdash A$.

Then one can prove in such a system the axioms for implication and negation in a Hilbert-style system and also give more perspicuous derivations of (H1)-(H3).