Infinite dimensional tilting theory

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Abstract. Infinite dimensional tilting modules are abundant in representation theory. They occur when studying torsion pairs in module categories, when looking for complements to partial tilting modules, or in connection with the Homological Conjectures. They share many properties with classical tilting modules, but they also give rise to interesting new phenomena as they are intimately related with localization, both at the level of module categories and of derived categories.

In these notes, we review the main features of infinite dimensional tilting modules. We discuss the relationship with approximation theory and with localization. Finally, we focus on some classification results and we give a geometric interpretation of tilting.

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1. Introduction

Tilting theory is a research area that has developed from representation theory of finite dimensional algebras [27, 54, 74] with applications going far beyond this context: tilting nowadays plays an important role in various branches of mathematics, ranging from Lie theory to combinatorics, algebraic geometry and topology.

Classical tilting modules are required to be finite dimensional. This survey focusses on tilting modules that need not be finitely generated, as first defined in [40, 2]. The aim is twofold. We first explain the main tools of infinite dimensional tilting theory and exhibit a number of examples. Then we discuss the interaction with localization theory, and we employ it to classify large tilting modules over several rings. We will see that such classification results also yield a classification of certain categories of finitely generated modules. Moreover, they lead to a classification of Gabriel localizations of the module category, or of an associated geometric category.

Large tilting modules occur in many contexts. For example, the representation type of a hereditary algebra is governed by the behaviour of certain infinite dimensional tilting modules (Theorem 3.1). Also the finitistic dimension of a noetherian ring is determined by a tilting module which is not finitely generated in general (Theorem 5.4). Large tilting modules further arise when looking for complements to partial tilting modules of projective dimension greater than one, or when computing intersections of tilting classes given by finite dimensional tilting modules

(Sections 5.1 and 5.3). Finally, over a commutative ring, every non-trivial tilting module is large (Section 3.2).

In fact, given a ring R, many important subcategories of mod-R can be studied by using tilting modules. Here mod-R denotes the category of modules admitting a projective resolution with finitely generated projectives, which is just the category of finitely generated modules when R is noetherian.

More precisely, for any set $\mathcal{S} \subset$ mod-R consisting of modules of bounded projective dimension, there is a tilting module T, not necessarily finitely generated, whose tilting class $T^{\perp} = \{M \in \text{Mod-}R \mid \text{Ext}^i_R(T, M) = 0 \text{ for all } i > 0\}$ coincides with $\mathcal{S}^{\perp} = \{M \in \text{Mod-}R \mid \text{Ext}^i_R(S, M) = 0 \text{ for all } i > 0,\ S \in \mathcal{S}\}$. Conversely, for any tilting module T, no matter whether finitely generated or not, the tilting class $T^{\perp}$ is determined by a set $\mathcal{S} \subset$ mod-R, in the sense that $T^{\perp} = \mathcal{S}^{\perp}$ (Corollary 5.1 and Theorem 6.1).

These results rely on work of Eklof and Trlifaj on the existence of approximations, on the relationship between tilting and approximation theory first discovered by Auslander and Reiten in the classical setup (Theorems 4.4 and 4.6), and on papers by Bazzoni, Eklof, Herbera, Šťovíček, and Trlifaj which also make use of some sophisticated set-theoretic techniques.

As a consequence, one obtains a bijection between the resolving subcategories of mod-R consisting of modules of bounded projective dimension and the tilting classes in Mod-R. For an artin algebra, one also gets a bijection between the torsion pairs in mod-R whose torsion class contains all indecomposable injectives and the tilting classes $T^{\perp}$ in Mod-R where T is a tilting module of projective dimension at most one (Corollary 6.4 and Theorem 6.8).

The results above show that large tilting modules share many properties with the classical tilting modules from representation theory. There is an important difference, however. If T is a classical tilting module over a ring R, and S is the endomorphism ring of T, then the derived categories of R and S are equivalent as triangulated categories. This is a fundamental result due to Happel. A generalization for large tilting modules holds true under a mild assumption, as recently shown in papers by Bazzoni, Mantese, Tonolo, Chen, Xi, and Yang. But instead of an equivalence, one has that the derived category $\mathcal{D}(\text{Mod-}R)$ is a quotient of $\mathcal{D}(\text{Mod-}S)$. When T has projective dimension one, $\mathcal{D}(\text{Mod-}R)$ is even a recollement of $\mathcal{D}(\text{Mod-}S)$ and $\mathcal{D}(\text{Mod-}\tilde{S})$, where $\tilde{S}$ is a localization of S (Theorem 6.11).

Tilting functors given by large tilting modules thus give rise to new phenomena and induce localizations of derived categories. Actually, localization already plays a role at the level of module categories. A typical example of a large tilting module is provided by the $\mathbb{Z}$-module $T = \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$. Following the same pattern, one can use localization techniques to construct tilting modules in many contexts. Indeed, every injective ring epimorphism $R \to S$ with nice homological properties gives rise to a tilting R-module of the form $S \oplus S/R$ (Theorem 7.1).

Over certain rings, tilting modules of this shape provide a classification of all tilting modules up to equivalence. Hereby, we say that two tilting modules are equivalent if they induce the same tilting class. Such identification is justified by the fact that the tilting class $T^{\perp}$ determines the additive closure $\text{Add}\, T$ of a tilting module T. Furthermore, recall that in this way, when we classify tilting modules, we are also classifying resolving subcategories of mod-R, and in case we restrict to projective dimension one, torsion pairs in mod-R.

The interaction between tilting and localization and its role in connection with classification problems is best illustrated by the example of a Dedekind domain R. The tilting modules over R are constructed as above from ring epimorphisms, or more precisely, from universal localizations of the ring R, which in turn are in bijection with the recollements of the unbounded derived category $\mathcal{D}(\text{Mod-}R)$. In other words, tilting modules are parametrized by the subsets $P \subset \text{Max-Spec}\, R$ of the maximal spectrum of R. Hereby, the two extreme cases yield the trivial tilting module R when $P = \emptyset$, and the tilting module $Q \oplus Q/R$ when $P = \text{Max-Spec}\, R$ (Section 8.1).

This result extends to Prüfer domains and to arbitrary commutative noetherian rings. Over the latter, tilting modules can be classified in terms of sequences of specialization closed subsets of the Zariski prime spectrum. In both cases, tilting modules of projective dimension one correspond to categorical localizations of Mod-R in the sense of Gabriel (Sections 8.3 and 8.4).

But a similar situation also occurs when R is the Kronecker algebra (Section 8.2). Actually, if we replace the maximal spectrum by the index set $\mathbb{X}$ of the tubular family $\mathbf{t} = \bigcup_{x\in\mathbb{X}} \mathcal{U}_x$ and restrict our attention to infinite dimensional tilting modules, we get a complete analogy to the Dedekind case. Indeed, the large tilting modules over R are parametrized by the subsets of $\mathbb{X}$. Here are the two extreme cases: $P = \emptyset$ yields the tilting module $\mathbf{L}$ corresponding to the resolving subcategory of mod-R formed by the preprojectives, and $P = \mathbb{X}$ the tilting module $\mathbf{W}$ corresponding to the resolving subcategory of mod-R formed by preprojective and regular modules. Moreover, all tilting modules are constructed from universal localization, with the only exception of the module $\mathbf{L}$, that is, of the set $P = \emptyset$. This analogy also allows a geometric interpretation of tilting. In fact, regarding $\mathbb{X}$ as an exceptional curve in the sense of [68], it turns out that large tilting modules correspond to Gabriel localizations of the category $\text{Qcoh}\mathbb{X}$ of quasi-coherent sheaves on $\mathbb{X}$, the same result as in the commutative case when replacing Mod-R by $\text{Qcoh}\mathbb{X}$.

For arbitrary tame hereditary algebras, the classification of large tilting modules is more complicated due to the possible presence of finite dimensional direct summands from non-homogeneous tubes. Infinite dimensional tilting modules are parametrized by pairs (Y, P) where Y determines the finite dimensional part, and P is a subset of $\mathbb{X}$. The infinite dimensional part is obtained as above from universal localization and from the module $\mathbf{L}$. And again, tilting modules correspond to Gabriel localizations of the category $\text{Qcoh}\mathbb{X}$ (Section 8.5).

Finally, these results lead to a classification of large tilting sheaves in the category $\text{Qcoh}\mathbb{X}$ on an exceptional curve $\mathbb{X}$. When $\mathbb{X}$ is of domestic type, the classification is essentially the same as for large tilting modules over tame hereditary algebras. For $\mathbb{X}$ of tubular type there are many more tilting sheaves. Indeed, for each rational slope we find the same tilting sheaves as in the domestic case, and in addition there is one tilting sheaf for each irrational slope. Since every large tilting sheaf has a slope, this yields a complete classification (Section 8.6).

The paper is organized as follows. In Section 2 we collect basic notation and definitions. In Section 3 we exhibit first examples of large tilting modules, e.g. over $\mathbb{Z}$ and over hereditary artin algebras. Further examples are presented in Section 5, after reviewing the relationship between tilting and approximation theory in Section 4. Section 6 is devoted to the interplay between finitely generated modules and large tilting modules. In Section 7 we explain the fundamental construction of tilting modules from ring epimorphisms or universal localizations. Finally, in Section 8 we illustrate the classification results for Dedekind domains, commutative noetherian rings, Prüfer domains, tame hereditary algebras and the category of quasi-coherent sheaves on an exceptional curve.

Many of the results presented here, in particular in Sections 1-6, are treated in detail in [52]. We also refer to the survey articles [88, 93, 94].

2. Large tilting modules

2.1. Notation. Throughout this note, let R be a ring (associative, with 1), Mod-R (respectively, R-Mod) the category of all right (respectively, left) R-modules, and mod-R (respectively, R-mod) the full subcategory of all modules M admitting a projective resolution

$$\cdots \to P_{k+1} \to P_k \to \cdots \to P_1 \to P_0 \to M \to 0$$

where all P_i are finitely generated. The modules $M \in$ mod-R are sometimes called of type FP_∞. They have the property that the functor $\mathrm{Ext}^i_R(M, -)$ commutes with direct limits for any $i \geq 0$ (see e.g. [52, 3.1.6]). Of course, mod-R is just the category of finitely presented (respectively, generated) modules when R is right coherent (respectively, noetherian).

If R is a k-algebra over a commutative ring k, we denote by $D = \mathrm{Hom}_k(-, I)$ the duality with respect to an injective cogenerator I of Mod-k. When R is an artin algebra, we take the usual duality D. For an arbitrary ring R, we can choose $k = \mathbb{Z}$ and $D = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$.

Given a class of modules $\mathcal{M} \subset$ Mod-R, we denote by $\mathrm{Add}\,\mathcal{M}$ the class of all modules isomorphic to a direct summand of a direct sum of modules of $\mathcal{M}$, and by $\mathrm{Prod}\mathcal{M}$ the class of all modules isomorphic to a direct summand of a direct product of modules of $\mathcal{M}$. The class of all modules isomorphic to a direct summand of a finite direct sum of modules of $\mathcal{M}$ is denoted by $\mathrm{add}\,\mathcal{M}$. Moreover, we write $\varinjlim \mathcal{M}$ for the class of all modules isomorphic to a direct limit of modules of $\mathcal{M}$. We set

$$\mathcal{M}^o = \{X \in \text{Mod-}R \mid \mathrm{Hom}_R(M, X) = 0 \text{ for all } M \in \mathcal{M}\}$$

$$\mathcal{M}^\perp = \{X \in \text{Mod-}R \mid \mathrm{Ext}^i_R(M, X) = 0 \text{ for all } i > 0 \text{ and } M \in \mathcal{M}\}$$

and we define correspondingly ${}^o\mathcal{M}$ and ${}^\perp\mathcal{M}$. When $\mathcal{M} = \{M\}$, we just write $\mathrm{Add}\,M$, $\mathrm{Prod}\,M$, M^o, $M^\perp$, ... All these classes are regarded as strictly full subcategories of Mod-R.

We denote by $\operatorname{pdim} M$ and $\operatorname{idim} M$ the projective and injective dimension of a module M, respectively, and for $\mathcal{M} \subset \text{Mod-}R$ we write

$$\operatorname{pdim}\mathcal{M} = \sup\{\operatorname{pdim}M \mid M \in \mathcal{M}\}.$$

Further, we set $\mathcal{P} = \{M \in \text{Mod-}R \mid \operatorname{pdim}M < \infty\}$, and $\mathcal{P}^{<\infty} = \mathcal{P} \cap \text{mod-}R$.

Finally, we recall that a short exact sequence $0 \to A \to B \to C \to 0$ in Mod-R is said to be *pure-exact* if the induced sequence $0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is exact for any left R-module M. A module $N \in \text{Mod-}R$ is *pure-injective* if the functor $\operatorname{Hom}_R(-, N)$ turns every pure-exact sequence into a short exact sequence of abelian groups. If $N^{(I)}$ is pure-injective for all sets I, then N is said to be Σ-*pure-injective.*

2.2. Tilting and cotilting modules. A (right) R-module T is called a *tilting module* provided it satisfies the following conditions

(T1) $\operatorname{pdim} T < \infty$;

(T2) $\operatorname{Ext}^i_R(T, T^{(I)}) = 0$ for each $i > 0$ and all sets I;

(T3) There exists a long exact sequence

$$0 \to R_R \to T_0 \to \cdots \to T_r \to 0$$

with $T_i \in \operatorname{Add} T$ for each $0 \leq i \leq r$.

The class $T^{\perp}$ is then called the *tilting class* induced by T. Further, T and $T^{\perp}$ are called *n-tilting* when $\operatorname{pdim} T \leq n$.

The tilting class determines the additive closure of T: two tilting modules T and T' induce the same tilting class $T^{\perp} = T'^{\perp}$ if and only if $\operatorname{Add} T = \operatorname{Add} T'$. In this case we say that T and T' are *equivalent.* Moreover, we say that a tilting module is *large* if it is not equivalent to any tilting module in mod-R.

Dually, a module C is called a *cotilting module* provided it satisfies the following conditions

(C1) $\operatorname{idim} C < \infty$;

(C2) $\operatorname{Ext}^i_R(C^I, C) = 0$ for each $i > 0$ and all sets I;

(C3) There exists an injective cogenerator Q and a long exact sequence

$$0 \longrightarrow C_r \longrightarrow \cdots \longrightarrow C_0 \longrightarrow Q \longrightarrow 0$$

with $C_i \in \operatorname{Prod} C$ for each $0 \leq i \leq r$.

The class ${}^{\perp}C$ is then called the *cotilting class* induced by C. Again, C and $C^{\perp}$ are called *n-cotilting* when $\operatorname{idim} C \leq n$.

Two cotilting modules C and C' are *equivalent* if ${}^{\perp}C = {}^{\perp}C'$, or equivalently, $\operatorname{Prod} C = \operatorname{Prod} C'$. A cotilting module is said to be *large* if it is not equivalent to any cotilting module in mod-R.

We will see in Proposition 6.2 that the dual $D(T)$ of a tilting module is always a cotilting module. However, not all cotilting modules arise in this way in general, see Remark 6.3. Furthermore, the dual of a cotilting module need not be a tilting module, cf. Section 3.

2.3. Faithful torsion pairs. Recall that a pair of classes $(\mathcal{T}, \mathcal{F})$ in Mod-R (or more generally, in an abelian category $\mathcal{A}$) is a *torsion pair* if $\mathcal{T} = {}^{o}\mathcal{F}$, and $\mathcal{F} = \mathcal{T}^{o}$. Every class $\mathcal{M}$ of modules (or of objects in $\mathcal{A}$) *generates* a torsion pair $(\mathcal{T}, \mathcal{F})$ by setting $\mathcal{F} = \mathcal{M}^{o}$ and $\mathcal{T} = {}^{o}(\mathcal{M}^{o})$. Similarly, the torsion pair *cogenerated by* $\mathcal{M}$ is given by $\mathcal{T} = {}^{o}\mathcal{M}$ and $\mathcal{F} = ({}^{o}\mathcal{M})^{o}$. We say that a torsion pair $(\mathcal{T}, \mathcal{F})$ is *split* if every short exact sequence $0 \to T \to M \to F \to 0$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$ splits. Finally, a torsion pair $(\mathcal{T}, \mathcal{F})$ in Mod-R, or in mod-R, is called *faithful* if $R \in \mathcal{F}$, and it is called *cofaithful* if $\mathcal{T}$ contains an injective cogenerator of Mod-R.

1-tilting modules generate cofaithful torsion pairs, and 1-cotilting modules cogenerate faithful torsion pairs.

Proposition 2.1 ([40, 36]). *(1) A module T is a 1-tilting module if and only if $T^{\perp}$ coincides with the class* $\operatorname{Gen} T$ *of all modules isomorphic to a quotient of a direct sum of copies of T. Then* $(\operatorname{Gen} T, T^{o})$ *is a cofaithful torsion pair in* Mod-R.

(2) A module C is a 1-cotilting module if and only if ${}^{\perp}C$ coincides with the class $\operatorname{Cogen} C$ *of all modules isomorphic to a submodule of a direct product of copies of C. Then* $({}^{o}C, \operatorname{Cogen} C)$ *is a faithful torsion pair in* Mod-R.

Torsion pairs as in statement (1) or (2) above are called *tilting*, respectively *cotilting torsion pairs*. They will be characterized in Corollaries 4.7 and 6.6, and in Theorem 6.8. For an extension of Proposition 2.1 to tilting or cotilting modules of arbitrary projective, respective injective, dimension, we refer to [18].

3. First examples

3.1. Classical tilting modules. If $T \in$ mod-R, then the functors $\operatorname{Ext}^{i}_{R}(T, -)$ commute with direct sums, so condition (T2) in definition 2.2 is equivalent to

(T2') $\operatorname{Ext}^{i}_{R}(T, T) = 0$ for each $i > 0$.

Moreover, it is easy to show that condition (T3) can be replaced by

(T3') There exists a long exact sequence $0 \to R_R \to T_0 \to \cdots \to T_r \to 0$ with $T_i \in \operatorname{add} T$ for each $0 \leq i \leq r$.

We thus recover the original definition of tilting module from [27, 54, 74]. Observe that a tilting module belongs to mod-R whenever it is finitely generated, see e.g. [32, 4.7].

3.2. Tilting modules over commutative rings. Over a commutative ring R, every finitely generated tilting module is projective. This was already observed in [39, 77]. It can also be derived from a more general statement on the unbounded derived category $\mathcal{D}(\text{Mod-}R)$. Indeed, using that over a commutative ring vanishing of $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}R)}(X, Y[n])$ is determined locally, one can show that every compact exceptional object $X \in \mathcal{D}(\text{Mod-}R)$ is projective up to shift, cf. [7]. So, all tilting modules are either equivalent to R or large. A classification of tilting and cotilting modules over commutative noetherian rings will be given in Section 8.3.

3.3. A tilting and cotilting abelian group. Let us focus on the special case $R = \mathbb{Z}$. A complete classification of the tilting $\mathbb{Z}$-modules will be given in Section 8.1. We start by considering the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$.

It is a tilting and cotilting module. Indeed, the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

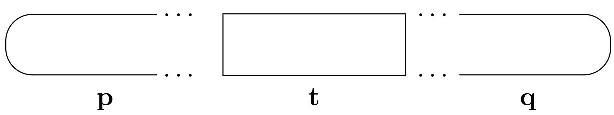
gives condition (T3) in definition 2.2, and the remaining conditions follow from the fact that $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of Mod-$\mathbb{Z}$.

The tilting class generated by $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ is the class $\mathcal{D}$ of *divisible* groups, i.e. the class of $\mathbb{Z}$-modules M such that $M = rM$ for all $0 \neq r \in \mathbb{Z}$. The corresponding torsion-free class $\mathcal{R}$ is the class of *reduced* groups, and the tilting torsion pair $(\mathcal{D}, \mathcal{R})$ is a split torsion pair. The cotilting class cogenerated by $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ is Mod-$\mathbb{Z}$.

The dual module $D(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is a cotilting $\mathbb{Z}$-module. Using Proposition 6.2, one proves that its cotilting class is the class of all *torsion-free* groups, i.e. the class of $\mathbb{Z}$-modules M such that for $0 \neq r \in \mathbb{Z}$ and $0 \neq m \in M$ always $rm \neq 0$. So, the cotilting torsion pair induced by $D(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})$ is the classical torsion pair whose torsion class is formed by the *torsion* groups, i.e. by the $\mathbb{Z}$-modules M such that for all $m \in M$ there is $0 \neq r \in \mathbb{Z}$ with $rm = 0$.

Observe that $D(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})$ is not tilting. Indeed, condition (T2) fails because the two direct summands $\mathbb{Q}$ and $D(\mathbb{Q}/\mathbb{Z})$ satisfy $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, D(\mathbb{Q}/\mathbb{Z})^{(\mathbb{N})}) \neq 0$, see [45, V.2].

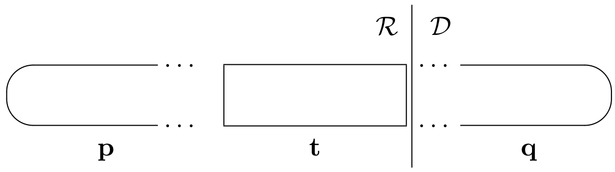
3.4. Two large tilting modules over hereditary algebras. Let R be a (connected) hereditary artin algebra of infinite representation type. The Auslander-Reiten-quiver of R is of the form



where $\mathbf{p}$ is the preprojective component, $\mathbf{q}$ is the preinjective component, and $\mathbf{t}$ consists of a family of regular components.

Let us consider the following torsion pairs in Mod-R:

(1) [80, 79] *The torsion pair* $(\mathcal{D}, \mathcal{R})$ *cogenerated by* $\mathbf{t}$ is a cofaithful torsion pair



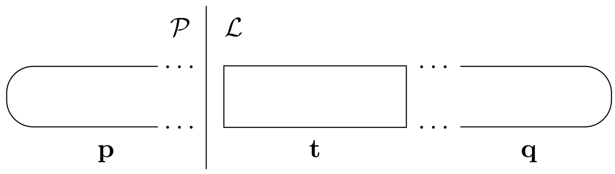
induced by a large tilting module $\mathbf{W} \in$ Mod-R. The shape of $\mathbf{W}$ in the tame case is described below, for the wild case we refer to [70, Section 4].

By the Auslander-Reiten formula $\operatorname{Gen} \mathbf{W} = \mathcal{D} = {}^o\mathbf{t} = \mathbf{t}^{\perp}$. The modules in $\mathcal{D}$ are called *divisible.*

(2) *The torsion pair* $(\mathcal{T}, \mathcal{F})$ *generated by* $\mathbf{t}$ is a faithful torsion pair induced by the cotilting module $D({}_R\mathbf{W})$, where ${}_R\mathbf{W} \in R$-Mod is the left version of $\mathbf{W}$. This follows from Proposition 6.2.

The modules in $\mathcal{F}$ are called *torsion-free*, the modules in $\mathcal{T}$ are called *torsion.* The class $\operatorname{Cogen} D({}_R\mathbf{W}) = \mathcal{F} = \mathbf{t}^o = {}^{\perp}\mathbf{t}$ consists of all direct limits of preprojective modules, and $\mathcal{T} = \operatorname{Gen} \mathbf{t}$, see [6, 5.4] or Lemma 6.7.

(3) [71, 61] *The torsion pair* $(\mathcal{L}, \mathcal{P})$ *cogenerated by* $\mathbf{p}$ is a cofaithful torsion pair



induced by a large tilting module $\mathbf{L}$. The class $\operatorname{Gen} \mathbf{L} = \mathcal{L} = {}^o\mathbf{p} = \mathbf{p}^{\perp}$ consists of the modules without indecomposable preprojective summands. The class $\mathcal{P}$ is the class of preprojective modules from [80, 2.7]: a module X belongs to $\mathcal{P}$ if and only if every non-zero submodule of X has a direct summand from $\mathbf{p}$.

The module $\mathbf{L}$ is called *Lukas tilting module.* It is constructed as follows. One defines inductively a chain in $\operatorname{add} \mathbf{p}$ starting with $A_0 = R$, and choosing at every step an embedding $A_n \subset A_{n+1} \in \operatorname{add} \mathbf{p}$ such that $A_{n+1}/A_n \in \operatorname{add} \mathbf{p}$ and $\operatorname{Hom}_R(A_{n+1}, \tau^{-n}R) = 0$. This is possible by [80, 2.5]. Now set $L_0 = \bigcup_{n \geq 0} A_n$ and take the short exact sequence $0 \to R \to L_0 \to L_1 \to 0$. The module $\mathbf{L} = L_0 \oplus L_1$ certainly satisfies (T1) and (T3), it is $\operatorname{add} \mathbf{p}$-filtered according to the definition in Section 4.2, and one verifies that it is a tilting module with the desired tilting class, see [70, 71] and [61, 3.3].

Notice that $\mathbf{L}$ has no finite-dimensional direct summands. Further, $\operatorname{Add} \mathbf{L} = \operatorname{Add} L_0$ by [71, 3.2], and when R is the Kronecker algebra or of wild representation type, then even $\operatorname{Add} \mathbf{L} = \operatorname{Add} M$ for any non-zero direct summand M of $\mathbf{L}$, see [71, 3.1] and [70, 6.1].

(4) *The torsion pair* $(\mathcal{Q}, \mathcal{C})$ *generated by* $\mathbf{q}$ is a faithful torsion pair induced by the cotilting module $D({}_R\mathbf{L})$, where ${}_R\mathbf{L} \in R$-Mod is the left version of $\mathbf{L}$. This follows again from Proposition 6.2.

The class $\operatorname{Cogen} D({}_R\mathbf{L}) = \mathcal{C} = \mathbf{q}^o = {}^{\perp}\mathbf{q}$ consists of the modules without indecomposable preinjective summands, and $\mathcal{Q} = \operatorname{Add} \mathbf{q}$ by [80, 3.3].

Notice that the tilting modules $\mathbf{L}$ and $\mathbf{W}$ determine the representation type of R:

Theorem 3.1 ([80, 3.7-3.9], [6, Theorem 18]). *Let R be a (connected) hereditary artin algebra. The following statements are equivalent.*

(1) *R is of tame representation type.*

(2) *The torsion pair $(\mathcal{Q}, \mathcal{C})$ splits.*

(3) *The module $\mathbf{L}$ is noetherian over its endomorphism ring.*

(4) *The class $\operatorname{Add} \mathbf{W}$ is closed under direct products.*

Assume now that R is of tame representation type. Then $\mathbf{W}$ is a cotilting module equivalent to $D({}_R\mathbf{L})$, that is, $\operatorname{Cogen} \mathbf{W} = \mathcal{C}$, see [79].

Moreover, the module $\mathbf{W}$ has an indecomposable decomposition

$$W = G \oplus \bigoplus \{\text{all Prüfer modules } S_\infty\}.$$

Here, G denotes the *generic module*, that is, the unique indecomposable infinite dimensional module, up to isomorphism, having finite length over its endomorphism ring (which is a division ring), or in other words, G is the unique indecomposable torsion-free divisible module, see [80, 5.3 and p.408]. Further, for each quasi-simple module $S \in \mathbf{t}$, we denote by S_m the module of regular length m on the ray

$$S = S_1 \subset S_2 \subset \cdots \subset S_m \subset S_{m+1} \subset \cdots$$

and let $S_\infty = \varinjlim S_m$ be the corresponding *Prüfer module.* The *adic module* $S_{-\infty}$ determined by S is defined dually as the inverse limit along the coray ending at S.

With similar arguments as in Section 3.3 one can now verify that $D({}_R\mathbf{W})$ is not a tilting module. Indeed, $D({}_R\mathbf{W})$ is isomorphic to the direct product of G and of all adic modules $S_{-\infty}$, and condition (T2) fails as $\operatorname{Ext}^1_R(G, {S_{-\infty}}^{(\mathbb{N})}) \neq 0$ by [76].

For more details on tilting and cotilting modules over tame hereditary algebras we refer to Example 6.9 and to Sections 8.2 and 8.5.

Examples of tilting modules of projective dimension bigger than one are discussed in Section 5, after providing some background on the connection between tilting and approximation theory.

4. Tilting and approximations

In this section, we review some fundamental results from approximation theory, and we describe tilting classes in terms of the existence of certain approximations.

4.1. Resolving and coresolving subcategories. Recall that a full subcategory $\mathcal{S} \subseteq \operatorname{Mod} R$ (or $\mathcal{S} \subseteq \operatorname{mod} R$) is *resolving* if it satisfies the following conditions:

(R1) $\mathcal{S}$ contains all (respectively, all finitely generated) projective modules, (R2) $\mathcal{S}$ is closed under direct summands and extensions,

(R3) $\mathcal{S}$ is closed under kernels of epimorphisms.

Coresolving subcategories of Mod$R$ (or of mod$R$) are defined by the dual conditions. For example, mod-R is resolving, cf. [4, 1.1], and $\mathcal{P}$ and $\mathcal{P}^{<\infty}$ are resolving subcategories of Mod-R and mod-R, respectively. Moreover, given any $\mathcal{M} \subseteq$ Mod$R$, the category ${}^{\perp}\mathcal{M}$ is always resolving and $\mathcal{M}^{\perp}$ is coresolving. If $\mathcal{S}$ is a resolving subcategory, then the class $\mathcal{S}^{\perp}$ coincides with

$$\mathcal{S}^{\perp_1} = \{X \in \text{Mod-}R \mid \operatorname{Ext}^1_R(S, X) = 0 \text{ for all } S \in \mathcal{S}\},$$

and the dual property holds true for coresolving subcategories.

4.2. Cotorsion pairs. Cotorsion pairs are the analog of torsion pairs where the functor Hom is replaced by Ext. A pair of classes $(\mathcal{A}, \mathcal{B})$ in Mod-R is a *cotorsion pair* if $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$. Every class of modules $\mathcal{M}$ *generates* a cotorsion pair $(\mathcal{A}, \mathcal{B})$ by setting $\mathcal{B} = \mathcal{M}^{\perp_1}$ and $\mathcal{A} = {}^{\perp_1}(\mathcal{M}^{\perp_1})$. Similarly, the cotorsion pair *cogenerated by* $\mathcal{M}$ is given by $\mathcal{A} = {}^{\perp_1}\mathcal{M}$ and $\mathcal{B} = ({}^{\perp_1}\mathcal{M})^{\perp_1}$.

In a cotorsion pair $(\mathcal{A}, \mathcal{B})$, the class $\mathcal{A}$ is always closed under coproducts and satisfies (R1) and (R2), while $\mathcal{B}$ is closed under products, contains all injective modules and satisfies (R2). Moreover, $\mathcal{A}$ is resolving if and only if $\mathcal{B}$ is coresolving, and this is further equivalent to $\operatorname{Ext}^i_R(A, B) = 0$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, $i \geq 1$. Cotorsion pairs with these equivalent properties are called *hereditary.*

A module M is said to be *filtered* by a class of modules $\mathcal{S}$ (or $\mathcal{S}$*-filtered*) provided that $M = M_\sigma$ is the union of a chain $(M_\alpha \mid \alpha \leq \sigma)$ of submodules such that $M_0 = 0$, $M_\alpha \subseteq M_{\alpha+1}$ for all $\alpha < \sigma$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for all limit ordinals $\alpha \leq \sigma$, and the consecutive factors $M_{\alpha+1}/M_\alpha$, $\alpha < \sigma$, are isomorphic to elements of $\mathcal{S}$.

The following result shows that the class $\mathcal{A}$ in a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is always closed under filtrations.

Lemma 4.1. ([52, 3.1.2]) *Let $A, B \in$ Mod-R. If A is ${}^{\perp_1}B$-filtered, then $A \in {}^{\perp_1}B$.*

Let us now turn to cotorsion pairs providing for approximations.

Lemma 4.2 ([81]). *Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. The following statements are equivalent.*

(1) *For every $M \in$ Mod-R there is a short exact sequence $0 \to M \xrightarrow{f} B \to A \to 0$ with $B \in \mathcal{B}$ and $A \in \mathcal{A}$.*

(2) *For every $M \in$ Mod-R there is a short exact sequence $0 \to B \to A \xrightarrow{g} M \to 0$ with $B \in \mathcal{B}$ and $A \in \mathcal{A}$.*

Under these conditions, the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called complete.

The sequence in 4.2(1) gives rise to a *left $\mathcal{B}$-approximation* (or *$\mathcal{B}$-preenvelope)* of M: every homomorphism $h : M \to B'$ with $B' \in \mathcal{B}$ factors through f. Similarly, the sequence in 4.2(2) gives rise to a *right $\mathcal{A}$-approximation* (or *$\mathcal{A}$-precover*) of M: every homomorphism $h : A' \to M$ with $A' \in \mathcal{A}$ factors through g.

Remark 4.3. (1) [48, 7.2.6] If $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair such that $\mathcal{A}$ is closed under direct limits, then every module $M \in \text{Mod-}R$ has a *minimal* left $\mathcal{B}$-approximation and a *minimal* right $\mathcal{A}$-approximation (also called $\mathcal{B}$-*envelope* and $\mathcal{A}$-*cover*).

(2) By Wakamatsu's Lemma [48, 7.2.3], a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is complete provided that every module $M \in \text{Mod-}R$ has a minimal left $\mathcal{B}$-approximation, or that every module $M \in \text{Mod-}R$ has a minimal right $\mathcal{A}$-approximation.

Cotorsion pairs, introduced by Salce [81] already in 1979, were rediscovered by Eklof and Trlifaj in 2000 with the following fundamental results that led, among other things, to the proof of the existence of flat covers, see e.g. [48, 7.4.4].

Theorem 4.4 ([46, 47]). *Let $\mathcal{M}$ be a class of modules.*

(1) *If the isomorphism classes of $\mathcal{M}$ form a set, then the cotorsion pair generated by $\mathcal{M}$ is complete.*

(2) *If $\mathcal{M}$ consists of pure-injective modules, then the cotorsion pair cogenerated by $\mathcal{M}$ is complete.*

Furthermore, if $(\mathcal{A}, \mathcal{B})$ is the cotorsion pair generated by a set $\mathcal{S} \subset \text{Mod-}R$ containing R, then $\mathcal{A} = {}^{\perp_1}(\mathcal{S}^{\perp_1})$ consists of all direct summands of $\mathcal{S}$-filtered modules [52, 3.2.4].

Given a module M, consider the set $\mathcal{S} = \{\Omega^n(M) \mid n \geq 0\}$ of all its syzygies. Using dimension shifting and the observations in Section 4.1, we see that $M^\perp = \mathcal{S}^{\perp_1}$ and ${}^\perp(M^\perp) = {}^{\perp_1}(\mathcal{S}^{\perp_1})$. Hence we obtain

Corollary 4.5. *Let M be a module. Then $({}^\perp(M^\perp),\, M^\perp)$ is a complete hereditary cotorsion pair.*

4.3. Cotorsion pairs induced by tilting modules.

Given a tilting module T of projective dimension n, let us collect some properties of the complete hereditary cotorsion pair $(\mathcal{A}, \mathcal{B}) = ({}^\perp(T^\perp), T^\perp)$ from Corollary 4.5.

(1) The modules in $\mathcal{A}$ have bounded projective dimension:

$$\text{pdim}\mathcal{A} = \sup\{\,\text{pdim}A \mid A \in \mathcal{A}\} = n.$$

Actually, this is true for any module T with $\text{pdim}\, T = n$. Indeed, let $A \in \mathcal{A}$ and let $M \in \text{Mod-}R$ be an arbitrary module with an injective coresolution $0 \to M \to E_0 \to E_1 \to \ldots$. The injective modules $E_0, E_1, \ldots, E_{n-1}$ are contained in $T^\perp$. By dimension shifting we see that the n-th cosysygy $\Omega^{-n}(M)$ also belongs to $T^\perp$ and we infer $\text{Ext}_R^{i+n}(A, M) = 0$ for all $i > 0$.

(2) Next, we claim that

$$\mathcal{A} \cap \mathcal{B} = \text{Add}\, T.$$

The inclusion $\supset$ is obvious. For $\subset$, observe first that the long exact sequence $0 \to R_R \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \longrightarrow \cdots \longrightarrow T_{r-1} \xrightarrow{f_r} T_r \to 0$ with $T_0, \ldots, T_r \in \text{Add}\, T$

from condition (T3) yields a kernel $\operatorname{Ker} f_i \in \mathcal{A}$ for each $1 \leq i \leq r$. In particular, $0 \to R \xrightarrow{f_0} T_0 \to \operatorname{Ker} f_1 \to 0$ is a short exact sequence as in 4.2(1), showing that f_0 is a left $\mathcal{B}$-approximation of R. This implies that for every $X \in \mathcal{B}$, any epimorphism $R^{(I)} \to X$ factors through ${f_0}^{(I)}$, and we infer $\mathcal{B} \subset \operatorname{Gen} T$.

So, if $X \in \mathcal{B}$ and $g : T' \to X$ is a right $\operatorname{Add} T$-approximation (given for instance by the codiagonal map $g : T^{(I)} \to X$ induced by all homomorphisms in $I = \operatorname{Hom}_R(T, X)$), then g must be surjective, and $0 \to K \to T' \xrightarrow{g} X \to 0$ is a sequence as in 4.2(2). In fact, by applying $\operatorname{Hom}_R(T, -)$, we obtain a long exact sequence $\cdots \to \operatorname{Hom}_R(T, T') \xrightarrow{\operatorname{Hom}_R(T,g)} \operatorname{Hom}_R(T, X) \to \operatorname{Ext}^1_R(T, K) \to \operatorname{Ext}^1_R(T, T') = 0$ showing that $K \in \mathcal{B}$.

Now, if we assume that $X \in \mathcal{A} \cap \mathcal{B}$, then id_X factors through the right $\mathcal{A}$-approximation g, so g is a split epimorphism and $X \in \operatorname{Add} T$.

(3) Finally, we conclude that a module X belongs to $\mathcal{B}$ if and only if it has an $\operatorname{Add} T$-resolution, that is, a long exact sequence

$$\ldots \longrightarrow T'_m \xrightarrow{g_m} \ldots \xrightarrow{g_1} T'_0 \xrightarrow{g_0} X \longrightarrow 0$$

with all $T'_i \in \operatorname{Add} T$. Indeed, the only-if-part follows by iterating the construction of the right $\operatorname{Add} T$-approximation above. For the if-part, note that by dimension shifting $\operatorname{Ext}^i_R(A, X) \cong \operatorname{Ext}^{i+n}_R(A, \operatorname{Ker} g_{n-1})$ for all $i > 0$ and all $A \in \mathcal{A}$. So $\operatorname{pdim} \mathcal{A} = n$ implies $X \in \mathcal{A}^{\perp}$.

In fact, the properties of $(\mathcal{A}, \mathcal{B})$ established above characterize the cotorsion pairs induced by tilting modules.

Theorem 4.6 ([2],[92]). *Let $\mathcal{B} \subseteq$ Mod-R and $\mathcal{A} = {}^{\perp}\mathcal{B}$. The following statements are equivalent.*

(1) *$\mathcal{B}$ is a tilting class.*

(2) *$(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair such that $\mathcal{A} \subset \mathcal{P}$ and $\mathcal{A} \cap \mathcal{B}$ is closed under coproducts.*

(3) *$(\mathcal{A}, \mathcal{B})$ is a hereditary cotorsion pair such that $\mathcal{A} \subset \mathcal{P}$ and $\mathcal{B}$ is closed under coproducts.*

Proof. (Sketch) As verified above, (1) implies that $\mathcal{A} \subset \mathcal{P}$ and that $\mathcal{A} \cap \mathcal{B}$ and $\mathcal{B}$ are closed under coproducts.

For (2)⇒(1), observe first that since $\mathcal{A} \subset \mathcal{P}$ and $\mathcal{A}$ is closed under coproducts, the projective dimensions attained on $\mathcal{A}$ are bounded by some $n \in \mathbb{N}$. Moreover, the completeness of $(\mathcal{A}, \mathcal{B})$ allows an iteration of left $\mathcal{B}$-approximations yielding a long exact sequence

$$0 \to R \xrightarrow{f_0} B_0 \xrightarrow{f_1} B_1 \to \ldots$$

with $B_i \in \mathcal{B}$ and $A_{i+1} = \operatorname{Coker} f_i \in \mathcal{A}$ for all i. Since $R \in \mathcal{A}$, we even have $B_i \in \mathcal{A} \cap \mathcal{B}$. But also $A_n \in \mathcal{A} \cap \mathcal{B}$, because $\operatorname{Ext}^i_R(A, A_n) \cong \operatorname{Ext}^{i+n}_R(A, R) = 0$ for all $A \in \mathcal{A}$ and $i > 0$. The module

$$T = B_0 \oplus \ldots \oplus B_{n-1} \oplus A_n$$

then satisfies $\operatorname{Add} T \subset \mathcal{A} \cap \mathcal{B}$, and one checks that it is a tilting module with tilting class $\mathcal{B}$.

For (3)⇒(2), one has to prove completeness. In fact, by refining set-theoretic methods originally developed by Eklof, Fuchs, Hill, and Shelah, it is proved in [92] that $\mathcal{A}$ is *deconstructible*, i.e. the modules in $\mathcal{A}$ are filtered by "small" modules from $\mathcal{A}$. Here "small" means that the modules admit a projective resolution consisting of projectives with a generating set of bounded cardinality, and the bound can be chosen as the least infinite cardinal κ such that every right ideal of R has a generating set of cardinality at most κ. Now the "small" modules from $\mathcal{A}$ form a set, and it follows from Lemma 4.1 that the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is generated by this set. So $(\mathcal{A}, \mathcal{B})$ is complete by Theorem 4.4. □

There is a dual characterization of cotilting classes, see [2, 4.2], and [84, 2.4]. Notice that these results generalize classical results for finitely generated tilting and cotilting modules over artin algebras due to Auslander and Reiten [15].

We obtain the following consequence for 1-tilting modules.

Corollary 4.7 ([13]). *The tilting torsion pairs are precisely the torsion pairs $(\mathcal{T}, \mathcal{F})$ in* Mod-R *such that for every R-module M (or equivalently, for $M = R$) there is a short exact sequence $0 \to M \to B \to A \to 0$ with $B \in \mathcal{T}$ and $A \in {}^{\perp}\mathcal{T}$.*

For a dual version of this result, see [13, 2.5] or Corollary 6.6.

5. Further examples

Here is an immediate application of Theorem 4.6.

Corollary 5.1. *Let $\mathcal{S}$ be a set of modules with $\operatorname{pdim} \mathcal{S} \leq n$. If $\mathcal{S}^{\perp}$ is closed under coproducts (for instance, if $\mathcal{S} \subset \text{mod-}R$), then it is an n-tilting class.*

5.1. Complements. As a consequence, we obtain the existence of complements to partial tilting modules. Notice that this fails when restricting the attention to finitely generated modules, as shown by the example in [78].

We say that a module $M \in \text{Mod-}R$ is a *partial tilting module* if it satisfies conditions (T1) and (T2). Dually, M is a *partial cotilting module* if it satisfies conditions (C1) and (C2).

Theorem 5.2 ([3]). *(1) Let $M \in \text{Mod-}R$ be a partial tilting module. There is $N \in \text{Mod-}R$ such that $T = M \oplus N$ is a tilting module with tilting class $T^{\perp} = M^{\perp}$ if and only if $M^{\perp}$ is closed under coproducts.*

(2) Let M be a pure-injective partial cotilting module. There is $N \in \text{Mod-}R$ such that $C = M \oplus N$ is a cotilting module with cotilting class ${}^{\perp}C = {}^{\perp}M$ if and only if ${}^{\perp}M$ is closed under products.

Proof. (Sketch of the if-part.) (1) The class $\mathcal{B} = M^{\perp}$ is a tilting class by Corollary 5.1. A tilting module T with $\mathcal{B} = T^{\perp}$ is obtained by taking a sequence $0 \to R \to N \to A \to 0$ with $N \in \mathcal{B}$ and $A \in \mathcal{A}$ and setting $T = M \oplus N$. Statement (2) is proven dually by employing Theorem 4.4(2). □

In particular, it follows that every partial tilting module $M \in$ mod-R has a complement N as in 5.2(1). Over an artin algebra R, also the dual result holds true. Indeed, every $M \in$ mod-R is pure-injective, and using the Auslander-Reiten formula, it is shown in [65, 6.4] that ${}^{\perp}M$ is closed under products.

Corollary 5.3 ([65]). *If R is an artin algebra, then every partial cotilting module $M \in$ mod-R has a complement N as in 5.2(2).*

In general, however, the complement N to a partial (co)tilting module $M \in$ mod-R will be infinite dimensional.

For the case of a hereditary artin algebra, see also [62].

5.2. A tilting module determining the finitistic dimension. In this section, let R be a right noetherian ring. The big and the little *finitistic dimension* of R are defined as $\operatorname{Findim} R = \operatorname{pdim} \mathcal{P}$ and $\operatorname{findim} R = \operatorname{pdim} \mathcal{P}^{<\infty}$. A long-standing open problem asks whether $\operatorname{findim} R < \infty$ for any artin algebra R. We are going to phrase this problem in terms of tilting modules.

Given a class of R-modules $\mathcal{X}$, we denote by $\mathcal{X}^{\wedge}$ the class of R-modules M admitting a long exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ with $X_0, \ldots, X_n \in \mathcal{X}$. The minimal length n of such $\mathcal{X}$-resolution is the *$\mathcal{X}$-resolution dimension* $\operatorname{resdim}_{\mathcal{X}}(M)$ of M.

Let now $(\mathcal{A}, \mathcal{B})$ be the complete hereditary cotorsion pair generated by the resolving subcategory $\mathcal{P}^{<\infty} \subset$ mod-R.

Theorem 5.4 ([14, 9]). *Let R be a right noetherian ring. The little finitistic dimension* $\operatorname{findim} R$ *is finite if and only if $\mathcal{B}$ is a tilting class. If T is a tilting module with $\mathcal{B} = T^{\perp}$, then*

(1) $\operatorname{findim} R = \operatorname{pdim} T = \operatorname{pdim} \mathcal{A}$,

(2) $\operatorname{Findim} R = \operatorname{pdim}(\operatorname{Add} T)^{\wedge}$,

(3) $\operatorname{Findim} R - \operatorname{findim} R \leq \operatorname{resdim}_{\mathcal{A}} \mathcal{P}$.

Proof. (Sketch.) (1) $\operatorname{findim} R = n < \infty$ if and only if $\operatorname{pdim} \mathcal{A} = n$, which means by Theorem 4.6 that there is a tilting module T of projective dimension n with $\mathcal{B} = T^{\perp}$.

For (2), one proves that $(\operatorname{Add} T)^{\wedge} = \mathcal{P} \cap \mathcal{B}$. Then $\operatorname{Findim} R \geq \operatorname{pdim}(\operatorname{Add} T)^{\wedge}$. For the reverse inequality, recall that for every $M \in \mathcal{P}$ there is an exact sequence $0 \to M \to B \to A \to 0$ where $B \in \mathcal{B}$ and $A \in \mathcal{A}$. Since $\mathcal{A} \subset \mathcal{P}$, we even have $B \in \mathcal{P} \cap \mathcal{B}$. Moreover, $\operatorname{pdim} A \leq n = \operatorname{pdim} T \leq \operatorname{pdim}(\operatorname{Add} T)^{\wedge}$. So, $\operatorname{pdim} M \leq \operatorname{pdim}(\operatorname{Add} T)^{\wedge}$.

(3) is a consequence of the inequality

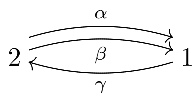
$$(*) \qquad \operatorname{resdim}_{\mathcal{A}}(\mathcal{P}) \leq \operatorname{Findim} R \leq \operatorname{pdim} \mathcal{A} + \operatorname{resdim}_{\mathcal{A}}(\mathcal{P})$$

obtained in [9, 4.3] for any complete hereditary cotorsion pair with $\mathcal{A} \subset \mathcal{P}$. □

Examples by Huisgen-Zimmermann and by Smalø show that the big and the little finitistic dimension need not coincide and that their difference can be arbitrarily large [58, 87]. From Theorem 5.4, we see that equality of the two dimensions holds true if and only if $\operatorname{pdim} T = \operatorname{pdim}(\operatorname{Add} T)^{\wedge}$. In particular, we recover the following results from [15] and [59].

Corollary 5.5 ([14]). *Assume that R is an artin algebra. There is a finitely generated tilting module T such that $\mathcal{B} = T^{\perp}$ if and only if the category $\mathcal{P}^{<\infty}$ is contravariantly finite in* mod-R. *In this case,* $(\operatorname{Add} T)^{\wedge} = \operatorname{Add} T$, *and* $\mathcal{A} = \mathcal{P}$. *In particular* $\operatorname{Findim} R = \operatorname{findim} R < \infty$.

Example 5.6. The inequality in Theorem 5.4(3) can be strict. For example, if R is the finite-dimensional algebra from [60] given by the quiver



with the relations $\alpha\gamma = \beta\gamma = \gamma\alpha = 0$, then $\operatorname{Findim} R = \operatorname{findim} R = 1$, but $\mathcal{P}^{<\infty}$ is not contravariantly finite in mod-R, and $\operatorname{resdim}_{\mathcal{A}} \mathcal{P} = 1$, see [9, 4.6]. The large 1-tilting module T from Theorem 5.4 is computed explicitly in [91].

Example 5.7 ([4]). Assume R is an (Iwanaga–)Gorenstein ring, i. e. R is noetherian, and the injective dimensions $\operatorname{idim} R_R$ and $\operatorname{idim}_R R$ are finite. Then $\operatorname{idim} R_R = \operatorname{idim}_R R = \operatorname{findim} R$, and the tilting module from Theorem 5.4 is $T = I_0 \oplus \ldots \oplus I_n$ where $0 \to R \to I_0 \to \ldots \to I_n \to 0$ is a minimal injective coresolution of R_R. In particular $(\operatorname{Add} T)^{\wedge} = \operatorname{Add} T$, and $\operatorname{Findim} R = \operatorname{findim} R$. In general (e.g. when R is a commutative Gorenstein ring), T is a large tilting module.

Theorem 5.8 ([9]). *If R is right noetherian, then for every tilting module T*

$$\operatorname{Findim} R \leq \operatorname{pdim} T + \operatorname{idim} T$$

Example 5.7 shows that the inequality above is sharp. Indeed, $\operatorname{Findim} R = \operatorname{pdim} T = \operatorname{pdim} T + \operatorname{idim} T$ in this case.

5.3. Limits of tilting or cotilting modules. It follows from Theorem 4.6 (or its dual version) that the intersection of a family of (co)tilting classes induced by (co)tilting modules of bounded projective (respectively, injective) dimension is again a (co)tilting class. A corresponding (co)tilting module can be constructed explicitly when the (co)tilting classes form a decreasing sequence.

Theorem 5.9 ([31]). *Let R be an artin algebra and let $(C_n)_{n\in\mathbb{N}}$ be a family of cotilting modules in* R-mod *with bounded injective dimension such that ${}^{\perp}C_n \supset {}^{\perp}C_{n+1}$ for all $n \in \mathbb{N}$. Then the cotilting class $\bigcap_{n\in\mathbb{N}} {}^{\perp}C_n$ is induced by a cotilting module C which can be realized as an inverse limit of an inverse system of cotilting modules equivalent to the C_n.*

The case of a decreasing sequence of tilting classes is treated in [28]. As shown in [31], even if starting with finite dimensional (co)tilting modules, in general one will get a large (co)tilting module. This will be illustrated in Example 6.9.

6. Large versus small

After exhibiting several examples of large tilting modules, let us now discuss differences and similarities with the classical setup. On one hand, we will see that tilting modules, even if infinite dimensional, still correspond to categories of finitely generated modules. On the other hand, a substantial difference will appear when considering tilting functors. In fact, the tilting functors given by a large tilting module induce localizations of derived categories rather than derived equivalences.

6.1. Resolving subcategories of mod-$\boldsymbol{R}$**.** The first result asserts that, surprisingly, every large tilting module is determined by a set of finitely presented modules. This has many important consequences as we will see below, and it is the key to the classification results in Section 8.

Theorem 6.1 ([21, 22, 92, 25]). *Every tilting class is of the form $\mathcal{B} = \mathcal{S}^{\perp_1}$ for some set $\mathcal{S} \subset$* mod-R *with* $\operatorname{pdim} \mathcal{S} < \infty$*. If $(\mathcal{A}, \mathcal{B})$ is the corresponding cotorsion pair, then one can choose the resolving subcategory $\mathcal{S} = \mathcal{A} \cap$* mod-$R$*.*

Let us point out that the statement can be rephrased as follows: every tilting class is a *definable* class, i.e. it is closed under direct limits, direct products and pure submodules.

Proof. (Sketch.) Theorem 6.1 was proved by Bazzoni, Eklof, Herbera, Sťovíček, and Trlifaj in a series of four papers. We briefly sketch the main steps of the proof and refer to [52, 5.2] for more details.

In a first step [21, 92], one uses set-theoretic methods as in the proof of the implication (3)⇒(2) in Theorem 4.6 to deconstruct the class $\mathcal{A}$ into countably presented modules, that is, one proves that the modules in $\mathcal{A}$ are $\mathcal{S}'$-filtered, where $\mathcal{S}'$ is the class of modules from $\mathcal{A}$ that admit a projective resolution with countably generated projectives. This implies that $\mathcal{B} = (\mathcal{S}')^{\perp_1}$ by Lemma 4.1.

In the next step [22], one shows that for a countably presented module A the vanishing of $\operatorname{Ext}^1_R(A, B^{(\mathbb{N})})$ can be translated into a Mittag-Leffler condition, and this Mittag-Leffler condition is preserved when passing to pure submodules of B. This implies that $\mathcal{B}$ is closed under pure submodules. But then $\mathcal{B}$ is also closed under direct limits, since it is coresolving and closed under coproducts. Further,

$\mathcal{B}$ is closed under direct products because the functors $\mathrm{Ext}^i_R(T,-)$ commute with products. So, we conclude that $\mathcal{B}$ is definable.

The last step is devoted to proving $\mathcal{B} = \mathcal{S}^{\perp_1}$. Notice that also $\mathcal{S}^{\perp_1}$ is definable, and two definable classes coincide if and only if they have the same pure-injective modules, cf. [43, 2.3 and 2.5]. As $\mathcal{B} \subset \mathcal{S}^{\perp_1}$, it remains to prove that every pure-injective module $M \in \mathcal{S}^{\perp_1}$ satisfies $\mathrm{Ext}^1_R(A,M) = 0$ for all $A \in \mathcal{S}'$. Now one shows [21, 25] that $A = \varinjlim S_n$ with $S_n \in \mathcal{S}$. The claim then follows from a well-known result by Auslander stating that $\mathrm{Ext}^1_R(\varinjlim S_n, M) \cong \varprojlim \mathrm{Ext}^1_R(S_n, M)$ when M is pure-injective. □

The dual of an n-tilting module is always an n-cotilting module whose cotilting class is then also determined by a set of finitely presented modules. Indeed, employing the Ext-Tor-relations [48, 3.2.1 and 3.2.13], one obtains

Proposition 6.2. ([4, 2.3]) *If T is an n-tilting right R-module and $\mathcal{S} \subset$ mod-R satisfies $T^\perp = \mathcal{S}^{\perp_1}$, then $C = D(T)$ is an n-cotilting left R-module with cotilting class*

$$\begin{aligned} {}^\perp C &= \{M \in R\text{-Mod} \mid D(M) \in T^\perp\} = \\ &= \{M \in R\text{-Mod} \mid \mathrm{Tor}_1^R(S,M) = 0 \text{ for all } S \in \mathcal{S}\} = {}^{\perp_1}D\mathcal{S} \end{aligned}$$

where $D\mathcal{S} = \{D(S) \mid S \in \mathcal{S}\}$.

Remark 6.3. In general not all cotilting modules arise in this way. Counterexamples are constructed in [17, 19] over valuation domains. Over commutative noetherian rings, however, every cotilting module is equivalent to the dual of a tilting module, see [10]. We will show in Corollary 6.10 that the same holds true for 1-cotilting left modules over left noetherian rings.

If $\mathcal{S}$ is a resolving subcategory of mod-R with $\mathrm{pdim}\,\mathcal{S} \le n$, then $\mathcal{B} = \mathcal{S}^{\perp_1} = \mathcal{S}^\perp$ is an n-tilting class by Corollary 5.1. Theorem 6.1 now shows that all tilting classes arise in this way.

Corollary 6.4 ([4]). *For every $n \in \mathbb{N}$ there is a bijection*

$$\{\textit{resolving } \mathcal{S} \subset \text{mod-}R \textit{ with } \mathrm{pdim}\,\mathcal{S} \le n\} \longrightarrow \{n\textit{-tilting classes in } \text{Mod-}R\}$$
$$\mathcal{S} \mapsto \mathcal{S}^{\perp_1}$$

Moreover, there is an injective map

$$\{\textit{resolving } \mathcal{S} \subset \text{mod-}R \textit{ with } \mathrm{pdim}\,\mathcal{S} \le n\} \longrightarrow \{n\textit{-cotilting classes in } R\textit{-Mod}\}$$
$$\mathcal{S} \mapsto {}^{\perp_1}D\mathcal{S}$$

which is a bijection if R is commutative noetherian, or if $n = 1$ and R is left noetherian.

Despite the lack of symmetry between tilting and cotilting modules discussed above, also cotilting classes are definable due to [52, 4.3.23], or to the following result of Bazzoni and Šťovíček.

Theorem 6.5 ([16, 90]). *Every cotilting module is pure-injective.*

In particular, cotilting classes are closed under direct limits. This property already allows to detect cotilting torsion pairs, thanks to the dual version of Corollary 4.7 and to an existence result for minimal right approximations given in [19, 2.6].

Corollary 6.6. ([19, 6.1]) *The cotilting torsion pairs are precisely the faithful torsion pairs* $(\mathcal{T}, \mathcal{F})$ *in* Mod-R *where* $\mathcal{F}$ *is closed under direct limits.*

6.2. Torsion pairs in mod-*R*. The next result asserts that over a noetherian ring R, the 1-cotilting classes are in bijection with the faithful torsion pairs in R-mod. A dual result holds true for 1-tilting classes over artin algebras. The following preliminary result is needed.

Lemma 6.7 ([42]). *Let* R *be a right noetherian ring.*

(1) *If* $(\mathcal{T}, \mathcal{F})$ *is a torsion pair in* Mod-R*, then its restriction* $(\mathcal{T} \cap \text{mod-}R, \mathcal{F} \cap \text{mod-}R)$ *to* mod-R *is a torsion pair in* mod-R.

(2) *If* $(\mathfrak{t}, \mathfrak{f})$ *is a torsion pair in* mod-R*, then its limit closure* $(\varinjlim \mathfrak{t}, \varinjlim \mathfrak{f})$ *in* Mod-R *is a torsion pair in* Mod-R*, which coincides with the torsion pair generated by* $\mathfrak{t}$.

Lemma 6.7(2) together with Corollary 6.6 yield that the limit closure of a faithful torsion pair in R-mod is a cotilting torsion pair in R-Mod, and all cotilting torsion pairs arise in this way. Over an artin algebra, the Auslander-Reiten formula allows to prove that a cofaithful torsion pair $(\mathfrak{t}, \mathfrak{f})$ in mod-R produces a tilting class ${}^o\mathfrak{f}$ in Mod-R, and applying Theorem 6.1 one can show that all tilting torsion pairs have this form.

Theorem 6.8 ([29, 61]). (1) *If* R *is left noetherian, there is a bijection*

$$\{\textit{cotilting torsion pairs in R-Mod}\} \leftrightarrow \{\textit{faithful torsion pairs in R-mod}\}$$
$$(\mathcal{T}, \mathcal{F}) \mapsto (\mathcal{T} \cap R\text{-mod}, \mathcal{F} \cap R\text{-mod})$$

The inverse map assigns to $(\mathfrak{t}, \mathfrak{f})$ *the torsion pair in* R-Mod *generated by* $\mathfrak{t}$.

(2) *If* R *is an Artin algebra, there is a bijection*

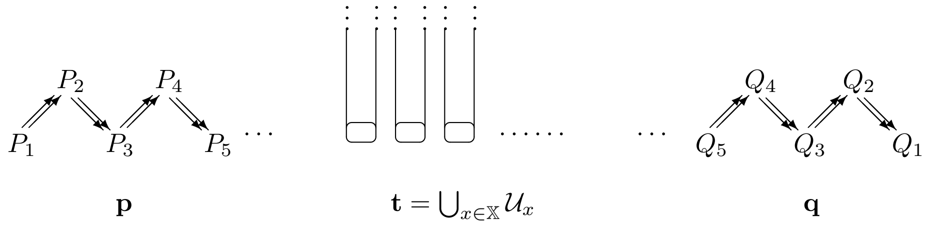
$$\{\textit{tilting torsion pairs in } \text{Mod-}R\} \leftrightarrow \{\textit{cofaithful torsion pairs in } \text{mod-}R\}$$
$$(\mathcal{T}, \mathcal{F}) \mapsto (\mathcal{T} \cap \text{mod-}R, \mathcal{F} \cap \text{mod-}R)$$

The inverse map assigns to $(\mathfrak{t}, \mathfrak{f})$ *the torsion pair in* Mod-R *cogenerated by* $\mathfrak{f}$.

Example 6.9. Every cofaithful torsion pair in mod-R over an artin algebra R is thus represented by a tilting module which is unique up to equivalence. Let us illustrate this for the Kronecker algebra, that is, is the path algebra of the quiver

$$\bullet \rightrightarrows \bullet$$

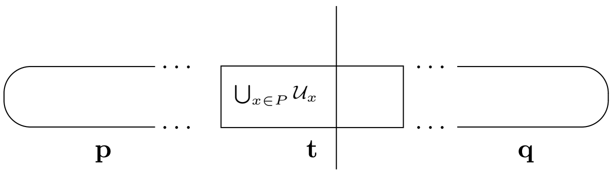
Its Auslander-Reiten quiver has the form



Here is a classification of the cofaithful torsion pairs in mod-R. It yields a classification of all tilting R-modules up to equivalence.

Cofaithful torsion pairs in mod-R	Tilting modules
$(\,\mathrm{add}(\{P_i \mid i > n\} \cup \mathbf{t} \cup \mathbf{q}),\ \mathrm{add}(P_1 \oplus \ldots \oplus P_n)\,),\ n \geq 0,$	$P_{n+1} \oplus P_{n+2}$
$(\,\mathrm{add}(\mathbf{t} \cup \mathbf{q}),\ \mathrm{add}(\mathbf{p})\,)$	$\mathbf{L}$
$(\,\mathrm{add}(\bigcup_{x\notin P} \mathcal{U}_x \cup \mathbf{q}),\ \mathrm{add}(\mathbf{p} \cup \bigcup_{x\in P} \mathcal{U}_x)\,),\ \emptyset \neq P \subsetneq \mathbb{X},$	T_P
$(\,\mathrm{add}(\mathbf{q}),\ \mathrm{add}(\mathbf{p} \cup \mathbf{t})\,)$	$\mathbf{W}$
$(\,\mathrm{add}(Q_1 \oplus \ldots \oplus Q_n),\ \mathrm{add}(\mathbf{p} \cup \mathbf{t} \cup \{Q_i \mid i > n\})\,),\ n \geq 2,$	$Q_{n-1} \oplus Q_n$

The modules $\mathbf{L}$ and $\mathbf{W}$ are the large tilting modules from Example 3.4. Further, for every subset $\emptyset \neq P \subsetneq \mathbb{X}$, we obtain a large tilting module T_P corresponding to the torsion pair below. The shape of T_P will be discussed in Section 8.2.



Observe that the tilting classes $\mathrm{Gen}(P_{n+1} \oplus P_{n+2}) = \mathrm{Gen}\, P_{n+1} = {}^{o}P_n$ form a decreasing sequence with intersection ${}^{o}\mathbf{p} = \mathrm{Gen}\,\mathbf{L}$, cf. Section 5.3.

As a consequence of Theorem 6.8, we can now prove the following.

Corollary 6.10 ([10]). *Over a left noetherian ring R, every 1-cotilting left R-module is equivalent to the dual of a tilting module.*

Proof. Let C be a 1-cotilting module with cotilting class $\mathcal{F}$. By Theorem 6.8 there is a set $\mathfrak{t} \subset R$-mod such that $\mathcal{F} = \mathfrak{t}^{o}$, and $R \in \mathcal{F}$. Then the modules $U \in \mathfrak{t}$ satisfy $\mathrm{Hom}_R(U, R) = 0$. Taking the *Auslander-Bridger transpose* $\mathrm{Tr}\, U$ of U, we see that $\mathrm{pdim}\, \mathrm{Tr}\, U \leq 1$ and that $\mathrm{Hom}_R(U, -) \cong \mathrm{Tor}_1^R(\mathrm{Tr}\, U, -)$ are isomorphic functors, see [10, 2.9]. By Corollary 5.1, the class $\mathcal{S} = \{\mathrm{Tr}\, U \mid U \in \mathfrak{t}\} \subset$ mod-R induces a 1-tilting class $\mathcal{S}^{\perp}$. If T is a tilting module with $T^{\perp} = \mathcal{S}^{\perp}$, then it follows from Proposition 6.2 that ${}^{\perp}D(T) = \{M \in R\text{-Mod} \mid \mathrm{Tor}_1^R(S, M) = 0 \text{ for all } S \in \mathcal{S}\} = \mathcal{F}$, and C is equivalent to $D(T)$. □

6.3. Tilting functors given by large tilting modules. Let us now turn to an important difference between large and classical tilting modules.

Let T_R be a tilting module and let $S = \operatorname{End} T_R$. When $T \in \text{mod-}R$, the tilting functors $j^* = - \otimes^{\mathbf{L}}_S T$ and $j_* = \mathbf{R}\operatorname{Hom}_R(T, -)$ define an equivalence between $\mathcal{D}(\text{Mod-}R)$ and $\mathcal{D}(\text{Mod-}S)$ by a well-known result of Happel [55].

This is no longer true for large tilting modules. However, a weaker result can be proven under the assumption that T satisfies

(T3') There exists a long exact sequence $0 \to R_R \to T_0 \to \cdots \to T_r \to 0$ with $T_i \in \operatorname{add} T$ for each $0 \le i \le r$.

A tilting module with this property is said to be *good*. Classical tilting modules are good, and it is easy to see that every tilting module is equivalent to a good one. The advantage of good tilting modules is that ${}_S T$ is then a finitely presented partial tilting module.

When T is good, the triangle functors $j^* = - \otimes^{\mathbf{L}}_S T$ and $j_* = \mathbf{R}\operatorname{Hom}_R(T, -)$ form an adjoint pair such that the counit adjunction morphism $j^* j_* \to \operatorname{Id}_{\mathcal{D}(\text{Mod}-R)}$ is invertible, and therefore j^* is fully faithful and j_* dense. Moreover, $D(\text{Mod-}R)$ is equivalent to the Verdier quotient of $\mathcal{D}(\text{Mod-}S)$ with respect to the kernel of the functor j^*. Notice that this kernel is zero if and only if $T \in \text{mod-}R$. For details we refer to [20, 23].

As shown in [95, 32, 33], the functors j^* and j_* actually belong to a recollement of derived categories in the sense of [26].

Theorem 6.11 ([20, 23, 95, 32, 33]). *Let T_R be a good n-tilting module, and $S = \operatorname{End} T_R$. Then there are a dg-algebra $\tilde{S}$ and a recollement of derived categories*

$$\mathcal{D}(\text{Mod-}\tilde{S}) \quad \begin{array}{c} \xleftarrow{\ i^*\ } \\ \xrightarrow{i_* = i_!} \\ \xleftarrow[\ i^!\ ]{} \end{array} \quad \mathcal{D}(\text{Mod-}S) \quad \begin{array}{c} \xleftarrow{\ j_!\ } \\ \xrightarrow{j^! = j^*} \\ \xleftarrow[\ j_*\ ]{} \end{array} \quad \mathcal{D}(\text{Mod-}R)$$

If $n = 1$, then $\tilde{S}$ is an ordinary ring that can be computed as universal localisation of S (as defined in Theorem 7.2).

Tilting functors given by large tilting modules thus give rise to new derived categories and to new phenomena. This was exploited in [32] to show that stratifications of derived categories in general are not unique, that is, they do not satisfy a Jordan-Hölder theorem.

For generalizations of Theorem 6.11 we refer to [24, 75].

7. Tilting modules arising from localization.

7.1. Ring epimorphisms. In order to find the shape of the tilting modules T_P from Example 6.9, we need a construction of tilting modules that is inspired by the tilting module $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ of Example 3.3. In fact, following the same pattern, we will obtain a big supply of tilting modules.

Recall that a ring homomorphism $\lambda : R \to S$ is said to be a *ring epimorphism* if it is an epimorphism in the category of rings, that is, whenever two ring homomorphisms $\varphi_i : S \to S_i$, $i = 1, 2$, satisfy $\varphi_1\lambda = \varphi_2\lambda$, then $\varphi_1 = \varphi_2$. Equivalently, $\lambda : R \to S$ induces a *full* embedding Mod-$S \hookrightarrow$ Mod-R.

Notice that every surjective ring homomorphism is an epimorphism, but the converse is not true, see for example $\mathbb{Z} \hookrightarrow \mathbb{Q}$.

A ring epimorphism $\lambda : R \to S$ is a *homological ring epimorphism* if it also induces a full embedding $\mathcal{D}(\text{Mod-}S) \hookrightarrow \mathcal{D}(\text{Mod-}R)$ at the level of derived categories. As shown in [51, 4.4], this can be expressed by the condition

$$\operatorname{Ext}^n_R(M, N) \cong \operatorname{Ext}^n_S(M, N) \text{ for all } n \in \mathbb{N} \text{ and } M, N \in \text{Mod-}S \tag{1}$$

or, equivalently, by the condition

$$\operatorname{Tor}^R_n(S, S) = 0 \text{ for all } n \in \mathbb{N}. \tag{2}$$

Assume now that $\lambda : R \to S$ is an injective ring epimorphism such that $\operatorname{Tor}^R_1(S, S) = 0$ and S_R is an R-module of projective dimension at most one.

By condition (2) above, λ is a homological ring epimorphism. In particular $\operatorname{Ext}^i_R(S, S^{(I)}) \cong \operatorname{Ext}^i_S(S, S^{(I)}) = 0$ for all $i > 0$ and all sets I, and as $\operatorname{pdim} S_R \le 1$, it follows $\operatorname{Gen} S_R \subset S^\perp$.

Consider now the short exact sequence

$$0 \to R \overset{\lambda}{\to} S \to S/R \to 0.$$

For $M \in$ Mod-R we obtain a long exact sequence

$$0 \to \operatorname{Hom}_R(S/R, M) \to \operatorname{Hom}_R(S, M) \overset{\operatorname{Hom}_R(\lambda, M)}{\longrightarrow} \operatorname{Hom}_R(R, M) \to \\ \to \operatorname{Ext}^1_R(S/R, M) \to \operatorname{Ext}^1_R(S, M) \to 0$$

The image of $\operatorname{Hom}_R(\lambda, M)$ under the identification $\operatorname{Hom}_R(R, M) \cong M$ is precisely the trace of S_R in M. Thus $M \in \operatorname{Gen} S_R$ if and only if $\operatorname{Ext}^1_R(S/R, M) = 0$. We conclude that the R-module

$$T = S \oplus S/R$$

satifies $\operatorname{Gen} T = \operatorname{Gen} S_R = (S/R)^\perp = T^\perp$ and is therefore a 1-tilting module.

Theorem 7.1 ([11]). (1) *If $\lambda : R \to S$ is an injective ring epimorphism such that $\operatorname{Tor}^R_1(S, S) = 0$ and $\operatorname{pdim} S_R \le 1$, then $S \oplus S/R$ is a 1-tilting R-module.*

(2) *A tilting R-module T is equivalent to a tilting module $S \oplus S/R$ as in (1) if and only if there is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ with $T_0, T_1 \in \operatorname{Add} T$ and $\operatorname{Hom}_R(T_1, T_0) = 0$.*

Proof. (Sketch of the if-part in (2).) Denote by $\mathcal{X} = T_1{}^o \cap T_1^\perp$ the *perpendicular category* of T_1. By [50] (see also [51, Proposition 3.8]) there is a ring epimorphism $\lambda : R \to S$ such that the category Mod-S, when viewed as a full subcategory of Mod-R, is equivalent to $\mathcal{X}$. More precisely, the inclusion functor $\iota : \mathcal{X} \to$ Mod-R

has a left adjoint $\ell : \text{Mod-}R \to \mathcal{X}$, and $\lambda : R \to S = \text{End}(\ell(R))$ is defined by the assignment $\lambda(r) = \ell(m_r)$, where $m_r : R \to R$ denotes the left multiplication with the element r.

Since T_0 belongs to $\mathcal{X}$, we see that the unit adjunction morphism $R \to \iota\ell(R)$ is given by the map f in the exact sequence $0 \to R \overset{f}{\to} T_0 \to T_1 \to 0$, and from the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & T_0 \\ {\scriptstyle m_r}\downarrow & & \downarrow{\scriptstyle \lambda(r)} \\ R & \xrightarrow{f} & T_0 \end{array}$$

we infer that $\lambda : R \to S$ is injective. Indeed, $\lambda(r) = 0$ implies $f\, m_r = 0$, hence $m_r = 0$ and $r = 0$. Moreover, $S_R \cong T_0$ has projective dimension at most one, and λ is even a homological ring epimorphism by [51, Corollary 4.8]. It follows from (1) that $S \oplus S/R$ is a tilting module with tilting class $\text{Gen}(S_R) = \text{Gen}(T_0)$. Moreover, the map $f : R \to T_0$ is a minimal left $T^{\perp}$-approximation of R, and as in Section 4.3, we deduce $\text{Gen}\, T_0 = T^{\perp}$. □

7.2. Universal localization. The technique of universal localization developed by Cohn and Schofield provides a large supply of ring epimorphisms as in Section 7.1.

Theorem 7.2 ([85]). *For any set of morphisms Σ between finitely generated projective right R-modules there is a ring homomorphism $\lambda\colon R \to R_\Sigma$ such that*

(1) λ *is* Σ-inverting: *if $\alpha\colon P \to Q$ belongs to Σ, then the R_Σ-homomorphism $\alpha \otimes_R 1_{R_\Sigma}\colon P \otimes_R R_\Sigma \to Q \otimes_R R_\Sigma$ is an isomorphism.*

(2) λ *is* universal *with respect to (1): any further Σ-inverting ring homomorphism $\lambda' : R \to R'$ factors uniquely through λ.*

The homomorphism $\lambda\colon R \to R_\Sigma$ is a ring epimorphism with $\text{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$, *called the* universal localization *of R at Σ.*

Let now $\mathcal{U} \subset \text{mod-}R$ be a set of modules of projective dimension one. For each $U \in \mathcal{U}$, we fix a projective resolution in mod-R

$$0 \to P \overset{\alpha_U}{\to} Q \to U \to 0$$

and we set $\Sigma = \{\alpha_U \mid U \in \mathcal{U}\}$. We denote by $R_{\mathcal{U}}$ the universal localization of R at Σ. Note that $R_{\mathcal{U}}$ does not depend on the chosen class Σ, cf. [34, Theorem 0.6.2].

If $U^* = \text{Hom}_R(U, R) = 0$ for all $U \in \mathcal{U}$, then we also have exact sequences $0 \to Q^* \overset{(\alpha_U)^*}{\to} P^* \to \text{Tr}\, U \to 0$, and $R_{\mathcal{U}} \cong R_{\text{Tr}\,\mathcal{U}}$ for $\text{Tr}\,\mathcal{U} = \{\text{Tr}\, U \mid U \in \mathcal{U}\}$. Moreover, the class $\mathcal{U}^{\perp}$ is a 1-tilting class by Corollary 5.1, and as in the proof of Corollary 6.10 one checks that the class $(\text{Tr}\,\mathcal{U})^o$ is the 1-cotilting class in R-Mod that corresponds to $\mathcal{U}^{\perp}$ under the bijection from Corollary 6.4.

Theorem 7.3. ([11, 4.13]) *Let $\mathcal{U} \subset$ mod-R be a set of modules of projective dimension one such that $U^* = 0$ for all $U \in \mathcal{U}$. Assume that R embeds in $R_\mathcal{U}$ and* $\operatorname{pdim} R_\mathcal{U} \leq 1$. *Then*

$$T_\mathcal{U} = R_\mathcal{U} \oplus R_\mathcal{U}/R$$

is a 1-tilting module. If $R_\mathcal{U}/R$ is a direct limit of $\mathcal{U}$-filtered right R-modules, then $\operatorname{Gen} T_\mathcal{U} = \mathcal{U}^\perp$, *and* $\operatorname{Cogen} D(T_\mathcal{U}) = (\operatorname{Tr}\mathcal{U})^o$.

Example 7.4. [11, 4.7 and 4.14] Let $\mathfrak{U}$ be a left Ore set of non-zero-divisors of R. Then the Ore localization $\mathfrak{U}^{-1}R$ coincides with the universal localization of R at $\mathcal{U} = \{R/uR \mid u \in \mathfrak{U}\}$ (or at $\operatorname{Tr}\mathcal{U} = \{R/Ru \mid u \in \mathfrak{U}\}$). Moreover, $\mathcal{U}^\perp$ is the class of $\mathfrak{U}$-*divisible* right R-modules, i.e. the modules M_R such that $M = Mu$ for all $u \in \mathfrak{U}$, and $(\operatorname{Tr}\mathcal{U})^o$ is the class of $\mathfrak{U}$-*torsion-free* left R-modules, i.e. the modules ${}_RM$ such that for $u \in \mathfrak{U}$ and $0 \neq m \in M$ always $um \neq 0$.

As an application of Theorem 7.3, we have that $\operatorname{pdim}(\mathfrak{U}^{-1}R_R) \leq 1$ if and only if $\operatorname{Gen}(\mathfrak{U}^{-1}R_R) = \mathcal{U}^\perp$, and in this case $T_\mathfrak{U} = \mathfrak{U}^{-1}R \oplus \mathfrak{U}^{-1}R/R$ is a tilting right R-module generating the $\mathfrak{U}$-divisible right R-modules, and $D(T_\mathfrak{U})$ is a cotilting module cogenerating the $\mathfrak{U}$-torsion-free left R-modules.

For a commutative ring R, the tilting module $T_\mathfrak{U}$ is used in [5] to prove that $\operatorname{pdim}(\mathfrak{U}^{-1}R_R) \leq 1$ if and only if the module $\mathfrak{U}^{-1}R/R$ is a direct sum of countably presented submodules. This generalizes classical results for domains due to Hamsher, Matlis, and Lee [53, 72, 67].

Example 7.5. Let R be a right hereditary ring, and let Σ be a set of morphisms between finitely generated projective right R-modules. It is shown in [86] that $R_\Sigma = R_\mathcal{U}$ for a subcategory $\mathcal{U} \subset$ mod-R which is closed under images, kernels, cokernels and extensions. More precisely, $\mathcal{U}$ consists of all $U \in$ mod-R such that the projective resolution $0 \to P \overset{\alpha}{\to} Q \to U \to 0$ is inverted by R_Σ, that is, $\alpha \otimes_R R_\Sigma$ is an isomorphism. Equivalently, $\mathcal{U}$ consists of all $U \in$ mod-R such that $\operatorname{Hom}_R(U, -)$ and $\operatorname{Ext}^1_R(U, -)$ vanish on all R_Σ-modules.

If $\mathcal{U}$ contains no projective module, then R embeds in $R_\mathcal{U}$, and $R_\mathcal{U}/R$ is a direct limit of modules in $\mathcal{U}$ by [86, 2.6]. It follows from Theorem 7.3 that $T_\mathcal{U}$ is a tilting module with tilting class $\mathcal{U}^\perp$.

8. Classification of tilting modules

Now we are ready for the first classification results. In fact, Theorem 7.3 is used in [11, 6.11] to prove that over certain hereditary rings all tilting modules are of the form $T_\mathcal{U}$ for some set of simple modules $\mathcal{U}$. In particular, this applies to maximal hereditary orders, to hereditary local noetherian prime rings which are not simple artinian (e.g. not necessarily commutative discrete valuation domains), and to Dedekind domains.

8.1. Dedekind domains. [21, 11]. Let R be a *Dedekind domain*, i.e. a commutative noetherian hereditary domain, for example $R = \mathbb{Z}$. Every tilting module is equivalent to a module of the form $T_\mathcal{U}$ where $\mathcal{U}$ is a set of simple R-modules.

Writing $\mathcal{U} = \mathcal{U}(P) = \{R/\mathfrak{m} \mid \mathfrak{m} \in P\}$ where P is a set of maximal ideals of R, we see that the equivalence classes of tilting modules are parametrized by the subsets of Max-Spec R. The trivial case $P = \emptyset$ corresponds to $T = R$, while the choice $P =$ Max-Spec R gives the tilting module $T = Q \oplus Q/R$ where Q is the field of quotients of R, cf. Example 3.3 for $R = \mathbb{Z}$.

Not all tilting R-modules, however, arise from Ore localization. Indeed, $T_{\mathcal{U}(P)}$ is of the form $T_{\mathfrak{U}}$ as in Example 7.4 if and only if the complement $X =$ Max-Spec $R \setminus P$ of P is compact. Here a subset X of Max-Spec(R) is said to be *compact* if every $\mathfrak{m} \in$ Max-Spec(R) such that $\mathfrak{m} \subseteq \bigcup_{\mathfrak{p} \in X} \mathfrak{p}$ must equal some $\mathfrak{p} \in X$.

Tilting classes in Mod-R correspond bijectively to universal localizations of R. In fact, by the structure of finitely generated modules over Dedekind domains, the non-trivial universal localizations of R are equivalent to localizations at sets of simple modules. By [64, 8.1], we infer that the tilting classes in Mod-R are in bijection with the recollements of $\mathcal{D}$(Mod-R).

8.2. The Kronecker algebra. [12] A similar classification as in the Dedekind case is obtained for large tilting modules when R is the Kronecker-algebra. Indeed, as we have seen in Example 6.9, the tilting classes corresponding to infinite dimensional tilting modules are of the form Gen $\mathbf{L} = {}^o\mathbf{p}$ or Gen $T_P = {}^o(\mathbf{p} \cup \bigcup_{x \in P} \mathcal{U}_x) = \bigcap_{x \in P} \mathcal{U}_x{}^{\perp}$ for some subset $\emptyset \neq P \subset \mathbb{X}$. So, the equivalence classes of tilting modules are parametrized by the subsets of $\mathbb{X}$. The trivial case $P = \emptyset$ corresponds to $T = \mathbf{L}$, while the choice $P = \mathbb{X}$ gives the module $T = \mathbf{W}$.

Applying Theorem 7.3, we see that $T_P = T_{\mathcal{U}(P)}$ can be obtained from universal localization at the set $\mathcal{U}(P)$ of quasi-simple modules in $\bigcup_{x \in P} \mathcal{U}_x$. So, every large tilting module is equivalent to one of the following:

- the tilting module $\mathbf{L}$
- a tilting module of the form $T_{\mathcal{U}}$ where $\mathcal{U} \neq \emptyset$ is a set of quasi-simple modules.

The rings $R_P = R_{\mathcal{U}(P)}$ are hereditary orders, and $R_{\mathbb{X}}$ is a simple artinian ring, see [41]. The tilting module $\mathbf{W}$ is equivalent to $T_{\mathbb{X}} = R_{\mathbb{X}} \oplus R_{\mathbb{X}}/R$. More generally, $T_P = R_P \oplus R_P/R$ has the form

$$T_P = R_P \oplus \bigoplus \{\text{all } S_\infty \text{ from tubes } \mathcal{U}_x, x \in P\}.$$

Here we say that S_∞ is a Prüfer module *from* $\mathcal{U}_x$ if $S \in \mathcal{U}_x$, and similarly for the adic modules.

From Corollary 6.10 we recover the classification of cotilting modules achieved by Buan and Krause in [29]. In fact, taking the dual modules $D(T_P)$, one obtains that every large cotilting R-module is equivalent to

$$C_P = G \oplus \bigoplus \{\text{all } S_\infty \text{ from tubes } \mathcal{U}_x, x \notin P\} \oplus \prod \{\text{all } S_{-\infty} \text{ from tubes } \mathcal{U}_x, x \in P\}$$

for some subset $P \subset \mathbb{X}$, where G again denotes the generic module.

Before we turn to arbitrary tame hereditary algebras, let us investigate more carefully the relationship between tilting modules and localization over commutative rings. In fact, the classification over Dedekind domains (which are precisely the noetherian Prüfer domains) extends both to commutative noetherian rings and to Prüfer domains.

8.3. Commutative noetherian rings. Recall that a torsion pair $(\mathcal{T}, \mathcal{F})$ in Mod-R (or more generally, in a Grothendieck category $\mathcal{A}$) is *hereditary* if $\mathcal{T}$ is closed under subobjects.

Let now R be a commutative noetherian ring. We are going to see that the cotilting torsion pairs over R are precisely the faithful hereditary torsion pairs in Mod-R, and they are parametrized in terms of certain sets of prime ideals.

A subset $P \subseteq \operatorname{Spec}(R)$ is said to be *closed under specialization* if for any $\mathfrak{p} \in P$ and any $\mathfrak{q} \in \operatorname{Spec}(R)$ we have $\mathfrak{q} \in P$ whenever $\mathfrak{q} \supseteq \mathfrak{p}$.

Every subset $P \subseteq \operatorname{Spec}(R)$ closed under specialization gives rise to a *Gabriel topology (or Gabriel filter)* $\mathcal{G}_P$ on R, which consists of those ideals I of R for which the set $V(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \supseteq I\}$ is contained in P. For details on Gabriel topologies we refer to [89]. Here we just recall that the Gabriel topologies on R correspond bijectively to the hereditary torsion pairs in Mod-R. In order to describe the torsion pair associated to $\mathcal{G}_P$ in terms of P, we need some further terminology.

Given $M \in$ Mod-R, a prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ is said to be *associated* to M if $R/\mathfrak{p}$ embeds in M. The set of all associated primes of M is denoted by Ass M. Further, denoting by $M_\mathfrak{p} = M \otimes_R R_\mathfrak{p}$ the localization of M at $\mathfrak{p}$, we write $\operatorname{Supp} M = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_\mathfrak{p} \neq 0\}$ for the *support* of M. For a category $\mathcal{M} \subseteq$ Mod$-R$, we set $\operatorname{Supp} \mathcal{M} = \bigcup_{M \in \mathcal{M}} \operatorname{Supp} M$.

The following result goes back to work of Gabriel. For details we refer to [89, Chapter VI, Theorem 5.1 and Example 6.6] and to [10, Proposition 2.3].

Proposition 8.1 ([49]). *Let R be a commutative noetherian ring. Every subset $P \subseteq \operatorname{Spec}(R)$ closed under specialization gives rise to a hereditary torsion pair $(\mathcal{T}(P), \mathcal{F}(P))$, where*

$$\mathcal{T}(P) = \{M \in \text{Mod-}R \mid \operatorname{Supp} M \subseteq P\},$$
$$\mathcal{F}(P) = \{M \in \text{Mod-}R \mid \operatorname{Ass} M \cap P = \emptyset\}.$$

The assignments $P \mapsto \mathcal{G}_P$ and $P \mapsto (\mathcal{T}(P), \mathcal{F}(P))$ define bijective correspondences between the subsets of $\operatorname{Spec}(R)$ *closed under specialization, the Gabriel topologies on R, and the hereditary torsion pairs in* Mod-R.

The class $\mathcal{F}(P)$ is called the class of *P-torsion-free modules*. The class of *P-divisible modules* is defined as $\mathcal{D}(P) = \{M \in \text{Mod-}R \mid \mathfrak{p}M = M \text{ for all } \mathfrak{p} \in P\}$.

From Theorem 6.8 we know that every cotilting torsion pair is of the form $(\mathcal{T}, \mathcal{F}) = (\varinjlim(\mathcal{T} \cap \text{mod-}R), \varinjlim(\mathcal{F} \cap \text{mod-}R))$. Moreover, it turns out that $\mathcal{F} =$

$\varinjlim(\mathcal{F} \cap \text{mod-}R) = \mathcal{F}(P)$ for some subset $P \subset \text{Spec}(R)$ closed under specialization. Finally, it follows from the definition of $\mathcal{F}(P)$ that the torsion pair $(\mathcal{T}(P), \mathcal{F}(P))$ is faithful if and only if P does not contain prime ideals that are associated to R. We thus obtain

Theorem 8.2 ([10]). *Let R be a commutative noetherian ring. There are bijections between the following sets:*

(i) *subsets $P \subseteq \text{Spec}(R)$ closed under specialization such that $\text{Ass } R \cap P = \emptyset$,*

(ii) *faithful hereditary torsion pairs in* Mod-R,

(iii) *1-cotilting classes in* Mod-R,

(iv) *1-tilting classes in* Mod-R

given by

Bijection	*Assignment*
(i) → (ii)	$P \mapsto (\mathcal{T}(P), \mathcal{F}(P))$
(ii) → (i)	$(\mathcal{T}, \mathcal{F}) \mapsto \text{Supp } \mathcal{T}$
(i) → (iii)	$P \mapsto \mathcal{F}(P)$
(i) → (iv)	$P \mapsto \mathcal{D}(P)$
(iv) → (iii)	$\text{Gen}\, T \mapsto \text{Cogen}\, D(T)$

Notice that over a Dedekind domain $\text{Ass } R = \{0\}$, so the set of maximal ideals Max-Spec(R) coincides with $\text{Spec}(R) \setminus \text{Ass } R$. The assignment $(i) \to (iv)$ thus becomes

$$\{\text{subsets of Max-Spec}(R)\} \to \{\text{tilting classes}\}, \;\; P \mapsto \text{Gen } T_{\mathcal{U}(P)}$$

and corresponds to the classification in Section 8.1.

More generally, for each $n \geq 1$, the n–tilting and n–cotilting classes over a commutative noetherian ring R are parametrized by finite sequences of subsets of Spec(R). For details we refer to [10]. In this way, by Corollary 6.4, one also obtains a classification of the resolving subcategories of mod-R of bounded projective dimension, which was obtained independently in [44].

8.4. Prüfer domains. [21, 82, 1] Recall that a *Prüfer domain* is a commutative domain that is semihereditary, i.e. every finitely generated ideal is projective. Then by a well-known result of Kaplansky, all finitely presented modules have projective dimension at most one.

Theorem 8.3 ([21, 82]). *Over a Prüfer domain R, the tilting classes in* Mod-R *correspond bijectively to the Gabriel topologies of R of finite type.*

Here a Gabriel topology $\mathcal{G}$ is *of finite type* if it has a *basis* of finitely generated ideals, that is, every $I \in \mathcal{G}$ contains a finitely generated ideal $I' \in \mathcal{G}$.

The bijection in Theorem 8.3 associates to a Gabriel topology of finite type $\mathcal{G}$ the tilting class of all $\mathcal{G}$-divisible modules, that is, of all modules X such that $IX = X$ for each $I \in \mathcal{G}$. Conversely, if $\mathcal{B}$ is a tilting class, then the non-zero finitely generated ideals I such that $R/I \in {}^{\perp}\mathcal{B}$ form a basis of the corresponding Gabriel topology.

The tilting modules over R were classified by Salce in [82]. Hereby, an important role is played by the localizations $R \to Q_{\mathcal{G}}$ with respect to a Gabriel topology $\mathcal{G}$.

Let us briefly recall the construction of $Q_{\mathcal{G}}$, for details we refer to [89, Chapter IX]. Every Gabriel topology $\mathcal{G}$ corresponds bijectively to a hereditary torsion pair $(\mathcal{T}, \mathcal{F})$. If $\mathcal{X} = \mathcal{T}^o \cap \mathcal{T}^{\perp}$ is the perpendicular category of $\mathcal{T}$, then the inclusion functor $\iota : \mathcal{X} \to \text{Mod-}R$ has a left adjoint $\ell : \text{Mod-}R \to \mathcal{X}$. As in the proof of Theorem 7.1, we see that the unit adjunction morphism $R \to \iota\ell(R)$ induces a ring structure on $Q_{\mathcal{G}} = \iota\ell(R)$ and a ring homomorphism $\lambda_{\mathcal{G}} : R \to Q_{\mathcal{G}}$. If $\mathcal{G}$ is a Gabriel topology of finite type over a Prüfer domain, then $\lambda_{\mathcal{G}} : R \to Q_{\mathcal{G}}$ is an injective ring epimorphism, and $Q_{\mathcal{G}}$ is a flat R-module. Further, $\lambda_{\mathcal{G}}$ is equivalent to the universal localization $\lambda_{\mathcal{U}}$ at the set $\mathcal{U}$ of all finitely presented modules from $\mathcal{T}$, that is, there is a ring isomorphism $\varphi : Q_{\mathcal{G}} \to R_{\mathcal{U}}$ such that $\varphi\lambda_{\mathcal{G}} = \lambda_{\mathcal{U}}$, see [1, 5.7 and 5.3].

Theorem 8.4 ([82, 1]). *Let R be a Prüfer domain. Let T be a tilting module, and let $\mathcal{G}$ be the associated Gabriel topology of finite type. The following statements are equivalent.*

(1) $\operatorname{pdim} Q_{\mathcal{G}} \leq 1$.

(2) *$Q_{\mathcal{G}} \oplus Q_{\mathcal{G}}/R$ is a tilting module equivalent to T.*

(3) *There is a set $\mathcal{U} \subset \text{mod-}R$ such that $T_{\mathcal{U}}$ is a tilting module equivalent to T.*

(4) *There is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ where $T_0, T_1 \in \operatorname{Add} T$ and $\operatorname{Hom}_R(T_1, T_0) = 0$.*

So, over Prüfer domains the tilting modules constructed from ring epimorphisms, from universal localizations and from Gabriel localizations are the same. But in general, not all tilting modules arise in this way: an example is provided by the Fuchs tilting module δ generating the class of all divisible modules when R is a Prüfer domain which is not a Matlis domain, that is, when the quotient field Q of R has projective dimension > 1 over R, cf. [52, 5.1.2] and [11, 3.11(4)].

8.5. Tame hereditary algebras. Throughout this section, R denotes a (connected) tame hereditary finite dimensional algebra over a field k. We use the notation from 3.4 and denote by $\mathbf{t} = \bigcup_{x \in \mathbb{X}} \mathcal{U}_x$ the regular components.

Our first aim is to rephrase the classification over the Kronecker algebra in terms of categorical localization. We are going to see that large tilting modules correspond to faithful hereditary torsion pairs, like in the commutative noetherian case, once we replace the module category Mod-R by the category Qcoh$\mathbb{X}$ of quasi-coherent sheaves on the *exceptional curve* $\mathbb{X}$. For details on the notion of an exceptional curve (which agrees with the notion of a *weighted projective line* if the field k is algebraically closed) we refer to [68, 69, 66].

Here we interpret Qcoh$\mathbb{X}$ as the *heart* of a *t-structure* in the bounded derived category $\mathcal{D}^b(\text{Mod-}R)$. Following [56] and [79, 11.1], we associate a t-structure to the split torsion pair $(\operatorname{Add}\mathbf{q}, \mathcal{C})$ in Mod-R from Section 3.4. We have a particularly nice situation since $(\operatorname{Add}\mathbf{q}, \mathcal{C})$ is a cotilting torsion pair given by the Σ-pure-injective cotilting module $\mathbf{W}$.

Theorem 8.5 ([79, 37, 38]). *The heart $\mathcal{A}$ of the t-structure in $\mathcal{D}^b(\text{Mod-}R)$ associated to the torsion pair $(\operatorname{Add}\mathbf{q}, \mathcal{C})$ is a hereditary locally noetherian Grothendieck-category with injective cogenerator $\mathbf{W}[1]$, and with a tilting object $V = R[1]$ inducing a split tilting torsion pair $(\mathcal{C}[1], \operatorname{Add}\mathbf{q})$. In particular, $\mathcal{A}$ is equivalent to Qcoh$\mathbb{X}$.*

The last statement follows from the axiomatic description of coh$\mathbb{X}$ given in [68], which yields an equivalence between coh$\mathbb{X}$ and the category $\operatorname{fp}\mathcal{A}$ of finitely presented objects in $\mathcal{A}$. Notice that the indecomposable injective objects in $\mathcal{A}$ are $G[1]$ and the objects $S_\infty[1]$, and the latter are uniserial objects with socle $S[1]$. The class $\operatorname{add}\mathbf{t}[1] = \operatorname{add}(\bigcup_{x\in\mathbb{X}}\mathcal{U}_x[1])$ consists precisely of the objects of $\operatorname{fp}\mathcal{A}$ that have finite length. The class $\operatorname{add}(\mathbf{q}\cup\mathbf{p}[1])$ of objects of infinite length in $\operatorname{fp}\mathcal{A}$ corresponds to the class vect$\mathbb{X}$ of all *vector bundles* in Qcoh$\mathbb{X}$.

According to [57, 63], the hereditary torsion pairs in the locally noetherian Grothendieck category $\mathcal{A}$ correspond bijectively to the *Serre subcategories* of $\operatorname{fp}\mathcal{A}$, that is, the subcategories $\mathcal{S}\subset\operatorname{fp}\mathcal{A}$ such that for every exact sequence $0\to A\to B\to C\to 0$ in $\operatorname{fp}\mathcal{A}$ we have $B\in\mathcal{S}$ if and only if $A, C\in\mathcal{S}$. The bijection assigns to a hereditary torsion pair $(\mathcal{T},\mathcal{F})$ the Serre subcategory $\mathcal{S} = \operatorname{fp}\mathcal{A}\cap\mathcal{T}$. Conversely, a Serre subcategory $\mathcal{S}$ is mapped to the (hereditary) torsion pair generated by $\mathcal{S}$.

We will say that a hereditary torsion pair $(\mathcal{T},\mathcal{F})$ in $\mathcal{A}$ is *faithful* if $\operatorname{add}(\mathbf{q}\cup\mathbf{p}[1])\subset\mathcal{F}$, or equivalently, if the corresponding Serre subcategory consists of finite length objects.

Recall that a torsion pair $(\mathcal{T},\mathcal{F})$ in a Grothendieck category is hereditary if and only if $\mathcal{F}$ is closed under injective envelopes. So, let us explain how to compute injective envelopes in $\mathcal{A}$. As shown in [79], the classes $\mathcal{C} = \operatorname{Cogen}\mathbf{W} = {}^\perp\mathbf{q}$ and $\mathcal{D} = \operatorname{Gen}\mathbf{W} = \mathbf{t}^\perp$ form a complete cotorsion pair $(\mathcal{C},\mathcal{D})$ in Mod-R. So, for every $M\in$ Mod-R there is a short exact sequence $0\to M\to D\to C\to 0$ where $D\in\mathcal{D}$ and $C\in\mathcal{C}$. In particular, if $M\in\mathcal{C}$, then the sequence above is a left $\operatorname{Add}\mathbf{W}$-approximation in Mod-$R$

$$0\to M\to\mathbf{W}_0\to\mathbf{W}_1\to 0$$

which gives rise to an injective coresolution in $\mathcal{A}$

$$0\to M[1]\to\mathbf{W}_0[1]\to\mathbf{W}_1[1]\to 0$$

For example, the injective coresolution of $S_{-\infty}[1]$ in $\mathcal{A}$ is

$$0\to S_{-\infty}[1]\to G'[1]\to(\tau^- S)_\infty[1]\to 0$$

where G' is a direct sum of copies of the generic module G, see [29, 2.4].

Now assume that R is the Kronecker algebra. By Section 8.2, every large cotilting R-module is equivalent to a module

$$C_P = G \oplus \bigoplus \{\text{all } S_\infty \text{ from tubes } \mathcal{U}_x, x \notin P\} \oplus \prod \{\text{all } S_{-\infty} \text{ from tubes } \mathcal{U}_x, x \in P\}$$

with $P \subset \mathbb{X}$. Then the injective envelope of $C_P[1]$ in $\mathcal{A}$ is a direct sum of copies of $G[1]$ and of objects $S_\infty[1]$ from tubes $\mathcal{U}_x, x \notin P$. Hence the class $\mathcal{F}(P) = \operatorname{Cogen} C_P[1]$ is closed under injective envelopes. We obtain a hereditary torsion pair $(\mathcal{T}(P), \mathcal{F}(P))$ in $\mathcal{A}$, which is faithful as the corresponding Serre subcategory is $\mathcal{S}(P) = \operatorname{add}(\bigcup_{x \in P} \mathcal{U}_x[1])$. Notice, moreover, that the quotient category $\mathcal{A}/\mathcal{T}(P)$ is equivalent to the category Mod-R_P.

We thus have an analog to Theorem 8.2.

Theorem 8.6. *Let R be the Kronecker algebra. There are bijections between the following sets:*

(i) *subsets $P \subset \mathbb{X}$,*

(ii) *faithful hereditary torsion pairs $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$,*

(iii) *equivalence classes of large cotilting left R-modules,*

(iv) *equivalence classes of large tilting right R-modules.*

Let us now turn to an arbitrary tame hereditary algebra. Here, in contrast to the Kronecker case, a large tilting module T can also have finite dimensional direct summands. They will be regular modules from non-homogeneous tubes.

We need some terminology in order to describe such summands. Given a tube $\mathcal{U}_x$ of rank $r > 1$ and a module $S_m \in \mathcal{U}_x$ of regular length $m < r$, we consider the full subquiver $\mathcal{W}_{S_m}$ of $\mathbf{t}_\lambda$ which is isomorphic to the Auslander-Reiten-quiver $\Theta(m)$ of the linearly oriented quiver of type $\mathbb{A}_m$ with S_m corresponding to the projective-injective vertex of $\Theta(m)$. The set $\mathcal{W}_{S_m}$ is called a *wing* of $\mathcal{U}_x$ of size m.

It turns out that the finite dimensional indecomposable summands of T are arranged in disjoint wings, and the number of summands in each wing equals the size of the wing. The direct sum Y of a complete irredundant set of finite dimensional indecomposable summands of T is thus a finite dimensional regular multiplicity-free R-module with $\operatorname{Ext}^1_R(Y, Y) = 0$ such that for every direct summand S_m of Y of regular length m there exist precisely m direct summands of Y that belong to $\mathcal{W}_{S_m}$. A module Y with these properties will be called a *branch module*.

By passing to a suitable universal localization $R_\mathcal{V}$ of R, one can reduce the classification problem to a situation similar to the Kronecker case. In fact, T is equivalent to a tilting R-module of the form $Y \oplus M$ where Y is a branch module and M is a tilting $R_\mathcal{V}$-module without finite dimensional indecomposable summands. This allows to conclude that T is equivalent either to $Y \oplus \mathbf{L}_\mathcal{V}$, where $\mathbf{L}_\mathcal{V}$ is the Lukas tilting module over $R_\mathcal{V}$, or it is equivalent to $Y \oplus R_\mathcal{V} \oplus R_\mathcal{V}/R_\mathcal{U}$ for a suitable subset $\mathcal{U}$ of $\mathcal{V}$. Here $R_{\mathcal{V}}/R_{\mathcal{U}}$ is a direct sum of Prüfer modules, and both $\mathbf{L}_{\mathcal{V}}$ and $R_{\mathcal{V}}$ can be replaced by $\mathbf{L}\otimes_R R_{\mathcal{V}}$, see [12, 5.6 and 5.7].

A complete list of all large tilting modules up to equivalence is then given by the modules

$$T_{(Y,P)} = Y \oplus (\mathbf{L}\otimes_R R_{\mathcal{V}}) \oplus \bigoplus\{\text{all } S_\infty \text{ in } {}^\perp Y \text{ from tubes } \mathcal{U}_x, x\in P\}$$

where Y is a branch module, $P\subset \mathbb{X}$, and $\mathcal{V}=\mathcal{V}_{(Y,P)}$ consists of all quasi-simple modules in $\bigcup_{\lambda\in P}\mathcal{U}_x$ and all regular composition factors of Y. By the Auslander-Reiten formula, the Prüfer modules S_∞ appearing in the direct sum are precisely those whose regular socle S does not occur as a regular composition factor of $\tau^- Y$.

Again, one can use Corollary 6.10 to recover the classification of large cotilting modules achieved in [29, 30]. So, every large cotilting module is equivalent to a module

$$\begin{aligned} C_{(Y,P)} = Y \oplus G \oplus \bigoplus\{\text{all } S_\infty \text{ in } {}^\perp Y \text{ from tubes } \mathcal{U}_x, x\notin P\}\oplus \\ \oplus \prod\{\text{all } S_{-\infty} \text{ in } Y^\perp \text{ from tubes } \mathcal{U}_x, x\in P\}\end{aligned}$$

Theorem 8.7 ([12]). *Let R be a connected tame hereditary artin algebra. There are bijections between the following sets:*

(i) *pairs (Y,P) where Y is a branch module and $P\subset\mathbb{X}$,*

(ii) *equivalence classes of large cotilting left R-modules,*

(iii) *equivalence classes of large tilting right R-modules.*

Moreover, given a pair (Y,P) as above, we can consider the Serre subcategory $\mathcal{S}_{(Y,P)}$ of $\operatorname{fp}\mathcal{A}$ consisting of the additive closure of the wings defined by $Y[1]$ and the tubes $\mathcal{U}_x[1], x\in P$. The torsion pair $(\mathcal{T},\mathcal{F})$ in $\mathcal{A}$ generated by $\mathcal{S}_{(Y,P)}$ is a faithful hereditary torsion pair, and all faithful hereditary torsion pairs in $\mathcal{A}$ arise in this way. Hence the equivalence classes of large (co)tilting modules correspond to Gabriel localizations of $\mathcal{A}$. It is easy to see, however, that this correspondence is not injective in general, as different branch modules can define the same wing.

Finally, we remark that by [73] all classical tilting modules over finite dimensional hereditary algebras (of any representation type) arise from universal localisations.

8.6. Tilting sheaves. The discussion above raises the problem of classifying large tilting sheaves in the category Qcoh$\mathbb{X}$ of quasi-coherent sheaves on an exceptional curve $\mathbb{X}$. As Qcoh$\mathbb{X}$ is a Grothendieck category, we can employ the tilting theory for Grothendieck categories from [35] and the background on cotorsion pairs developed in [83]. We can also use the interplay between sheaves on $\mathbb{X}$ and modules over the derived equivalent canonical algebra R.

Considering tilting sheaves $T_{(Y,P)}$ analogous to the modules constructed in Section 8.5, one obtains

Theorem 8.8 ([8]). *Let $\mathbb{X}$ be an exceptional curve of domestic type. There are bijections between the following sets:*

(i) *pairs (Y, P) where Y is a branch object and $P \subset \mathbb{X}$,*

(ii) *equivalence classes of large tilting sheaves in* Qcoh$\mathbb{X}$.

When $\mathbb{X}$, or the associated canonical algebra R, is of tubular type, there are many more tilting sheaves on $\mathbb{X}$, or tilting modules over R. In fact, there are tilting modules of the form $T_{(Y,P)}$ for each rational slope, and in addition there are tilting modules of irrational slope.

Let us first recall the definition of slope. If R is a (connected) canonical algebra of tubular type, the AR-quiver of R consists of a preprojective component $\mathbf{p}_0$, a preinjective component $\mathbf{q}_\infty$ and a countable number of sincere separating tubular families $\mathbf{t}_\alpha, \alpha \in \mathbb{Q}_0^\infty = \mathbb{Q}^+ \cup \{0, \infty\}$, where $\mathbf{t}_\alpha$ is stable for $\alpha \in \mathbb{Q}^+$.

We fix $w \in \mathbb{R}_0^\infty = \mathbb{R}^+ \cup \{0, \infty\}$ and set

$$\mathbf{p}_w = \mathbf{p}_0 \cup \bigcup_{\alpha < w} \mathbf{t}_\alpha, \qquad \mathbf{q}_w = \bigcup_{w < \gamma} \mathbf{t}_\gamma \cup \mathbf{q}_\infty$$

For rational w we thus obtain a trisection $(\mathbf{p}_w, \mathbf{t}_w, \mathbf{q}_w)$ of the category of indecomposable finitely generated modules as in Section 3.4, while for irrational w the finitely generated indecomposable modules all belong either to $\mathbf{p}_w$ or to $\mathbf{q}_w$.

In [79, §13], the classes $\mathbf{p}_w$ and $\mathbf{q}_w$ are used to construct torsion pairs in Mod-R in the same way as in Section 3.4. The class

$$\mathcal{B}_w = {}^o(\mathbf{p}_w) = (\mathbf{p}_w)^\perp,$$

is a 1-tilting class by Corollary 5.1. We denote by $\mathbf{L}_w$ a tilting module generating $\mathcal{B}_w$. It is an infinite dimensional module that can be obtained by a similar construction as in Section 3.4.

Dually, the class

$$\mathcal{C}_w = (\mathbf{q}_w)^o = {}^\perp(\mathbf{q}_w),$$

is a cotilting class, and we denote by $\mathbf{W}_w$ a cotilting module cogenerating $\mathcal{C}_w$.

According to [79], for any $w \in \mathbb{R}_0^\infty$ we say that a module has *slope w* if it belongs to the class $\mathcal{C}_w \cap \mathcal{B}_w$. Notice that every indecomposable module not in $\mathbf{p}_0 \cup \mathbf{q}_\infty$ has a slope by [79, Theorem 13.1].

If w is rational, the situation is completely analogous to the tame hereditary case. Indeed, the classes $\mathcal{C}_w$ and $\mathcal{D}_w = {}^o(\mathbf{t}_w) = (\mathbf{t}_w)^\perp$ form a cotorsion pair $(\mathcal{C}_w, \mathcal{D}_w)$, and $\mathbf{W}_w$ is also a tilting module which generates $\mathcal{D}_w$ and which can be chosen as the direct sum of a set of representatives of the Prüfer modules and the generic module G_w from the family $\mathbf{t}_w$. Moreover, Add $\mathbf{W}_w$ is closed under direct products, and $(\mathcal{Q}_w, \mathcal{C}_w)$ is a split torsion pair with $\mathcal{Q}_w = \operatorname{Gen} \mathbf{q}_w$, see [79, 3.1 and 13.1]. The heart $\mathcal{A}_w$ of the t-structure in $\mathcal{D}^b$(Mod-R) associated to this torsion pair is again of the form Qcoh$\mathbb{X}_w$ for a tubular exceptional curve $\mathbb{X}_w$ which parametrizes the family $\mathbf{t}_w$, see [69],[66, 8.1.6]. So, with the corresponding torsion pairs and the corresponding notion of slope in Qcoh$\mathbb{X}$ (which coincides with the definition in [69, 66]), we can define tilting modules or sheaves $T_{(Y,P)}$ as above.

If w is irrational, then there is just one tilting module and one cotilting module of that slope. In fact, any tilting module of slope w is equivalent to $\mathbf{L}_w$, and any cotilting module of slope w is equivalent to $\mathbf{W}_w$. Moreover, a module has slope w if and only if it is a pure submodule of a product of copies of $\mathbf{W}_w$, or equivalently, it is a pure-epimorphic image of a direct sum of copies of $\mathbf{L}_w$.

It is well known that coherent tilting sheaves must have direct summands of different slopes. Indeed, in a coherent tilting sheaf T, the number of pairwise non-isomorphic indecomposable summands from a given tubular family $\mathbf{t}_w$ is bounded by $\sum_{i=1}^{t}(p_i - 1)$, where $p_1, \dots, p_t$ are the ranks of the non-homogeneous tubes in $\mathbf{t}_w$, and it is therefore strictly smaller than the rank of the Grothendieck group $\operatorname{rk} K_0(\mathbb{X}) = \sum_{i=1}^{t}(p_i - 1) + 2$.

In contrast, large tilting sheaves have a well-defined slope. So, we obtain a complete classification as follows.

Theorem 8.9 ([8]). *Let $\mathbb{X}$ be an exceptional curve of tubular type.*
(1) Every large tilting sheaf has a slope w.
(2) If w is irrational, then $\mathbf{L}_w$ is the only tilting sheaf of slope w up to equivalence.
(3) For rational w there are bijections between the following sets:

(i) *pairs (Y, P) where Y is a branch object in $\mathbf{t}_w$ and $P \subset \mathbb{X}_w$,*

(ii) *equivalence classes of large tilting sheaves in* $\operatorname{Qcoh}\mathbb{X}$ *of slope w.*

References

[1] L. Angeleri Hügel, M. Archetti, Tilting modules and universal localization. *Forum Math.* **24** (2012), 709–731.

[2] L. Angeleri Hügel, F.U. Coelho, Infinitely generated tilting modules of finite projective dimension. *Forum Math.* **13** (2001), 239–250.

[3] L. Angeleri Hügel, F. U. Coelho: Infinitely generated complements to partial tilting modules. *Math. Proc. Camb. Phil. Soc.* **132** (2002), 89–96.

[4] L. Angeleri Hügel, D. Herbera, J. Trlifaj, Tilting modules and Gorenstein rings. *Forum Math.*, **18** (2006), 217–235.

[5] L. Angeleri Hügel, D. Herbera, J. Trlifaj, Divisible modules and localization. *J. Algebra*, **294** (2005) 519–551.

[6] L. Angeleri Hügel, O. Kerner, J. Trlifaj, Large tilting modules and representation type. *Manuscr. Math.* **132** (2010), 483–499.

[7] L. Angeleri Hügel, S. Koenig, Q. Liu, D. Yang, Derived simple algebras and restrictions of recollements of derived module categories. In preparation.

[8] L. Angeleri Hügel, D. Kussin, Large tilting modules and sheaves for canonical algebras. In preparation.

[9] L. Angeleri Hügel, O. Mendoza, Homological dimensions in cotorsion pairs. *Illinois J. Math* **53** (2009), 251–263.

[10] L. Angeleri Hügel, D. Pospíšil, J. Šťovíček, J. Trlifaj, Tilting, cotilting, and spectra of commutative noetherian rings. *Trans. Amer. Math. Soc.*, to appear

[11] L. Angeleri Hügel, J. Sánchez, Tilting modules arising from ring epimorphisms. *Algebr. Repres. Theory* **14** (2011), 217–246.

[12] L. Angeleri Hügel, J. Sánchez, Tilting modules over tame hereditary algebras. *Journal Reine Angew. Math.*, to appear.

[13] L. Angeleri Hügel, A. Tonolo, J. Trlifaj, Tilting preenvelopes and cotilting precovers. *Algebr. Repres. Theory* **4** (2001), 155–170.

[14] L. Angeleri Hügel, J. Trlifaj, Tilting theory and the finitistic dimension conjectures. *Trans. Amer. Math. Soc.* **354** (2002), 4345–4358.

[15] M. Auslander, I. Reiten, Applications of contravariantly finite subcategories. *Adv. Math.* **86** (1991), 111–152.

[16] S. Bazzoni, Cotilting modules are pure-injective. *Proc. Amer. Math. Soc.* **131** (2003), 3665–3672.

[17] S. Bazzoni, Cotilting and tilting modules over Pruefer domains. *Forum Math.*, **19** (2007), 1005–1027.

[18] S. Bazzoni, A characterization of n-cotilting and n-tilting modules. *J. Algebra* **273** (2004), 359–372.

[19] S. Bazzoni, When are definable classes tilting and cotilting classes? *J. Algebra* **320** (12) (2008), 4281–4299.

[20] S. Bazzoni, Equivalences induced by infinitely generated tilting modules. *Proc. Amer. Math. Soc.* **138** (2010), no. 2, 533–544.

[21] S. Bazzoni, P. C. Eklof, J. Trlifaj, Tilting cotorsion pairs. *Bull. London Math. Soc.* **37** (2005) 683–696.

[22] S. Bazzoni, D. Herbera, One dimensional tilting modules are of finite type. *Algebras and Repres. Theory* **11** (2008), 43–61.

[23] S. Bazzoni, F. Mantese, A. Tonolo, Derived equivalence induced by n-tilting modules. *Proc. Amer. Math. Soc.* **139**, (2011), 4225–4234.

[24] S. Bazzoni, A. Pavarin, Recollements from partial tilting complexes. arxiv:1206.5650.

[25] S. Bazzoni, J. Šťovíček, All tilting modules are of finite type. *Proc. Amer. Math. Soc.* **135** (2007), 3771–3781.

[26] A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers. *Astérisque* **100** (1982).

[27] S.Brenner, M.Butler, Generalizations of the Bernstein-Gelfand-Ponomarev reflections functors. In *Proc. of ICRA II*, Springer Lecture Notes in Mathematics 832 (1980), 103–169.

[28] C. Braga, F.U. Coelho, Limits of tilting modules. *Colloq. Math.* **115** (2009), 207–217.

[29] A.B. Buan, H. Krause, Cotilting modules over tame hereditary algebras. *Pacific J. Math.* **211** (2003), 41–60.

[30] A.B. Buan, H. Krause, Tilting and cotilting for quivers of type $\widetilde{A}_n$. *J. Pure Appl. Algebra* **190** (2004), 1–21.

[31] A.B. Buan, Ø. Solberg, Limits of pure-injective cotilting modules. *Algebr. Represent. Theory* **8** (2005), 621–634.

[32] H.Chen, C.Xi, Good tilting modules and recollements of derived module categories. *Proc. London Math. Soc.* **104** (2012), 959–996.

[33] H.Chen, C.Xi, Ringel modules and homological subcategories. arxiv:1206.0522.

[34] P. M. Cohn, *Free Rings and Their Relations*, second ed., London Math. Soc. Monographs 19, Academic Press Inc. London, 1985.

[35] R. Colpi, Tilting in Grothendieck categories. *Forum Math.* **11** (1999), 735–759.

[36] R. Colpi, G. D' Este, A. Tonolo, Quasi-tilting modules and counter equivalences. *J. Algebra* **191** (1997), 461–494. Corrigendum in *J.Algebra* **206** (1998), 370.

[37] R. Colpi, E.Gregorio, F. Mantese, On the heart of a faithful torsion theory. *J. Algebra* **307** (2007) 841–863.

[38] R. Colpi, F. Mantese, A. Tonolo, Cotorsion pairs, torsion pairs, and Σ-pure-injective cotilting modules. *J. Pure Appl. Algebra* **214** (2010), 519–525.

[39] R. Colpi, C. Menini, On the structure of ∗-modules. *J. Algebra*, **158** (1993), 400–419.

[40] R.Colpi, J.Trlifaj, Tilting modules and tilting torsion theories. *J. Algebra* **178** (1995), 614–634.

[41] W. Crawley-Boevey. Regular modules for tame hereditary algebras. *Proc. London Math. Soc.*, **62** (1991), 490–508.

[42] W. Crawley-Boevey, Locally finitely presented additive categories. *Comm. Algebra* **22** (1994), 1644–1674.

[43] W. Crawley-Boevey, Infinite-dimensional modules in the representation theory of finite-dimensional algebras. In *Workshop on Representations of Algebras and Related Topics, Trondheim 1996*, CMS Conf. Proc. 23 (1998), 29–54.

[44] H. Dao, R. Takahashi, Classification of resolving subcategories and grade consistent functions. arxiv:1202.5605.

[45] P. C. Eklof, A. H. Mekler, *Almost Free Modules*, North Holland, Amsterdam, 1990.

[46] P. Eklof, J. Trlifaj, How to make Ext vanish. *Bull. London Math. Soc.* **33** (2001), 41–51.

[47] P. Eklof, J. Trlifaj, Covers induced by Ext. *J. Algebra* **231** (2000), 640–651.

[48] E. Enochs, O. Jenda, *Relative Homological Algebra*, de Gruyter, Berlin, 2000.

[49] P. Gabriel. Des catégories abéliennes. *Bulletin de la S. M. F.* **90** (1962), 323–448.

[50] P. Gabriel, J. A. de la Peña. Quotients of representation-finite algebras. *Comm. Algebra*, **15** (1987), 279–307.

[51] W. Geigle, H. Lenzing. Perpendicular categories with applications to representations and sheaves. *J. Algebra* **144** (1991), 273–343.

[52] R. Goebel, J.Trlifaj, *Approximations and Endomorphism Algebras of Modules.* W. de Gruyter, Berlin 2006 (see also Second Edition 2012).

[53] R.M. Hamsher, On the Structure of a One-Dimensional Quotient Field, *J. Algebra* **19** (1971), 416–425.

[54] D. Happel, C.M. Ringel, Tilted algebras. *Trans. Amer. Math. Soc.* **274** (1982), 399–443.

[55] D. Happel, On the derived category of a finite-dimensional algebra. *Comment. Math. Helv.* **62** (1987), 339–389.

[56] D. Happel, I. Reiten, S.O. Smalø, Tilting in Abelian Categories and Quasitilted Algebras. *Mem. Amer. Math. Soc.* **575** (1996).

[57] I. Herzog, The Ziegler spectrum of a locally coherent Grothendieck catgeory. *Proc. London Math. Soc.* **74** (1997), 503–558.

[58] B. Huisgen-Zimmermann, Homological domino effects and the first Finitistic Dimension Conjecture. *Invent. Math.* **108** (1992), 369–383.

[59] B. Huisgen-Zimmermann, S.O.Smalø, A homological bridge between finite and infinite dimensional representations. *Algebr. Represent. Theory* **1** (1998), 169–188.

[60] K. Igusa, S.O.Smalø, G. Todorov, Finite projectivity and contravariant finiteness. *Proc. Amer. Math. Soc.* **109** (1990), 937–941.

[61] O. Kerner, J. Trlifaj, Tilting classes over wild hereditary algebras. *J. Algebra* **290** (2005), 583–556.

[62] O. Kerner, J. Trlifaj, Constructing tilting modules. *Trans. Amer. Math. Soc.* **360** (2008), 1907–1925.

[63] H. Krause. The spectrum of a locally coherent category. *J. Pure Appl. Algebra* **114** (1997), 259–271.

[64] H. Krause, J. Šťovíček, The telescope conjecture for hereditary rings via Ext-orthogonal pairs. *Adv. Math.* **225** (2010) 2341–2364.

[65] H. Krause, Ø. Solberg, Filtering modules with finite projective dimension. *Forum Math.* 15 (2003), 377–393.

[66] D. Kussin, Noncommutative curves of genus zero: related to finite dimensional algebras. *Mem. Amer. Math. Soc.* **201** (2009), no. 942, x+128.

[67] S.B. Lee, On divisible modules over domains, *Arch. Math.* **53** (1989), 259–262.

[68] H. Lenzing, Representations of finite dimensional algebras and singularity theory. In *Trends in ring theory (Miskolc, 1996)* (V. Dlab et al., ed.), CMS Conf. Proc. 22, Amer. Math. Soc., Providence (1998), 71–97.

[69] H. Lenzing, J. A. de la Peña, Concealed-canonical algebras and separating tubular families. *Proc. London Math. Soc.* **78** (1999), 513–540.

[70] F. Lukas, Infinite-dimensional modules over wild hereditary algebras. *J. London Math. Soc.* **44** (1991), 401–419.

[71] F. Lukas, A class of infinite-rank modules over tame hereditary algebras. *J. Algebra*, **158** (1993), 18–30.

[72] E. Matlis, 1-Dimensional Cohen-Macaulay Rings, *Lecture Notes Math.* **327**, Springer, New York 1973.

[73] F. Marks, Universal localisations and tilting modules for finite dimensional algebras. In preparation.

[74] Y. Miyashita, Tilting modules of finite projective dimension. *Math. Z.* **193** (1986), 113–146.

[75] P. Nicolás, M. Saorín, Generalized tilting theory. arXiv:1208.2803.

[76] F. Okoh, Cotorsion modules over tame finite-dimensional hereditary algebras. In *Springer Lecture Notes in Math.* **903** (1980), 263–269.

[77] D. Pospíšil, J. Trlifaj, Tilting for regular rings of Krull dimension two. *J. Algebra*, **336** (2011), 184–199.

[78] J. Rickard, A. Schofield, Cocovers and tilting modules. *Math. Proc. Camb. Phil. Soc.* **106** (1989), 1–5.

[79] I.Reiten, C.M. Ringel, Infinite dimensional representations of canonical algebras. *Canad. J. Math.*, **58** (2006), 180–224.

[80] C. M. Ringel, Infinite dimensional representations of finite dimensional hereditary algebras. *Symposia Math.* **XXIII** (1979), 321–412.

[81] L. Salce, Cotorsion theories for abelian groups. *Symposia Math.* **XXIII** (1979), 11–32.

[82] L. Salce, $\mathcal{F}$-divisible modules and tilting modules over Prüfer domains. *J. Pure Appl. Algebra* **199** (2005), 245-259.

[83] M. Saorín, J. Šťovíček, On exact categories and applications to triangulated adjoints and model structures. *Adv. Math.* **228** (2011), 968–1007.

[84] J. Šaroch, J. Trlifaj, Completeness of cotorsion pairs. *Forum Math.* **19** (2007), 749–760.

[85] A. H. Schofield, *Representation of rings over skew fields*, London Math. Soc. Lecture Note Ser. 92. Cambridge University Press, Cambridge, 1985.

[86] A. H. Schofield, Universal localization of hereditary rings. arxiv:0708.0257.

[87] S.O.Smalø, Homological difference between finite and infinite dimensional representations of algebras. In *Trends in Mathematics*, Birkhäuser, Basel 2000, 81–93.

[88] Ø. Solberg, Infinite dimensional tilting modules over finite dimensional algebras. In *Handbook of tilting theory*, London Math. Soc. Lecture Note Ser. 332. Cambridge Univ. Press, Cambridge, 2007, 323–344.

[89] B. Stenström, *Rings of quotients*, Springer-Verlag, New York, 1975, Die Grundlehren der Mathematischen Wissenschaften, Band 217.

[90] J. Šťovíček, All n-cotilting modules are pure-injective. *Proc. Amer. Math. Soc.* **134** (2006), 1891–1897.

[91] J. Šťovíček, Tilting modules over an algebra by Igusa, Smalø and Todorov. *J. Algebra* **311** (2007) 299–318.

[92] J. Šťovíček, J.Trlifaj, All tilting modules are of countable type. *Bull. Lond. Math.* Soc. **39** (2007), 121–132.

[93] J. Trlifaj, Infinite dimensional tilting modules and cotorsion pairs. In *Handbook of tilting theory*, London Math. Soc. Lecture Note Ser. 332. Cambridge Univ. Press, Cambridge, 2007, 279–321.

[94] J. Trlifaj, Filtrations for the roots of Ext. *Milan J. Math.* **75** (2007), 61–90

[95] D. Yang, Recollements from generalized tilting. *Proc. Amer. Math. Soc.* **140** (2012), 83–91.

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