Some dual characteristics of convex sets in a Hilbert space and their applications to a time optimal control problem with a constant dynamics

Vladimir V. Goncharov, Fátima F. Pereira

Departamento de Matemática, Universidade de Évora

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Plan

Part I Dual properties of convex sets

- Rotundity and Smoothness in general topological setting
- Moduli of Rotundity associated to a convex solid
- Moduli of Smoothness. Lindenstrauss' duality
- Curvatures. Connection with the second order derivative

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Part II Minimum Time problem

- Equivalent formulations
- History of the question
- Local Well-posedness conditions
- Regularity results

Part I. Basic notions

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It is well-known that under the standing assumptions above $(F^0)^0 = F$ and $\sigma_{F^0}(\xi) = \rho_F(\xi)$, $\xi \in H$.

F is rotund iff F^0 is smooth

Topological setting

Let us assume that X be a Banach space and X^* be its topologically dual with the respective norms denoted by $\|\cdot\|$ and $\|\cdot\|_*$. Let $\langle\cdot,\cdot\rangle$ be the bilinear form stabilizing the duality between X and X^*. For the simplicity assume that X is reflexive, i.e., $X = X^{**}$. As well-known

$$\|\xi\|_* = \sup_{\xi \neq 0} \frac{\langle \xi, \xi^* \rangle}{\|\xi\|},$$

and, clearly

If F is the closed unit ball in X then the respective ball in X^* is F^0

So, we have various dual properties of the spaces through the respective properties of unit balls. In particular, X is (uniformly) rotund iff X^* is (uniformly) smooth.

Topological setting (comments)

During 20-th century the dual property above surely was refined in various directions: the concepts of local, weak etc. rotundity and smoothness are appeared.

The classic Lindenstrauss formula connects the modulus of the (uniform) rotundity of the unit ball in X with the modulus of the (uniform, or Fréchet) smoothness of the unit ball in the dual space X^*.

Our outlook

We consider instead an arbitrary convex solid with $0 \in \operatorname{int} F$ in a Hilbert space, which is not symmetric ($-F \neq F$, in general).
We should acchieve three main goals:

1. Introduce the quantitative characteristics, which would describe the rotundity and the smoothness of an arbitrary set satisfying the standing assumptions (not necessarily symmetric one)
2. Find a formula, which would establish quantitatively the duality between the rotundity and the smoothness and which would be not uniform but localized at a fixed point of the boundary
3. Graduate (quantitatively) various degrees of the rotundity (of the smoothness) and obtain duality formulas connecting them

Duality maping

In order to achieve the **second** goal (localization) let us introduce the notion of duality mapping $\mathfrak{J}_F : \partial F^0 \to \partial F$,

$$\mathfrak{J}_F(\xi^*) := \{\xi \in \partial F : \langle \xi, \xi^* \rangle = 1\}$$

Let us observe the other characterizations of this mapping:

$$\mathfrak{J}_F(\xi^*) = \partial \rho_{F^0}(\xi^*)$$

(subdifferential of the convex function)

$$\mathfrak{J}_{F^0}(\xi^*) = \mathbf{N}_F(\xi) \cap \partial F^0$$

($\mathbf{N}_F(\xi)$ stands for the normal cone to the convex set)

$$\mathfrak{J}_{F^0}(\xi^*) = \mathfrak{J}_F^{-1}(\xi^*)$$

Duality maping (comments)

The lack of smoothness of F^0 at a point $\xi^* \in \partial F^0$ means the multiplicity of the normal vectors at this point (geometrically), or, in other words, the multiplicity of $\mathfrak{J}_F(\xi^*)$. But this is the same to ask that at each $\eta \in \mathfrak{J}_F(\xi^*)$ (various points from the boundary ∂F) the normal vectors to F coincide (with ξ^*), i.e., at each η the set F is not rotund.

This dual property is in fact reflected in the formulas above.

Moduli of rotundity

Assume that $\xi^* \in \partial F^0$ and $\xi \in \mathfrak{J}_F(\xi^*)$ (we say that (ξ, ξ^*) is a dual pair). For $r > 0$ let us define

$$\mathfrak{C}_F^+(r, \xi, \xi^*) := \inf \{ \langle \xi - \eta, \xi^* \rangle : \eta \in F, \ \rho_F(\eta - \xi) \geq r \}$$

$$\mathfrak{C}_F^-(r, \xi, \xi^*) := \inf \{ \langle \xi - \eta, \xi^* \rangle : \eta \in F, \ \rho_F(\xi - \eta) \geq r \}$$

$$\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) := \inf \{ \langle \xi - \eta, \xi^* \rangle : \eta \in F, \ \|\xi - \eta\| \geq r \}$$

These three moduli can be different, in general, taking into account the asymmetry of F.

Definition

F is said to be rotund (locally) at the **point** ξ w.r.t. the **direction** ξ^* if $\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) > 0$ for all $r > 0$.

Moduli of rotundity (comments)

Observe that the rotundity of F depends only on the direction ξ^* but not on the point ξ on the boundary, because in this case the set $\mathfrak{J}_F(\xi^*)$ is a singleton (which we denote further simply by ξ), and we may speak about the rotundity w.r.t. ξ^* (do not specifying a point where this property is verified).

On the other hand, if for some $\xi \in \partial F$ the set $\mathfrak{J}_{F^0}(\xi)$ is not a singleton then F can be rotund w.r.t. some $\eta^* \in \mathfrak{J}_{F^0}(\xi)$ and not rotund w.r.t. others.

Modulus of smoothness

We introduce the modulus of smoothness for the polar set F^0 taking into account that each set F satisfying the standing assumptions is the polar of F^0. Let t be any real parameter. Then we define

$$\mathfrak{S}_{F^0}(t, \xi^*, \xi) := \sup \left\{ \rho_{F^0}(\xi^* + t\eta^*) - \rho_{F^0}(\xi^*) - t\langle \xi, \eta^* \rangle : \ \eta^* \in F^0 \right\}$$

Definition

F^0 is said to be smooth (locally) at the **point** ξ^* w.r.t. the **direction** ξ if

$$\lim_{t \to 0+} \frac{\mathfrak{S}_{F^0}(t, \xi^*, \xi)}{t} = 0$$

From this definition we see that the local smoothness of F^0 is connected with the (Fréchet) differentiability of the gauge function $\rho_{F^0}(\cdot)$.

Lindenstrauss duality formulas

For each $t > 0$ we have

$$\mathfrak{S}_{F^0}(\pm t, \xi^*, \xi) = \sup\left\{ tr - \mathfrak{C}_F^{\pm}(r, \xi, \xi^*) : \ r > 0 \right\}$$

By employing the notion of the Legendre-Fenchel transform we can write

$$\mathfrak{S}_{F^0}(\cdot, \xi^*, \xi) = \mathfrak{C}_F^{\bigstar}(\cdot, \xi, \xi^*),$$

where

$$\mathfrak{C}_F(r, \xi, \xi^*) := \begin{cases} \mathfrak{C}_F^{+}(r, \xi, \xi^*) & \text{if} \quad r > 0; \\ 0 & \text{if} \quad r = 0; \\ \mathfrak{C}_F^{-}(-r, \xi, \xi^*) & \text{if} \quad r < 0 \end{cases}$$

Lindenstrauss duality formulas (comments)

Let us remind the well-known definition of the Legendre-Fenchel transform of a lower semicontinuous function $f : H \to \mathbb{R}$ (H is a Hilbert space):

$$f^{\bigstar}(x) := \sup_{y \in H} \{\langle x, y \rangle - f(y)\}$$

Definition

The set F is said to be rotund of the order $\alpha > 0$ at the point ξ w.r.t. ξ^* if

$$\hat{\gamma}_{F,\alpha}(\xi,\xi^*) := \liminf_{\substack{(r,\eta,\eta^*)\to(0+,\xi,\xi^*) \\ x\eta\in\mathfrak{J}(\eta^*),\, \eta^*\in\partial F^0}} \frac{\widehat{\mathfrak{C}}_F(r,\eta,\eta^*)}{r^\alpha} > 0$$

The numbers $\gamma^{\pm}_{F,\alpha}(\xi,\xi^*)$ are defined similarly as $\hat{\gamma}_{F,\alpha}$ but with $\mathfrak{C}^{\pm}_F$, respectively, in the place of $\widehat{\mathfrak{C}}_F$.

Definition

The number

$$\hat{\varkappa}_F(\xi,\xi^*) := \frac{1}{\|\xi^*\|}\hat{\gamma}_{F,2}(\xi,\xi^*)$$

is said to be (square) curvature of F at ξ w.r.t. ξ^*

Curvature radius

The geometric sense of the curvature can be seen through the so called curvature radius

$$\widehat{\mathfrak{R}}_F(\xi, \xi^*) := \frac{1}{2\hat{\gamma}_{F,2}(\xi, \xi^*)},$$

or, in another form,

$$\widehat{\mathfrak{R}}_F(\xi, \xi^*) = \limsup_{\substack{(\varepsilon,\eta,\eta^*)\to(0+,\xi,\xi^*) \\ \eta\in\mathfrak{J}(\eta^*),\, \eta^*\in\partial F^0}} \inf\left\{r > 0 : F \cap \left(\eta + \varepsilon\overline{B}\right) \subset \subset \eta - r\eta^* + r\|\eta^*\|\,\overline{B}\right\}$$

Roughly speaking, the curvature radius means the radius of the "extreme" sphere centred on the half-line opposite to the direction ξ^*, which touches the boundary of F only at ξ, and each ball of the larger radius contains a part of F near this point.

Invariantness and semicontinuity properties

- $\hat{\varkappa}_F(\xi,\xi^*)$ and $\widehat{\mathfrak{R}}_F(\xi,\xi^*)$ characterize the structure of the boundary of F near the point ξ and do not depend on the position of 0 in the interior of F
- The characteristics $\frac{1}{\|\xi^*\|}\gamma^{\pm}_{F,\alpha}(\xi,\xi^*)$ instead depend also on the point of view of an observer placed at the origin. They are connected with the "true" curvature through the inequalities

$$\frac{1}{\|F^0\|^2}\hat{\varkappa}_F(\xi,\xi^*) \le \frac{\gamma^{\pm}_F(\xi,\xi^*)}{\|\xi^*\|} \le \|F\|^2\,\hat{\varkappa}_F(\xi,\xi^*),$$

  where $\|F\| := \sup\{\|\xi\| : \xi \in F\}$
- $\hat{\varkappa}_F$, $\gamma^{\pm}_F$ are lower semicontinuous w.r.t. their arguments, while $\widehat{\mathfrak{R}}_F$ is upper semicontinuous

Properties of the curvatures (comments)

Intuitively, if an observer stands at the origin, which is closer to the boundary, then the boundary seems less rotund than in the case when an observer (in the origin) stands far from it (although objectively the curvature is the same). This subjective point of view is reflected in the definition of "one-sided curvatures" $\gamma_F^{\pm}(\xi,\xi^*)/\|\xi^*\|$.

Example

Let, for instance, $F := \{\xi \in H : \|\xi - a\| \leq 1\}$ where $a \in H$ is fixed with $\|a\| < 1$. Assume that $\xi^* \in \partial F^0$ is a vector colinear to a, and ξ is the unique point on ∂F belonging to $\mathfrak{J}_F(\xi^*)$. Then $\gamma_F^{\pm}(\xi,\xi^*)$ is equal to $\frac{1-\|a\|}{2}$ if ξ^* is closer to ∂F, and to $\frac{1+\|a\|}{2}$ otherwise. Thus $\gamma_F^{\pm}(\xi,\xi^*) \to 0$ whenever the origin tends to belong to ∂F.

The passage to limits in the definitions of the curvatures and of the radius of curvature is necessary to guarantee the (lower or upper) semicontinuity.

Dual connection with the second order derivative

Theorem

If the boundary of F^0 is of the class $\mathcal{C}^2$ in a neighbourhood of ξ then we have

$$\gamma_F^+(\xi,\xi^*) = \gamma_F^-(\xi,\xi^*) = \frac{1}{2\left\|\nabla^2\rho_{F^0}(\xi^*)\right\|_{F^0}},$$

where

$$\left\|\nabla^2\rho_{F^0}(\xi^*)\right\|_{F^0} := \sup_{v^*\in F^0}\left\langle\nabla^2\rho_{F^0}(\xi^*)\,v^*, v^*\right\rangle$$

and $\xi^* := \nabla\rho_{F^0}(\xi)$.

Notice that if F^0 is smooth of the second order we have not only that F is second order rotund (the curvature is positive) but that the left and right curvatures coincide.

Part II. Standing assumptions. Setting of the problem

Let H be a Hilbert space, $F \subset H$ be a convex closed bounded subset with $0 \in int\, F$ (called further dynamics) and $C \subset H$ be just closed (target set). Consider the following minimum time control problem with a constant convex dynamics and a closed target set:

$$\min \{ T > 0 : \exists x(\cdot),\ x(T) \in C,\ x(0) = x, \\ \textit{and } \dot{x}(t) \in F \textit{ a.e. in } [0, T] \}$$

Let us denote by $\mathfrak{T}_C^F(x)$ the value function in the above problem (called time-minimum function, and by $\pi_C^F(x)$ the set of point on C (may be empty), which are achieved from x for a minimal time (called further time-minimum projection).

Equivalent formulations

Due to convexity of the dynamics F this time-minimum control problem can be treated as

- the mathematic programming problem

$$\text{Minimize}\,\{\rho_F(y-x) : y \in C\}$$

  and then

$$\pi_C^F(x) = \left\{ y \in C : \rho_F(y-x) = \mathfrak{T}_C^F(x) \right\}$$

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- viscosity solution of the Hamilton-Jacobi Equation

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We are interested in the local well-posedness of this problem, i.e., conditions guaranteeing that $\pi_C^F(x)$ is a singleton, defined and continuous on a neighbouhood of the target, and in the regularity of the time-minimum function (differentiability and Hölder continuity of its (Fréchet) derivative).

- In the case $F = \overline{B}$ (unit closed ball in H) the function $\mathfrak{T}_C^F(\cdot)$ is usual distance from the closed set C, and $\pi_C^F(\cdot)$ is the metric projection to C. It is well known that

  $x \mapsto \pi_C^F(x)$ is single-valued and continuous on H iff C is convex

  (it is even Lipschitzean with the Lipschitz constant 1)
- Local version of this property was studied by many authors (**H. Federrer, J.-P. Vial, A. Canino, A. Shapiro, F. Clarke, R. Stern, P. Wolenski, R. Rockafellar, L. Thibault, R. Poliquin, G. Colombo, V.G.** and others). Exaustive result in this direction says that the following two properties are equivalent:

- there exists an open set $\mathcal{U} \supset C$ such that the metric projection $\pi_C^{\overline{B}}(\cdot)$ is well-defined and continuous on $\mathcal{U}$
- the set C is φ-convex

History (continuation)

Definition

The set $C \subset H$ is said to be φ-convex (proximally smooth, proximally regular or with a positive reach) if there exists a continuous function $\varphi : C \to \mathbb{R}^+$ such that for all $x, y \in C$ and $v \in \mathbf{N}_C^p(x)$ the inequality

$$\langle v, y - x \rangle \leq \varphi(x) \|x - y\|^2$$

holds. Here $\mathbf{N}_C^p(x)$ stands for the proximal normal cone to C at $x \in C$.

- The first result on the local well-posedness for $F \neq \overline{B}$ obtained by G. Colombo and P. Wolenski in 2004 states that $\pi_C^F(\cdot)$ is a continuous singleton defined on an open ball around C if both C is φ-convex with a constant $\varphi(x) = \varphi > 0$ and F is uniformly strictly convex (strict convexity is controlled by a parameter $\gamma > 0$). The radius of that ball is determined through ballance between φ and γ.

φ-convexity (comments)

In the definition of φ-convexity the proximal normal cone $\mathbf{N}_C^p(x)$ can be substituted, for instance, to the Fréchet normal cone

$$\mathbf{N}_C^f(x) := \left\{ v \in H : \limsup_{x \neq y \to x, y \in C} \frac{\langle v, y - x \rangle}{\|y - x\|} \leq 0 \right\}$$

It is known that for each φ-convex set all the basic normal cones coincide (similarly as for a convex set).

Smoothness assumption. Scaled curvature

- For the sake of simplicity we consider only the case of the smooth target C. Namely, we assume that at each point $x \in \partial C$ there exists a unique normal vector (e.g., in the **Fréchet** sense) $\mathfrak{n}(x)$ continuously depending on x ($\mathbf{N}_C^f(x) \cap \overline{B} = \{\mathfrak{n}(x)\}$). Let us denote by

$$\mathfrak{v}(x) := -\frac{\mathfrak{n}(x)}{\rho_{F^0}(-\mathfrak{n}(x))}$$

- To formulate the well-posedness result we need the curvature notion but not local one $\hat{\varkappa}_F(\xi, \xi^*)$ defined above. We will utilize the following "scaled curvature" (it depends not only on the structure of the boundary near ξ but also on the size of F, in particular, it can not be too large):

$$\varkappa_F(\xi, \xi^*) = \frac{1}{\|\xi^*\|} \liminf_{\substack{(\eta,\eta^*)\to(\xi,\xi^*) \\ \eta\in\mathfrak{J}(\eta^*),\, \eta^*\in\partial F^0}} \inf_{r>0} \frac{\widehat{\mathfrak{C}}_F(r, \eta, \eta^*)}{r^2}$$

The well-posedness result

Theorem

Assume that C is φ-convex with the continuous function $\varphi : C \to \mathbb{R}^+$, and that F is strictly convex of the second order w.r.t. each vector $\mathfrak{v}(x)$, $x \in C$ (observe that $\mathfrak{J}_F(\mathfrak{v}(x))$ is a singleton). Then $\pi_C^F(\cdot)$ is well-defined and continuous on the open set

$$\mathfrak{A}(C) = \left\{ z : \liminf_{\substack{\rho_F(x-z)\to\mathfrak{T}_C^F(z)+ \\ \xi^*-\mathfrak{v}(x)\to 0 \\ x\in\partial C,\, \xi^*\in\partial F^0}} \left\{ \varkappa_F\left(\mathfrak{J}_F\left(\xi^*\right), \xi^*\right) - \mathfrak{T}_C^F(z)\,\varphi(x) \right\} > 0 \right\}$$

If $\dim H < +\infty$ then this set can be represented in a simpler form:

$$\left\{ z : \liminf_{\rho_F(x-z)\to\mathfrak{T}_C^F(z)+} \left\{ \varkappa_F\left(\mathfrak{J}\left(\mathfrak{v}(x)\right), \mathfrak{v}(x)\right) - \mathfrak{T}_C^F(z)\,\varphi(x) \right\} > 0 \right\}$$

The well-posedness result (comments)

- This result has a theoretical interest since it gives the exact formula for a neighbourhood of C, where the well-posedness (and as a consequence certain regularity of the value function) takes place
- It is not difficult to obtain from this formula some test showing whether the problem is well-posed near a given point $x \notin C$, and to find a neighbourhood of x, where the well-posedness holds (as illustrated by an example below)
- The above result can be extended to the general case where the target set C is not necessary smooth
- Another (independent) set of hypotheses guaranteeing the local well-posedness of the time-minimum projection can be proposed, which do not use the second order rotundity of F (as well as φ-convexity of C) but only the properties of the duality mapping $\mathfrak{J}_F(\cdot)$ (so called the first order conditions)

Local version of the well-posedness result

Given $x_0 \in \partial C$ we have two conditions guaranteeing the well-posedness of the time-minimum projection in some neighbourhood $U(x_0)$:

1. the set C should is φ-convex near x_0 with $\varphi(x) \leq M$ for some constant $M > 0$
2. F is rotund of the second order w.r.t. to each ξ^* from a neighbourhood of $\mathfrak{v}(x_0)$ with $\varkappa_F(\xi, \xi^*) \geq K$. Here as usual $\xi = \mathfrak{J}_F(\xi^*)$, and $K > 0$ is some constant

Example

Let $F := \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \leq 1 - \xi_1^4,\ -1 \leq \xi_1 \leq 1\right\}$ and $C := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq x_2^2\right\}$.

Here C is φ-convex (unbounded), but F is not uniformly rotund of the second order (its curvature in the points $(0, \pm 1)$ is equal to 0). However, there is an open set $\mathfrak{A}$ around C (not a ball), where $\pi_C^F(\cdot)$ is well-posed.

Well-posedness through the true curvature

Under the same conditions as above we can express the neighbourhood, where the well-posedness of the time-minimum projection takes place, through the "true" curvature $\hat{\varkappa}_F(\xi, \xi^*)$, or reminding the duality result, through the second order gradient of the gauge function $\rho_{F^0}(\cdot)$. To this end we use some kind of the continuity modulus of $\nabla^2 \rho_{F^0}(\xi^*)$. Namely, let us set

$$\beta(\delta, \xi^*) := \sup\left\{\|\eta^* - \xi^*\| : \|\nabla^2 \rho_{F^0}(\eta^*) - \nabla^2 \rho_{F^0}(\xi^*)\| \leq \delta\right\},$$

$\delta > 0$. In particular, if $\nabla^2 \rho_{F^0}(\cdot)$ is Lipschitzean near ξ^* (with a Lipschitz constant L_{ξ^*} in a ε_{ξ^*}-neighbourhood of ξ^*) then we can put

$$\beta(\delta, \xi^*) = \frac{L_{\xi^*}}{\varepsilon_{\xi^*}}$$

Well-posedness result through the second derivative

Theorem

Assume that F^0 has the boundary of the class $\mathcal{C}^2$ near each vector $\mathfrak{v}(x)$, $x \in C$. Then given $\delta > 0$ the time-minimum projection $\pi_C^F(\cdot)$ is well-defined on the open set $\mathfrak{A}(C)$ given above, where the number $\mathfrak{Q}(\delta, \xi^*)$ stands in the place of the scaled curvature $\varkappa_F(\xi, \xi^*)$

$$\mathfrak{Q}(\delta, \xi^*) := \frac{1}{2\|F\|^2 \|\xi^*\|} \min \left[\frac{\beta^2(\delta, \xi^*) \left(\left\| \nabla^2 \rho_{F^0}(\xi^*) \right\|_{F^0} + \delta \left\| F^0 \right\|^2 \right)}{4\|F\|^2 \|F^0\|^2}, \frac{1}{\left\| \nabla^2 \rho_{F^0}(\xi^*) \right\|_{F^0} + \delta \left\| F^0 \right\|^2} \right]$$

Notice that we can slightly control the size of the neighbourhood $\mathfrak{A}(C)$ by the suitable choice of the parameter $\delta > 0$.

Regularity of the time-minimum projection

Let us start with the regularity of $\pi_C^F(\cdot)$. In the case of ball we have (local) Lipschitzianity of the (metric) projection, while, in general, only the Höldereanity of the order $1/2$ always takes place.

In order to have more regularity we should impose some suplementary regularity assumptions on C or/and on F.

Theorem

Assume that either the target set C has smooth boundary near x_0 and the normal vector $\mathfrak{n}_C(\cdot)$ is Hölder continuous with an exponent $0 < \alpha \leq 1$, or the dynamics F is uniformly smooth w.r.t. each ξ^* near the normal vectors to C at x_0, and the gradient $\nabla \rho_F(\cdot)$ is Hölder continuous near $\xi = \mathfrak{J}_F(\xi^*)$ with an exponent α. Then $\pi_C^F(\cdot)$ is (locally) Hölder continuous near x_0 with the exponent $1/(2-\alpha)$.

Regularity of the value function

We conclude with a result on regularity of the time-minimum function.

Theorem

Let $x_0 \in \partial C$. Assume that

- the set C is φ-convex near x_0
- the dynamics F is the second order rotund w.r.t. each ξ^* near the normals to C at x_0 with the scaled curvature $\varkappa_F(\xi, \xi^*)$ bounded away from zero
- for each $x \notin C$ enough close to x_0 one of the properties below holds:
  - C has smooth boundary at $\bar{x} := \pi_C^F(x)$ and the unit normal vector $\mathfrak{n}_C(\cdot)$ is Hölder continuous with an exponent $0 < \alpha \leq 1$
  - F is uniformly smooth at $\xi := \frac{\bar{x}-x}{\rho_F(\bar{x}-x)}$, and $\nabla \rho_F(\cdot)$ is Hölder continuous with an exponent $0 < \alpha \leq 1$

Then the value function $\mathfrak{T}_C^F(\cdot)$ is of class $C_{loc}^{1,\frac{\alpha}{2-\alpha}}$ in a neighbourhood of x_0.

Final remarks

- It is known that in the case $F = \overline{B}$ the well-posedness of the (metric) projection is equivalent to the Lipschitz continuity of the value function (distance from the closed set)
- For an arbitrary dynamics F this is false in general, and we have only the Hölderianity of the gradient of the time-minimum function as seen from the above regularity result
- Under our rotundity assumptions we have the Lipschitz continuity of $\mathfrak{T}_C^F(\cdot)$ if either the target set, or the dynamics admits the Lipschitz continuous normal vector at the respective point