Direct limits of modules of finite projective dimension

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Dedicated to Paul Eklof on his 60th birthday

Abstract

We describe in homological terms the direct limit closure of a class $\mathcal{C}$ of modules over a ring $R$. We also determine the closure of the cotorsion pair $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ cogenerated by $\mathcal{C}$. As an application, we solve a problem of Fuchs and Salce on the structure of direct limits of modules of projective dimension at most one over commutative domains. Then we consider the case when $R$ is a right coherent ring and $\mathcal{C} = \mathcal{P}^{<\infty}$, the class of all finitely presented modules of finite projective dimension. If $\text{findim} R < \infty$ then $\mathcal{C}$ is a tilting cotorsion pair induced by a tilting module $T$. We characterize closure properties of $\mathcal{A}$ in terms of properties of $T$. Finally, we discuss an example where $\mathcal{A}$ is not closed under direct limits.

Let $R$ be a ring. Denote by $\mathcal{P}$ the class of all modules of finite projective dimension, and by $\mathcal{P}^{<\infty}$ the class of all finitely presented modules in $\mathcal{P}$. For $n < \omega$ let $\mathcal{P}_n$ be the class of all modules of projective dimension at most $n$, and let $\mathcal{P}_n^{<\infty}$ be the corresponding subclass of $\mathcal{P}^{<\infty}$.

In this paper, we study the categories $\varinjlim \mathcal{P}_n$ and $\varinjlim \mathcal{P}_n^{<\infty}$ of all direct limits of modules in $\mathcal{P}_n$ and $\mathcal{P}_n^{<\infty}$, respectively. To this end, we consider the complete cotorsion pair $(\mathcal{A}_n, \mathcal{B}_n)$ cogenerated by $\mathcal{P}_n^{<\infty}$ and investigate the class $\mathcal{A}_n$.

Our main tool is a homological description of $\varinjlim \mathcal{A}_n$. We show that in many cases the limit closure $\varinjlim \mathcal{A}$ of the first component in a cotorsion pair $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ consists of all modules $X$ satisfying $\text{Ext}_R^1(X, B) = 0$ for each pure-injective module $B \in \mathcal{B}$; there is also a characterization of $\varinjlim \mathcal{A}$ in terms of vanishing of $\text{Tor}$ (see Section 2).

This result allows us to discuss a more general question: What is the smallest complete cotorsion pair $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ containing $\mathcal{C}$ with $\mathcal{A}$ being closed under direct limits? The question is of particular interest because of a classical result of Enochs saying that in this case $\mathcal{A}$ is a covering class, and $\mathcal{B}$ an enveloping class, in $\text{Mod}R$.

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In Corollary 2.4, we answer the question for cotorsion pairs cogenerated by classes of finitely presented modules over right coherent rings.

Section 3 deals with an application to the particular case when \( n = 1 \) and \( R \) is a commutative domain. In Theorem 3.5, we solve a problem of Fuchs and Salce ([15, Problem 22 on p.246]) by showing that a module belongs to \( \varprojlim \mathcal{P}_1 \) if and only if it has flat dimension at most one. Furthermore, we investigate the divisible modules of projective dimension at most one and answer a question related to the Fuchs’ divisible module \( \delta \) ([14, Problem 6 in Chapter VI]).

In Section 4, we consider right coherent rings and continue our investigation started in [4] of the complete cotorsion pair \((\mathcal{A}, \mathcal{B})\) cogenerated by \( \mathcal{P}^{< \infty} \). If \( \text{findim} R < \infty \), then we know from [4] that there is a tilting module \( T \) such that \( \mathcal{B} = T^{\perp} \). Furthermore, \( \mathcal{A} = \mathcal{P} \) if and only if the category \( \text{Add} T \) is closed under cokernels of monomorphisms, and in this case the little and the big finitistic dimensions of \( R \) coincide: \( \text{findim} R = \text{Findim} R \).

Our focus here is on the category \( \varprojlim \mathcal{P}^{< \infty} \). Note that \( \mathcal{A} \) is always contained in \( \varprojlim \mathcal{P}^{< \infty} \). Moreover, if \( \mathcal{P} \) is closed under direct limits, e. g. if \( R \) is right perfect and \( \text{Findim} R < \infty \), then \( \mathcal{A} \subseteq \varprojlim \mathcal{P}^{< \infty} \subseteq \mathcal{P} \). Using the tilting module \( T \) from [4], we investigate closure properties of \( \mathcal{A} \). We characterize the cases \( \mathcal{A} = \varprojlim \mathcal{P}^{< \infty} \) and \( \varprojlim \mathcal{P}^{< \infty} = \mathcal{P} \) in Theorems 4.2 and 4.7. In Theorem 4.3, we determine when \( \mathcal{A} \) is a definable class.

Finally, in Section 5, we study an important example in detail, namely the artin algebra introduced by Igusa, Smalø, and Todorov [19]. In this case, we show that \( \mathcal{A} = \varprojlim \mathcal{P}^{< \infty} \) fails while \( \varprojlim \mathcal{P}^{< \infty} = \mathcal{P} \) holds true.

1 Preliminaries

First, we fix our terminology and notation.

Let \( R \) be an arbitrary ring, \( \text{Mod} R \) be the category of all (right) \( R \)-modules, and \( \text{mod} R \) the subcategory of all finitely presented modules. For a subcategory \( \mathcal{M} \) of \( \text{Mod} R \), we denote by \( \text{Add} \mathcal{M} \) (respectively \( \text{add} \mathcal{M} \)) the subcategory of all modules isomorphic to a direct summand of a (finite) direct sum of modules of \( \mathcal{M} \).

Following [24, p.210], we will say that a module \( M \) is \( FP_n \) provided that \( M \) has a projective resolution

\[
\cdots \rightarrow P_{k+1} \rightarrow P_k \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0
\]

where all \( P_i \) with \( i \leq n \) are finitely generated. So \( FP_0 \) stands for finitely generated,
FP_1 for finitely presented, and a module M is FP_2 if and only if there are \( n < \omega \), a finitely presented module \( K \), and a short exact sequence \( 0 \to K \to R^n \to M \to 0 \). Notice that \( \mathcal{P}_1^{\infty} \) coincides with the class of all FP_2 modules of projective dimension \( \leq 1 \). If \( R \) is right coherent and \( M \) is finitely presented then \( M \) is FP_n for all \( n \geq 1 \).

A. Precovers and Preenvelopes. Let \( \mathcal{M} \) be a subcategory of \( \text{Mod} R \), and let \( A \) be a right \( R \)-module. A morphism \( f \in \text{Hom}_R(A, X) \) with \( X \in \mathcal{M} \) is an \( \mathcal{M} \)-preenvelope (or a left \( \mathcal{M} \)-approximation) of \( A \) provided that the abelian group homomorphism \( \text{Hom}_R(f, M): \text{Hom}_R(X, M) \to \text{Hom}_R(A, M) \) is surjective for each \( M \in \mathcal{M} \). An \( \mathcal{M} \)-preenvelope \( f \in \text{Hom}_R(A, X) \) of \( A \) is said to be special if \( f \) is a monomorphism and \( \text{Ext}_R^1(\text{Coker} f, M) = 0 \) for all \( M \in \mathcal{M} \). An \( \mathcal{M} \)-envelope of \( A \) is an \( \mathcal{M} \)-preenvelope \( f \in \text{Hom}_R(A, X) \) which is left minimal, that is, \( h \) is an automorphism of \( X \) whenever \( h \in \text{End}_R(X) \) satisfies \( hf = f \). If it exists, an \( \mathcal{M} \)-envelope is unique up to isomorphism.

The notions of an \( \mathcal{M} \)-cover and a (special) \( \mathcal{M} \)-precovar are defined dually.

Finally, a subcategory \( \mathcal{C} \) of \( \text{mod}R \) is said to be covariantly (respectively, contravariantly) finite in \( \text{mod}R \) if every finitely presented module has a \( \mathcal{C} \)-preenvelope (respectively, a \( \mathcal{C} \)-precovar).

B. Closure under direct limits. Let \( \mathcal{C} \) be a class of modules. Denote by \( \varinjlim \mathcal{C} \) the class of all modules \( D \) such that \( D = \varinjlim_{i \in I} C_i \) where \( \{C_i \mid i \in I\} \) is a direct system of modules from \( \mathcal{C} \).

In general, the class \( \varinjlim \mathcal{C} \) is not closed under direct limits:

**Example 1.1** Let \( R = \mathbb{Z} \) and let \( \mathcal{C} = \{A\} \) where \( A \) is an indecomposable torsion-free abelian group of rank \( r \geq 2 \) such that \( \text{End}(A) \cong \mathbb{Z} \). (There is a proper class of such groups: by [10, XII.3.5], there exist arbitrarily large indecomposable torsion-free abelian groups such that \( \text{End}(A) \cong \mathbb{Z} \).)

Consider the direct system \( \{C_n \mid n < \infty\} \) where \( C_n = A \) for all \( n < \infty \) and \( f_n : C_n \to C_{n+1} \) is the multiplication by \( n \). Then \( \varinjlim_{n<\infty} C_n \) coincides with the injective envelope \( E(A) \cong \mathbb{Q}^0 \) of \( A \).

Now, consider the direct system \( \{D_n \mid n < \infty\} \) where \( D_n = E(A) \) for all \( n < \infty \) and \( g_n : D_n \to D_{n+1} \) is the projection on a fixed copy of \( \mathbb{Q} \) in \( E(A) \). Then \( \varinjlim_{n<\infty} D_n \cong \mathbb{Q} \).

On the other hand, \( \mathbb{Q} \notin \varinjlim \mathcal{C} \). Namely, let \( \{C_i \mid i \in I\} \) be a direct system with \( C_i = A \) for all \( i \in I \), and let \( D = \varinjlim_{i \in I} C_i \). Since \( \text{End}(A) \cong \mathbb{Z} \) and \( A \) is torsion-free, all maps in the direct system are either monomorphisms or zero. It follows that either \( D \) contains a copy of \( A \), or else \( D = 0 \). Anyway, \( D \) has rank \( \neq 1 \), so \( D \not\cong \mathbb{Q} \). \( \square \)
There is an important case when \( \varinjlim \mathcal{C} \) is always closed under direct limits. The characterization goes back to Lenzing:

**Lemma 1.2** [23] Let \( R \) be a ring, and let \( \mathcal{C} \) be a full additive subcategory of \( \text{mod} \, R \) which is closed under isomorphisms and direct summands. The following statements are equivalent for a module \( A \).
1. \( A \in \varinjlim \mathcal{C} \).
2. There is a pure epimorphism \( \prod_{i \in I} X_i \to A \) for a sequence \( (X_i | i \in I) \) of modules from \( \mathcal{C} \).
3. Every homomorphism \( h : F \to A \) where \( F \) is finitely presented factors through a module in \( \mathcal{C} \).

In particular, \( \varinjlim \mathcal{C} \) is closed under direct limits, and \( \varinjlim \mathcal{C} \cap \text{mod} \, R = \mathcal{C} \).

Crawley-Boevey and Krause observed that Lenzing’s result implies a characterization of when \( \varinjlim \mathcal{C} \) is a definable class of modules. Recall that a subcategory \( \mathcal{M} \) of \( \text{Mod} \, R \) is *definable* provided it is closed under direct limits, direct products and pure submodules.

**Theorem 1.3** [9, 4.2] [21, 3.11] Let \( R \) be a ring, and let \( \mathcal{C} \) be a full additive subcategory of \( \text{mod} \, R \) which is closed under isomorphisms and direct summands. The following statements are equivalent.
1. \( \mathcal{C} \) is covariantly finite in \( \text{mod} \, R \).
2. \( \varinjlim \mathcal{C} \) is closed under products.
3. Every right \( R \)-module has a \( \varinjlim \mathcal{C} \)-preenvelope.
4. \( \varinjlim \mathcal{C} \) is definable.

For example, if \( R \) is a left coherent and right perfect ring, then \( \varinjlim \mathcal{P}_1^{<\infty} = \mathcal{P}_1 \) is closed under products, so \( \mathcal{P}_1^{<\infty} \) is covariantly finite in \( \text{mod} \, R \), cf. [4], [18].

C. **Cotorsion pairs.** Next, we recall the notion of a cotorsion pair. This is the analog of the classical (non-hereditary) torsion pair where \( \text{Hom} \) is replaced by \( \text{Ext}^1 \).

For a class of modules \( \mathcal{M} \subseteq \text{Mod} \, R \), we set \( \mathcal{M}^{\perp 1} = \{ X \in \text{Mod} \, R \mid \text{Ext}^1_R(M, X) = 0 \text{ for all } M \in \mathcal{M} \} \) and \( ^{\perp 1} \mathcal{M} = \{ X \in \text{Mod} \, R \mid \text{Ext}^1_R(X, M) = 0 \text{ for all } M \in \mathcal{M} \} \).

By the well-known properties of \( \text{Ext} \) collected below in Lemma 1.4, the class \( \mathcal{M}^{\perp 1} \) is definable if \( \mathcal{M} \) consists of finitely presented modules over a right coherent ring, and the class \( ^{\perp 1} \mathcal{M} \) is closed under direct limits if \( \mathcal{M} \) consists of pure-injective modules.
Lemma 1.4 [13, Lemma 10.2.4], [5, Chap 1, Proposition 10.1] Let $R$ be a ring, $M$ an $R$-module, and $\{(N_{a}, f_{a}) | \alpha \leq \beta \in I\}$ an arbitrary direct system of modules. Then the following hold true for each $n < \omega$.

1. $\text{Ext}^n_R(M, \lim_{\alpha \in I} N_{a}) \cong \lim_{\alpha \in I} \text{Ext}^n_R(M, N_{a})$ provided that $M$ is FP$_{n+1}$.

2. $\text{Ext}^n_R(\lim_{\alpha \in I} N_{a}, M) \cong \lim_{\alpha \in I} \text{Ext}^n_R(N_{a}, M)$ provided that $M$ is pure-injective.

Let $\mathcal{A}, \mathcal{B} \subseteq \text{Mod}R$ be classes of modules. Then $(\mathcal{A}, \mathcal{B})$ is said to be a cotorsion pair if $\mathcal{A} = \perp^{\perp} \mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$. The class $\mathcal{A} \cap \mathcal{B}$ is called the kernel of the cotorsion pair $(\mathcal{A}, \mathcal{B})$.

The basic relation between cotorsion pairs and approximations goes back to Salce [27]. It may be viewed as a substitute for the non-existence of a duality for arbitrary modules:

Lemma 1.5 [27, Corollary 2.4] Let $R$ be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. The following are equivalent:

1. Every module has a special $\mathcal{A}$-precover.

2. Every module has a special $\mathcal{B}$-preenvelope.

In this case, the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called complete.

Moreover, we say that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is closed if $\mathcal{A}$ is closed under direct limits. The importance of this notion comes from the following result of Enochs: If $(\mathcal{A}, \mathcal{B})$ is a complete and closed cotorsion pair, then every module has an $\mathcal{A}$-cover and a $\mathcal{B}$-envelope [13, 7.2.6].

Complete and/or closed cotorsion pairs occur quite frequently. For a class of modules $\mathcal{C}$, let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair cogenerated by $\mathcal{C}$, that is, let $\mathcal{B} = \mathcal{C}^{\perp}$ and $\mathcal{A} = \perp^{\perp}(\mathcal{C}^{\perp})$. Then we know from [11] that $(\mathcal{A}, \mathcal{B})$ is complete provided that the isomorphism classes of modules in $\mathcal{C}$ form a set.

In this case, there is a useful description of the modules in $\mathcal{A}$. Recall that for an ordinal $\sigma$, a chain of modules $(M_{\alpha} | \alpha \leq \sigma)$ is said to be continuous provided that $M_{\alpha} \subseteq M_{\alpha+1}$ for all $\alpha < \sigma$ and $M_{\alpha} = \bigcup_{\beta < \alpha} M_{\beta}$ for all limit ordinals $\alpha \leq \sigma$. Moreover, if $\mathcal{S}$ is a class of modules, a module $M$ is $\mathcal{S}$-filtered provided that there is a continuous chain $(M_{\alpha} | \alpha \leq \sigma)$ consisting of submodules of $M$ such that $M = M_{\sigma}$, and each of the modules $M_{0}, M_{\alpha+1}/M_{\alpha} (\alpha < \sigma)$, is isomorphic to an element of $\mathcal{S}$. Now, if the isomorphism classes of $\mathcal{C}$ form a set $\mathcal{S}$, then $\mathcal{A} = \perp^{\perp}(\mathcal{C}^{\perp})$ consists of all direct summands of $\mathcal{S} \cup \{R\}$-filtered modules, [28, Theorem 2.2].
Dually, let \((\mathcal{A}, \mathcal{B})\) be the cotorsion pair generated by \(\mathcal{C}\), that is, let \(\mathcal{A} = \perp \mathcal{C}\) and \(\mathcal{B} = (\perp \mathcal{C})\perp\). Then we know from [12] that \((\mathcal{A}, \mathcal{B})\) is complete and closed provided that \(\mathcal{C}\) consists of pure injective modules.

D. Examples of Complete Cotorsion Pairs. Let \(n < \omega\). Denote by \(\mathcal{P}_n\) (\(\mathcal{I}_n\)) the class of all modules of projective (injective) dimension at most \(n\), and by \(\mathcal{F}_n\) the class of all modules of flat (= weak) dimension at most \(n\). Then \((\mathcal{P}_n, (\mathcal{P}_n)^\perp)\) and \((\mathcal{I}_n, \mathcal{I}_n)\) are complete cotorsion pairs, cf. [13, 7.4.6] and [28, 2.1]. Moreover, \((\mathcal{F}_n, (\mathcal{F}_n)^\perp)\) is complete and closed, [12].

For \(N \in \text{Mod-}\mathcal{R}\), let \(N^c = \text{Hom}_\mathcal{R}(N, \mathbb{Q}/\mathbb{Z})\) be the dual module of \(N\). Denote by \(\mathcal{H}\) the class of all dual modules of all left \(\mathcal{R}\)-modules. It is well-known that the class \(\mathcal{P}\) of all pure-injective modules consists of all direct summands of modules in \(\mathcal{H}\), and that the cotorsion pair \((\mathcal{F}_0, (\mathcal{F}_0)^\perp)\) is generated by \(\mathcal{H}\) (and by \(\mathcal{P}\)), cf. [13].

Next, we give a criterion for equality of two cotorsion pairs. Recall that a class \(\mathcal{M} \subseteq \text{Mod}\mathcal{R}\) is resolving (coresolving) if it is closed under extensions, kernels of epimorphisms (cokernels of monomorphisms), and it contains all projective (injective) modules. For example, the class \(\perp \mathcal{M} = \{X \in \text{Mod}\mathcal{R} \mid \text{Ext}^i_{\mathcal{R}}(X, M) = 0\text{ for all }M \in \mathcal{M}\text{ and all }i > 0\}\) is resolving, while \(\mathcal{M}^\perp = \{X \in \text{Mod}\mathcal{R} \mid \text{Ext}^i_{\mathcal{R}}(M, X) = 0\text{ for all }M \in \mathcal{M}\text{ and all }i > 0\}\) is coresolving.

**Lemma 1.6** Let \((\mathcal{E}, \mathcal{D})\) be a complete cotorsion pair such that \(\mathcal{E}\) is resolving. Let further \((\mathcal{X}, \mathcal{Y})\) be a cotorsion pair with \(\mathcal{E} \subseteq \mathcal{X}\). Then the two cotorsion pairs coincide if and only if \(\mathcal{X} \cap \mathcal{D} \subseteq \mathcal{E} \cap \mathcal{D}\).

**Proof:** It is enough to verify \(\mathcal{X} \subseteq \mathcal{E}\) in case \(\mathcal{X} \cap \mathcal{D} \subseteq \mathcal{E} \cap \mathcal{D}\). Take \(X \in \mathcal{X}\) and consider a special \(\mathcal{D}\)-preenvelope \(0 \rightarrow X \rightarrow D \rightarrow E \rightarrow 0\). Since \(E \in \mathcal{E} \subseteq \mathcal{X}\) and \(\mathcal{X}\) is closed under extensions, we have \(D \in \mathcal{X} \cap \mathcal{D} \subseteq \mathcal{E} \cap \mathcal{D}\). Thus \(X \in \mathcal{E}\) because \(\mathcal{E}\) is resolving. □

Finally, we consider cotorsion pairs induced by a tilting module. Recall from [3] that a module \(T\) is a tilting module provided that

1. \(\text{pd} T < \infty\);
2. \(\text{Ext}^i_{\mathcal{R}}(T, T(I)) = 0\) for each \(i > 0\) and all sets \(I\);
3. There is \(r \geq 0\) and a long exact sequence \(0 \rightarrow R \rightarrow T_0 \rightarrow \cdots \rightarrow T_r \rightarrow 0\) with \(T_i \in \text{Add} T\) for each \(0 \leq i \leq r\).

In this case \((\perp (T\perp), T\perp)\) is a complete cotorsion pair with the kernel \(\text{Add} T\), and \((T\perp) \subseteq \mathcal{P}_n\) where \(n = \text{pd} T\), see [3, Section 2].
2 The closure of a cotorsion pair

We now consider the natural partial order \( \leq \) on the class of all cotorsion pairs induced by inclusion of the first components. Observe that \( \leq \) is a complete lattice order, the least element being \( \mathcal{G} = (\mathcal{P}_0, \text{Mod-}R) \), the largest \( \mathcal{E} = (\text{Mod-}R, \mathcal{I}_0) \), and the meet of the cotorsion pairs \( \{(A_\alpha, B_\alpha) \mid \alpha \in I\} \) being \( (\bigcap_{\alpha \in I} A_\alpha, (\bigcap_{\alpha \in I} A_\alpha)^{-1}) \).

Since \( \mathcal{E} \) is closed, and meets of closed cotorsion pairs are likewise closed, each cotorsion pair \( \mathcal{C} \) is contained in the smallest closed one, the closure of \( \mathcal{C} \).

The interesting case is when the closure is complete, hence provides for envelopes and covers of modules. We will show that this always occurs when \( \mathcal{C} \) is cogenerated by a class of finitely presented modules over a right coherent ring (see Corollary 2.4 below).

For a class of modules \( \mathcal{C} \) we denote by \( \widetilde{\mathcal{C}} \) the class of all pure epimorphic images of elements of \( \mathcal{C} \). Clearly, \( \mathcal{C} \cap \text{mod} \, R = \widetilde{\mathcal{C}} \cap \text{mod} \, R \) provided that \( \mathcal{C} \) is closed under direct summands.

For example, if \( (\mathcal{A}, \mathcal{B}) \) is a complete cotorsion pair, then the class \( \widehat{\mathcal{A}} \) is easily seen to coincide with the class of all modules \( M \) such that each (or some) special \( \mathcal{A} \)-precover of \( M \) is a pure epimorphism.

Define \( \mathcal{C}^r = \ker \text{Tor}_1^R(\mathcal{C}, -) \) for a class \( \mathcal{C} \subseteq \text{Mod} \, R \), and \( \mathcal{D}^r = \ker \text{Tor}_1^R(-, \mathcal{D}) \) for a class \( \mathcal{D} \subseteq \text{RMod} \). For a class \( \mathcal{C} \subseteq \text{Mod} \, R \), we define \( \widehat{\mathcal{C}} = \mathcal{C}^r \).

Note that \( \varprojlim \mathcal{C} \subseteq \widehat{\mathcal{C}} \), and \( \mathcal{C} \subseteq \widehat{\mathcal{C}} \), since \( \widehat{\mathcal{C}} \) is obviously closed under direct limits and pure epimorphic images. Moreover, we have

**Lemma 2.1** Let \( R \) be a ring, \( \mathcal{C} \) be a class of modules, and \( (\mathcal{A}, \mathcal{B}) \) be the cotorsion pair cogenerated by \( \mathcal{C} \).

1. \( \widehat{\mathcal{C}} = \mathcal{C}^r (\mathcal{B} \cap \mathcal{H}) = \widehat{\mathcal{A}} \).

2. Assume that \( \mathcal{C} \) is closed under arbitrary direct sums. Then \( \varprojlim \mathcal{C} \subseteq \widetilde{\mathcal{C}} \subseteq \widehat{\mathcal{C}} \).

3. Assume that \( \mathcal{C} \) consists of \( \text{FP}_2 \) modules. Then \( M \in \mathcal{B} \) if and only if \( M^{cn} \in \mathcal{B} \) for any module \( M \). In particular, \( \widehat{\mathcal{C}} = \mathcal{C}^r (\mathcal{B} \cap \mathcal{P} \mathcal{I}) \).

**Proof:** 1. Let \( M \in \text{Mod} \, R \). The well-known Ext-Tor relations yield \( M \in \widehat{\mathcal{C}} \) iff \( M \in (N^c) \) for all \( N \in \mathcal{C}^r \). Moreover, \( N \in \mathcal{C}^r \) iff \( N^c \in \mathcal{C}^{11} \cap \mathcal{H} = \mathcal{B} \cap \mathcal{H} \). Taking \( \mathcal{C} = \mathcal{A} \), we get in particular \( \widehat{\mathcal{A}} = \mathcal{C}^r (\mathcal{B} \cap \mathcal{H}) \).

2. This is clear since \( \widetilde{\mathcal{C}} \) is closed under direct limits in this case.

3. Let \( M \in \text{Mod} \, R \). In this setting, the Ext-Tor relations yield \( M \in \mathcal{B} \) iff \( M^c \in \mathcal{C}^r \).


iff $M^{cc} \in \mathcal{B}$. Since each pure-injective module $M$ is a direct summand in $M^{cc}$, 
$\downarrow^1(B \cap H) = \downarrow^1(\mathcal{B} \cap \mathcal{P} \mathcal{T})$, and the assertion follows by part 1. □

**Lemma 2.2** Let $R$ be a ring. Let $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair such that 
$\mathcal{B}$ is closed under taking double dual modules. Then $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}$.

**Proof:** Let $M \in \widehat{\mathcal{A}}$. By Lemma 2.1.1, $M \in \downarrow^1(B \cap H)$. Let $0 \to B \xrightarrow{\mu} A \to M \to 0$ be an exact sequence with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Consider the canonical pure embedding $\nu : B \to B^{cc}$, and take the push-out of $\mu$ and $\nu$:

$$
\begin{array}{c}
0 \to B \xrightarrow{\mu} A \to M \to 0 \\
\nu \downarrow \quad \quad \eta \downarrow \\
0 \to B^{cc} \xrightarrow{\tau} N \to M \to 0
\end{array}
$$

By assumption, $B^{cc} \in \mathcal{B} \cap H$, so the bottom row splits. It follows that $\nu$ factors through $\mu$, hence $\mu$ is pure, and $M \in \widehat{\mathcal{A}}$. □

By Lemma 2.1, each cotorsion pair $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ is contained in the complete and 
closed cotorsion pair $(\widehat{\mathcal{A}}, \widehat{\mathcal{A}}^\perp)$ generated by the class $\mathcal{B} \cap H$. We now investigate 
whether the latter is the closure of $\mathcal{C}$.

**Theorem 2.3** Let $R$ be a ring. Let $\mathcal{C}$ be a class consisting of FP$_2$ modules such that 
$\mathcal{C}$ is closed under extensions and direct summands and $R \in \mathcal{C}$. Then $\varinjlim \mathcal{C} = \widehat{\mathcal{C}}$.

If $(\mathcal{A}, \mathcal{B})$ denotes the cotorsion pair cogenerated by $\mathcal{C}$ then $\varinjlim \mathcal{C} = \varinjlim \widehat{\mathcal{A}} = \widehat{\mathcal{A}} = \widehat{\mathcal{A}}$.

**Proof:** From Lemmas 2.1.1, 2.1.3 and 2.2 we get that $\widehat{\mathcal{A}} = \widehat{\mathcal{A}} = \widehat{\mathcal{C}}$.

Next, we show that $\mathcal{A} \subseteq \varinjlim \mathcal{C}$. The proof is a generalization of a particular case 
considered in [4, 2.1]. First, the isomorphism classes of $\mathcal{C}$ form a set, so $\mathcal{A}$ consists 
of all direct summands of $\mathcal{C}$-filtered modules. By Lemma 1.2, $\varinjlim \mathcal{C}$ is closed under 
direct limits, hence under direct summands. So it suffices to prove that $\varinjlim \mathcal{C}$ contains 
all $\mathcal{C}$-filtered modules.

We proceed by induction on the length, $\delta$, of the filtration. The cases when $\delta = 0$ 
and $\delta$ is a limit ordinal are clear (the latter by Lemma 1.2). Let $\delta$ be non-limit, so 
we have an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ with $A \in \varinjlim \mathcal{C}$ and $C \in \mathcal{C}$. 
We will apply Lemma 1.2 to prove that $B \in \varinjlim \mathcal{C}$.

Let $h : F \to B$ be a homomorphism with $F$ finitely presented. Since $C$ is FP$_2$, 
there is a presentation $0 \to G \to P \xrightarrow{p} C \to 0$ with $P$ finitely generated.
projective and $G$ finitely presented. There is also $q : P \to B$ such that $p = gq$. We have the commutative diagram

$$
\begin{array}{c}
0 \rightarrow F' \xrightarrow{f'} F \oplus P \xrightarrow{(gh) \oplus p} C \rightarrow 0 \\
h' \downarrow & & h \oplus q \downarrow & & \parallel \\
0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0
\end{array}
$$

Considering the pull-back of $p$ and $(gh) \oplus p$, we see that the pull-back module $U$ is an extension of $G$ by $F \oplus P$, and $F'$ is isomorphic to a direct summand in $U$. So $U$, and $F'$, are finitely presented. Since $A \in \lim \mathcal{C}$, Lemma 1.2 provides for a module $C' \in \mathcal{C}$ and maps $\sigma' : F' \to C'$, $\tau' : C' \to A$ such that $h' = \tau' \sigma'$. Consider the push-out of $f'$ and $\sigma'$:

$$
\begin{array}{c}
0 \rightarrow F' \xrightarrow{f'} F \oplus P \xrightarrow{g(h \oplus p)} C \rightarrow 0 \\
\sigma' \downarrow & & \sigma \downarrow & & \parallel \\
0 \rightarrow C' \xrightarrow{\rho} D \xrightarrow{\sigma} C \rightarrow 0
\end{array}
$$

By assumption, $D \in \mathcal{C}$. By the push-out property, there is $\tau : D \to B$ such that $\tau \sigma = h \oplus q$, hence $\tau \sigma \upharpoonright F = h$. So $h$ factors through $D$, and $B \in \lim \mathcal{C}$.

Now, since $\lim \mathcal{C}$ is closed under pure epimorphic images by Lemma 1.2, we infer that $\tilde{A} \subseteq \lim \mathcal{C}$. So $\lim \mathcal{C} = \tilde{A} = \lim \mathcal{A}$. □

**Corollary 2.4** Let $R$ be a ring and $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair cogenerated by a class of FP$_2$ modules. (For example, let $R$ be right coherent and $\mathcal{C}$ be cogenerated by a subclass of mod$R$.) Then the closure $\overline{\mathcal{C}} = (\overline{\mathcal{A}}, \overline{\mathcal{B}})$ of $\mathcal{C}$ is generated by the class $\mathcal{B} \cap \mathcal{P}$. In particular, $\overline{\mathcal{C}}$ is complete, and $\overline{\mathcal{A}} = \lim \overline{\mathcal{A}} = \tilde{\mathcal{A}}$.

**Proof:** If $\mathcal{C}$ is cogenerated by a class of FP$_2$ modules $\mathcal{D}$, we let $\mathcal{C}$ be the smallest class of modules closed under extensions and direct summands which contains $\mathcal{D} \cup \{R\}$. Then $\mathcal{C}$ also consists of FP$_2$ modules, and it cogenerates $\mathcal{C}$, so Lemma 2.1.3 and Theorem 2.3 apply. □

**Corollary 2.5** Let $R$ be a ring and $\mathcal{C}$ be a class consisting of FP$_2$ modules. Assume $R \in \mathcal{C}$. Then the class $\mathcal{A}(\mathcal{C})$ consists of all direct summands of $\mathcal{C}$-filtered modules while $\mathcal{C}(\mathcal{C})$ consists of all pure-epimorphic images of $\mathcal{C}$-filtered modules.

One of the ingredients in the proof of Theorem 2.3 was Lemma 1.4.1. Part 2 of that Lemma yields another case of coincidence of the classes $\lim \mathcal{A}$ and $\tilde{\mathcal{A}}$: 9
Proposition 2.6 Let $R$ be a ring and $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair generated by a class of dual modules. Then $\mathcal{A} = \lim\limits_\leftarrow \mathcal{A} = \widehat{\mathcal{A}}$.

Proof: By Lemma 1.4.2, $\mathcal{A} = \lim\limits_\leftarrow \mathcal{A}$. By Lemma 2.1.2, it suffices to prove that $\widehat{\mathcal{A}} \subseteq \mathcal{A}$. But $\widehat{\mathcal{A}} = \perp_1(\mathcal{B} \cap \mathcal{H})$ by Lemma 2.1.1. Since $\mathcal{C} \subseteq \mathcal{B} \cap \mathcal{H}$, we have $\widehat{\mathcal{A}} \subseteq \perp_1 \mathcal{C} = \mathcal{A}$. □

As an application, we consider the classes of modules of bounded flat dimension:

Corollary 2.7 Let $R$ be a ring and $n < \infty$. Let $\mathcal{H}_n = \{\Omega^{-n}(M) \mid M \in \mathcal{H}\}$ where $\Omega^{-n}(M)$ denotes an $n$-th cosyzygy of $M$. Then $\mathcal{F}_n = \perp_1 \mathcal{H}_n = \widehat{\mathcal{F}_n}$.

Proof: First, $\mathcal{F}_0 = \tau(R\text{Mod}) = \perp_1 \mathcal{H}_0$. Let $n > 0$. For a module $N$, denote by $\Omega_n(N)$ the $n$-th syzygy (in a projective resolution) of $N$. We have $N \in \mathcal{F}_n$ iff $\Omega_n(N) \in \mathcal{F}_0$ iff $\text{Ext}_R^1(\Omega_n(N), \mathcal{H}_0) = 0$ iff $\text{Ext}_R^{n+1}(N, \mathcal{H}_0) = 0$ iff $\text{Ext}_R^1(N, \mathcal{H}_n) = 0$. This proves that $\mathcal{F}_n = \perp_1 \mathcal{H}_n$. Finally, all cosyzygies of a dual module can be taken to be dual modules as well, so Proposition 2.6 applies. □

3 Modules of projective dimension at most one over commutative domains

In this section, $R$ will denote a commutative domain and $Q$ its quotient field.

We start by reviewing some properties of the class $\mathcal{DI}$ of all divisible modules. Recall that $\mathcal{DI} = \mathcal{CP}^{\perp_1}$ where $\mathcal{CP} = \{R/rR \mid r \in R\}$ denotes a set of representatives of all cyclically presented modules. It is well-known that the complete cotorsion pair $(\perp_1 \mathcal{DI}, \mathcal{DI})$ is cogenerated by a tilting module of projective dimension one, namely the Fuchs' divisible module $\delta$, cf. [15, VII.1], [28].

Denote by $\mathcal{HD}$ the class of all h-divisible modules, that is, of all modules that are homomorphic images of direct sums of copies of $Q$. Clearly, $\mathcal{HD} \subseteq \mathcal{DI}$, and the equality holds true if and only if $R$ is a Matlis domain, that is, pd$Q = 1$, cf. [15, VII.2].

For any domain $R$, we have $\mathcal{P}_1 = \perp_1 \mathcal{HD}$ by [15, VII.2.5], so the complete cotorsion pair $(\mathcal{P}_1, (\mathcal{P}_1)^{\perp_1})$ is generated by the class of all h-divisible modules.

Let us look more closely at the case when $R$ is a Prüfer domain.

Example 3.1 Assume $R$ is Prüfer. Then $(\perp_1 \mathcal{DI}, \mathcal{DI})$ coincides with $(\mathcal{P}_1, (\mathcal{P}_1)^{\perp_1})$ and also with the cotorsion pair $(\mathcal{A}, \mathcal{B})$ cogenerated by $\mathcal{P}^{\perp_\infty}$, so the Fuchs' divisible module $\delta$ is just the tilting module considered in [4].
Indeed, in this case we have $\mathcal{P}^{<\infty} = \mathcal{P}_1^{<\infty} = \text{mod } R$, hence $\mathcal{A} \subseteq \mathcal{P}_1$, cf. [4, Section 2]. So, we have a chain of complete cotorsion pairs $(\mathcal{A}^1, \mathcal{D}^1) \leq (\mathcal{A}, \mathcal{B}) \leq (\mathcal{P}_1, (\mathcal{P}_1)^{\perp_1})$. On the other hand, $\mathcal{P}_1 \subseteq \mathcal{D}^1$ by [14, VI.3.9], and the claim follows.

Note that, in contrast to the artin algebra case [4, 4.2], the fact that $\mathcal{P}^{<\infty}$ is contravariantly finite in $\text{mod } R$ does not force $\mathcal{A}$ to coincide with $\varprojlim \mathcal{P}^{<\infty}$. Indeed, $\mathcal{A}$ is not closed under direct limits unless $R$ is a Dedekind domain. □

In particular, the above observations show that for all Matlis and all Prüfer domains, all divisible modules of projective dimension at most one belong to $\text{Add } \delta$, cf. [14, VI.3.10]. Problem 6 of [14, Chapter VI] asks whether this is true for any domain. The following result provides for an answer.

**Proposition 3.2** The following are equivalent for a commutative domain $R$.

1. All divisible modules of projective dimension at most one belong to $\text{Add } \delta$.
2. The cotorsion pairs $(\mathcal{A}^1, \mathcal{D}^1)$ and $(\mathcal{P}_1, (\mathcal{P}_1)^{\perp_1})$ coincide.
3. $\mathcal{D}^1 = (\mathcal{A}^1)^{\perp_1}$.
4. A module has projective dimension at most one if and only if it is a direct summand of a $\mathcal{CP}$-filtered module.

**Proof:** We always have $\mathcal{D} = (\mathcal{A}^1, \mathcal{D}^1) \leq (\mathcal{P}_1, (\mathcal{P}_1)^{\perp_1}) = \mathfrak{P}$. Condition (1) states that $\mathcal{P}_1 \cap \mathcal{D} = \text{Add } \delta = \mathcal{A}^1 \cap \mathcal{D}^1$. This is equivalent to condition (2) by Lemma 1.6. Further, (2) is equivalent to (3), since the cotorsion pair $\mathfrak{P}$ is generated by $\mathcal{H} \mathcal{D}$. On the other hand, the cotorsion pair $\mathcal{D}$ is cogenerated by the set $\mathcal{CP}$, hence $\mathcal{A}^1 \mathcal{D}^1$ consists of all direct summands of $\mathcal{CP}$-filtered modules, cf. section 1.C. So (2) is equivalent to (4). □

Next, we consider the class $\mathcal{T} \mathcal{F}$ of all torsion-free modules, that is, $\mathcal{T} \mathcal{F} = (\mathcal{CP})^\perp$. There is a duality between torsion-free and divisible modules: a module $N$ is torsion-free iff $N^c$ is divisible, [28, §4]. By a result of Warfield, $(\mathcal{T} \mathcal{F}, Q^{\perp_1} \cap \mathcal{I}_1)$ is a closed cotorsion pair, cf. [28, §2].

First, we show that $(\mathcal{T} \mathcal{F}, Q^{\perp_1} \cap \mathcal{I}_1)$ is generated by the class of all pure-injective modules of injective dimension at most one:

**Lemma 3.3** Let $R$ be a commutative domain. Then $\mathcal{T} \mathcal{F} = \mathcal{I}_1 \cap \mathcal{P} \mathcal{I} = (\mathcal{F}_1)^\perp$.

**Proof:** Since $Q$ is a flat module, $Q^{\perp_1}$ contains all pure-injective modules, so $Q^{\perp_1} \cap \mathcal{I}_1 \cap \mathcal{P} \mathcal{I} = \mathcal{I}_1 \cap \mathcal{P} \mathcal{I}$ and $\mathcal{T} \mathcal{F} \subseteq \mathcal{I}_1 \cap \mathcal{P} \mathcal{I}$. Consider $M \in \mathcal{I}_1 \cap \mathcal{P} \mathcal{I}$. Let
$N \in F_1$. Then $N^c \subseteq I_1$, so $0 = \text{Ext}^1_R(M,N^c) \cong (\text{Tor}_n^R(M,N))^c$. It follows that $\perp^1(I_1 \cap PI) \subseteq (F_1)^\perp$. Finally, since $R/\perp R \in F_1$, we get $(F_1)^\perp \subseteq T F_1$. □

Similarly, the cotorsion pair $(F_1, (F_1)^\perp)$ is generated by the class of all divisible pure-injective modules. In fact, there is a more general result (where, for a class of modules $C$, $\perp^1 C$ denotes the class $\{M \in \text{Mod}R \mid \text{Ext}^1_R(M,C) = 0 \text{ for all } C \in C\}$):

**Lemma 3.4** Let $R$ be a commutative domain and $n > 0$. Then $F_n = \perp^1(\mathcal{DI} \cap PI)$. In particular, $F_1 = \widehat{\mathcal{P}} = \perp^1(\mathcal{DI} \cap PI)$.

**Proof:** First, $F_1 = \widehat{\mathcal{P}}$ by Corollary 2.7 and Lemma 3.3. Let $n > 0$. By Corollary 2.7, $F_n = \perp^1 \mathcal{H}_n = \perp^1 \mathcal{H}_1$. Since $\mathcal{H}_1 \subseteq \mathcal{DI}$, we have $F_n \subseteq \perp^1(\mathcal{DI} \cap PI)$.

Conversely, let $M \in F_n$, and let $N$ be a module such that $N^c \subseteq \mathcal{DI}$. Then $N \in T F_1 = (F_1)^\perp$ by Lemma 3.3. So $0 = \text{Tor}_n^R(\Omega^{n-1}M,N) \cong \text{Tor}_n^R(M,N)$. This shows that $M \in \perp^1(\mathcal{DI} \cap \mathcal{H})$. Finally, by Lemma 2.1.3, $\perp^1(\mathcal{DI} \cap \mathcal{H}) = \perp^1(\mathcal{DI} \cap PI)$. □

Altogether, we have the following chain of complete cotorsion pairs

$$\perp^1(\mathcal{DI}, \mathcal{DI}) \subseteq (\mathcal{P}_1, (\mathcal{P}_1)^\perp) \subseteq (F_1, (F_1)^\perp) = (\widehat{\mathcal{P}}, (\widehat{\mathcal{P}})^\perp) = (\perp^1(\mathcal{DI}, \mathcal{DI}))^{-1}.$$

By Corollary 2.4 and Lemma 3.4, $(F_1, (F_1)^\perp)$ is the closure of $\perp^1(\mathcal{DI}, \mathcal{DI})$ and hence of $(\mathcal{P}_1, (\mathcal{P}_1)^\perp)$.

Problem 22 in [15, p.246] asks for the structure of the modules which are direct limits of modules in $\mathcal{P}_1$. The answer has already been known for Prüfer domains: $\varprojlim \mathcal{P}_1 = \text{Mod}R$, cf. Example 3.1. Since Prüfer domains are characterized as the domains of weak global dimension at most one, $\varprojlim \mathcal{P}_1$ then coincides with the class of all modules of flat dimension at most one. The latter description extends to any commutative domain:

**Theorem 3.5** Let $R$ be a commutative domain. Then

$$\varprojlim \mathcal{P}_1^\perp = \varprojlim \mathcal{P}_1 = F_1.$$  

**Proof:** Theorem 2.3 applied to the cotorsion pair $(A_1, B_1)$ cogenerated by $\mathcal{P}_1^\perp$ and combined with Lemma 2.1.3 yields $\varprojlim \mathcal{P}_1^\perp = \mathcal{P}_1^\perp = \perp^1(B_1 \cap \mathcal{PI})$.

We now claim that $\perp^1(B_1 \cap \mathcal{PI}) = F_1$. By Lemma 3.4, it suffices to show that $B_1 \cap \mathcal{H} = DI \cap \mathcal{H}$. Now, for a module $N \in R\text{-Mod}$, we have $N^c \subseteq B_1 = (\mathcal{P}_1^\perp)^{\perp^1}$ iff $N \in (\mathcal{P}_1^\perp)^{\perp^1}$. By Lemma 3.3, the latter is equivalent to $N \in T F_1$, and hence to $N^c \in DI$. 

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So, the claim is proven, and we obtain \( \mathcal{F}_1 = \lim_{\rightarrow} P_1^{<\infty} \subseteq \lim_{\rightarrow} P_1 \). On the other hand, \( P_1 \subseteq \mathcal{F}_1 \), and \( \lim_{\rightarrow} P_1^{<\infty} \) is closed under direct limits by Lemma 1.2. Thus \( \lim_{\rightarrow} P_1 = \mathcal{F}_1 \). \( \square \)

We don’t know whether this result can be extended to higher dimensions, that is, whether the limit closure of \( \mathcal{P}_n \) always coincides with \( \mathcal{F}_n \).

### 4 Direct limits of finitely presented modules of finite projective dimension

Throughout this section, \( R \) will denote a right coherent ring, and \( (\mathcal{A}, \mathcal{B}) \) will be the (complete) cotorsion pair cogenerated by \( \mathcal{P}^{<\infty} \). From Section 2 we obtain

\[(\mathcal{A}, \mathcal{B}) \leq (\lim_{\rightarrow} \mathcal{P}^{<\infty}, (\lim_{\rightarrow} \mathcal{P}^{<\infty})^\perp)\]

where the right-hand term is the closure. In general, \( (\mathcal{A}, \mathcal{B}) \) is not closed.

**Example 4.1** Let \( R \) be a von-Neumann-regular ring. Then \( \mathcal{P}^{<\infty} \) consists of the finitely generated projective modules, so \( \mathcal{A} = \mathcal{P}_0 \), while \( \lim_{\rightarrow} \mathcal{P}^{<\infty} = \text{Mod} R \). \( \square \)

In order to investigate when \( (\mathcal{A}, \mathcal{B}) \) is a closed cotorsion pair, we will assume \( \text{findim} R < \infty \) and use the tilting module \( T \) from [4] satisfying \( \mathcal{B} = T^\perp \).

Moreover, we will deal with the property that all pure submodules of a given module \( M \) are direct summands. Modules \( M \) with such property are called \( \sum\text{-pure-split} \) in [7]. We will say that \( M \) is \( \sum\text{-pure-split} \) if all modules in \( \text{Add} M \) are pure-split. For example, every \( \sum\text{-pure-injective} \) module is \( \sum\text{-pure-split} \).

**Theorem 4.2** Let \( R \) be a right coherent ring with \( \text{findim} R < \infty \), and let \( T \) be a tilting module with \( \mathcal{B} = T^\perp \). Then the following statements are equivalent.

1. \( \mathcal{A} \) is closed under direct limits (that is, \( \mathcal{A} = \lim_{\rightarrow} \mathcal{P}^{<\infty} \)).
2. \( T \) is \( \sum\text{-pure-split} \).

**Proof:** (1)\( \Rightarrow \) (2): If \( \mathcal{A} \) is closed under direct limits then it is closed under pure-epimorphic images by Theorem 2.3. Recall further that \( \mathcal{B} \) is always closed under pure submodules as it is definable. So, if \( 0 \rightarrow X \rightarrow T' \rightarrow Y \rightarrow 0 \) is a pure-exact sequence with \( T' \in \text{Add} T \), we have \( Y \in \mathcal{A} \) and \( X \in \mathcal{B} \), which shows that the sequence splits.

For the implication (2)\( \Rightarrow \) (1), we first observe that \( (\lim_{\rightarrow} \mathcal{P}^{<\infty}, (\lim_{\rightarrow} \mathcal{P}^{<\infty})^\perp) \) is a cotorsion pair such that \( \lim_{\rightarrow} \mathcal{P}^{<\infty} = \mathcal{A} \) contains the resolving class \( \mathcal{A} \), cf. Theorem 2.3 and
Corollary 2.4. So, by Lemma 1.6, it suffices to prove that \( \lim \mathcal{P}^{<\infty} \cap \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B} = \text{Add} T \). For this purpose, consider \( M \in \lim \mathcal{P}^{<\infty} \cap \mathcal{B} \) and let \( 0 \to B \to A \to M \to 0 \) be an exact sequence with \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \). Then \( A \in \text{Add} T \). Further, since \( M \in \mathcal{A} \), the exact sequence above is pure-exact. Hence it splits by assumption, and \( M \in \text{Add} T \). \( \square \)

Closure under direct limits of \( \mathcal{A} \) can thus be interpreted as a pure-injectivity property of the module \( T \). We now exhibit a further closure property of \( \mathcal{A} \) that can be tested on \( T \). Recall that a module \( M \) with \( \text{Add} M \) being closed under products is said to be \emph{product-complete}, see [22].

**Theorem 4.3** Let \( R \) be a right coherent ring with \( \text{findim} R < \infty \), and let \( T \) be a tilting module with \( \mathcal{B} = T^\perp \). The following statements are equivalent.

1. \( \mathcal{A} \) is definable.
2. \( \mathcal{A} \) is closed under direct products.
3. \( T \) is product-complete.
4. \( \mathcal{A} \) is closed under direct limits, and \( \mathcal{P}^{<\infty} \) is covariantly finite in \( \text{mod} R \).

**Proof:** (1)\( \Rightarrow \) (2) is clear. Further, (2) implies (3) because \( \text{Add} T = \mathcal{A} \cap \mathcal{B} \), see Section 1.D. The implication (4)\( \Rightarrow \) (1) holds by Theorems 1.3 and 2.3.  

(3)\( \Rightarrow \) (4): Observe that a module \( A \) belongs to \( \mathcal{A} \) if and only if there is a long exact sequence \( 0 \to A \to T_0 \to T_1 \to \ldots \to T_n \to 0 \) with \( n = \text{findim} R \) and \( T_i \in \text{Add} T \) for all \( 0 \leq i \leq n \). In fact, the if-part holds since \( \mathcal{A} \) is resolving, and the only-if-part is shown as for the special case \( \mathcal{A} = R \) in the proof of [3, 4.1]. This shows that \( \mathcal{A} \) is closed under direct products if so is \( \text{Add} T \). Moreover, \( T \) is \( \Sigma \)-pure-injective by [22] and thus \( \Sigma \)-pure-split, hence \( \mathcal{A} = \lim \mathcal{P}^{<\infty} \) by Theorem 4.2. From Theorem 1.3 it now follows that the category \( \mathcal{P}^{<\infty} \) is covariantly finite in \( \text{mod} R \). \( \square \)

Note that with the results above we can refine [4, 3.4 and 3.7]:

**Corollary 4.4** Let \( R \) be a right coherent ring with \( \text{findim} R < \infty \), and let \( T \) be a tilting module with \( \mathcal{B} = T^\perp \). Assume that \( \mathcal{P}^{<\infty} \) is covariantly finite (this happens when \( R \) is a two-sided coherent and right perfect ring with \( \text{Findim} R = 1 \), or more generally, with \( \mathcal{P} = \lim \mathcal{P}^{<\infty} \), see [4]). Then \( \mathcal{A} \) is definable if and only if it is closed under direct limits. In other words, \( T \) is product-complete if and only if \( T \) is \( \Sigma \)-pure-split.
Recall that a family \((M_i)_{i \in I}\) is called \emph{locally} (right) \emph{t-nilpotent} if for each sequence of non-isomorphisms \(M_{i_1} \overset{f_1}{\to} M_{i_2} \overset{f_2}{\to} M_{i_3} \ldots\) with indices \((i_n)_{n \in \mathbb{N}}\) from \(I\), and each element \(x \in M_{i_1}\), there exists \(m = m_x \in \mathbb{N}\) such that \(f_m f_{m-1} \ldots f_1(x) = 0\).

**Corollary 4.5** Let \(R\) be a right coherent ring with \(\text{findim} \ R < \infty\), and assume that \(\mathcal{A}\) is closed under direct limits. If \(T\) is a tilting module with \(\mathcal{B} = T^\perp\), then there is a locally t-nilpotent family \((T_i)_{i \in I}\) of modules with local endomorphism ring such that \(T = \bigoplus_{i \in I} T_i\). In particular, the direct sum \(M\) of a representative set of all indecomposable modules from \(\mathcal{A} \cap \mathcal{B}\) is a tilting module with \(\mathcal{B} = M^\perp\).

**Proof:** Recall that a submodule \(X\) of a module \(Y\) is called a local direct summand of \(Y\) if there is a decomposition \(X = \bigsqcup_{k \in K} X_k\) with the property that \(\bigsqcup_{k \in K_0} X_k\) is a direct summand of \(Y\) for every finite subset \(K_0 \subset K\). Of course, \(X\) then is a pure submodule of \(Y\). So, we can now conclude from Theorem 4.2 that all local direct summands of modules in \(\text{Add} T\) are direct summands. By [16, 2.3] this implies that \(T\) has a decomposition \(T = \bigoplus_{i \in I} T_i\) in modules with local endomorphism ring. By [17, 7.3.5] and [1, 2.7] we further know that the family \((T_i)_{i \in I}\) is locally t-nilpotent. The statement on \(M\) then follows from Azumaya’s Decomposition Theorem. □

The next result shows that we can test contravariant finiteness of \(\mathcal{P}^{< \infty}\) on the indecomposable modules in \(\mathcal{P}^{< \infty} \cap \mathcal{B}\). Note however that in general \(\mathcal{P}^{< \infty} \cap \mathcal{B}\) can be zero, see Remark 5.6.

**Corollary 4.6** The following statements are equivalent for an artin algebra \(R\) with \(\text{findim} \ R < \infty\).

\(\text{(a)}\) \(\mathcal{P}^{< \infty}\) is contravariantly finite in \(\text{mod} \ R\).

\(\text{(b)}\) \(\mathcal{P}^{< \infty}\) is covariantly finite in \(\text{mod} \ R\), and the direct sum \(U\) of a representative set of all indecomposable modules from \(\mathcal{P}^{< \infty} \cap \mathcal{B}\) is a \(\Sigma\)-pure-split tilting module with \(U^\perp = \mathcal{B}\).

**Proof:** Of course, if \(\mathcal{P}^{< \infty} \cap \mathcal{B} \neq 0\), then \(U \in \mathcal{A} \cap \mathcal{B} = \text{Add} \ T\). Now, we know from [4, 4.2 and 4.3] that \(\mathcal{P}^{< \infty}\) is contravariantly finite if and only if \(T\) is finitely presented, and that in this case \(\mathcal{A} = \mathcal{P}\) is closed under direct limits. This proves that (a) implies the second statement in (b), while the first is shown in [18]. Conversely, if (b) is satisfied, then we know from Theorems 4.2 and 4.3 that \(U\) is product-complete, which implies by [2, 5.2] that \(U\) is finitely presented, hence (a) holds true. □
If $\text{Findim} \, R < \infty$ and $\mathcal{P}$ is closed under direct limits (e.g. if $R$ is right perfect), we also have
\[
(\lim_{\to} \mathcal{P}^{< \infty}, (\lim_{\to} \mathcal{P}^{< \infty})^\perp) \leq (\mathcal{P}, \mathcal{P}^\perp).
\]
Our next aim is a criterion for $\mathcal{P} = \lim_{\to} \mathcal{P}^{< \infty}$. More generally, for an integer $n \geq 0$, we will consider the cotorsion pair $(\mathcal{A}_n, \mathcal{B}_n)$ cogenerated by $\mathcal{P}^{< \infty}_n$ and describe when $\mathcal{P}_n = \lim_{\to} \mathcal{P}^{< \infty}_n$. Our criterion is in terms of properties of pure-injective modules:

**Theorem 4.7** Let $R$ be a right perfect and right coherent ring.

1. Let $n < \infty$. Then $\mathcal{P}_n = \mathcal{P}^{< \infty}_n \cap \mathcal{H}$.
   Moreover, $\mathcal{P}_n = \lim_{\to} \mathcal{P}^{< \infty}_n$ iff $\mathcal{B}_n \cap \mathcal{P} \mathcal{I} = \mathcal{P}_n^{\perp} \cap \mathcal{P} \mathcal{I}$ iff $\mathcal{P}_n = (\mathcal{P}^{< \infty}_n)^\tau$.

2. Assume $\text{Findim} \, R < \infty$. Then $\mathcal{P} = \mathcal{P}^{< \infty} \cap \mathcal{H}$.
   Moreover, $\mathcal{P} = \lim_{\to} \mathcal{P}^{< \infty}$ iff $\mathcal{B} \cap \mathcal{P} \mathcal{I} = \mathcal{P}^{\perp} \cap \mathcal{P} \mathcal{I}$ iff $\mathcal{P}^\tau = (\mathcal{P}^{< \infty})^\tau$.

**Proof:** 1. By assumption, $\mathcal{P}_n = \mathcal{F}_n$, so Corollary 2.7 and Lemma 2.1.1 give the first assertion.
   Assume $\mathcal{P}_n = \lim_{\to} \mathcal{P}^{< \infty}_n$. Then $\mathcal{B}_n \cap \mathcal{P} \mathcal{I} = \mathcal{P}_n^{\perp} \cap \mathcal{P} \mathcal{I}$ by Lemma 1.4.2. Furthermore, if $\mathcal{B}_n \cap \mathcal{P} \mathcal{I} = \mathcal{P}_n^{\perp} \cap \mathcal{P} \mathcal{I}$, then $\mathcal{B}_n \cap \mathcal{H} = \mathcal{P}_n^{\perp} \cap \mathcal{H}$, so $\mathcal{P}_n^{\tau} = (\mathcal{P}^{< \infty}_n)^\tau$. The latter implies $\mathcal{P}_n^{\tau} = (\mathcal{P}^{< \infty}_n)^\tau = \lim_{\to} \mathcal{P}^{< \infty}_n$ by Corollary 2.7 and Theorem 2.3.

2. By assumption, $\mathcal{P} = \mathcal{P}_n$ and $\mathcal{P}^{< \infty} = \mathcal{P}^{< \infty}_n$ for some $n < \infty$, and part 1. applies. □

5 **On an example of Igusa, Smalø, and Todorov**

Examples 3.1 and 4.1 already yield cases where $\mathcal{P} = \lim_{\to} \mathcal{P}^{< \infty}$ and $\mathcal{A} = \mathcal{P}^{< \infty}$ is not closed under direct limits. We now show that the same can happen over an artin algebra. To this end, we study the following example from [19].

Let $k$ be an algebraically closed field and $R$ the finite-dimensional algebra given by the quiver

\[
\begin{array}{ccc}
1 & \xrightarrow{\gamma} & 2 \\
\alpha & \parallel & \beta
\end{array}
\]

with the relations $\alpha \gamma = \beta \gamma = \gamma \alpha = 0$.

It was shown in [19] that $\text{Findim} \, R = \text{findim} \, R = 1$, but $\mathcal{P}^{< \infty}$ is not contravariantly finite in $\text{mod} \, R$. We then know from [4] that there is a $\mathcal{P}^{< \infty}$-filtered tilting module $T$ of projective dimension 1 which is not finitely generated and satisfies $T^\perp = B$. 

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Moreover, each module of finite projective dimension is a direct limit of elements of $\mathcal{P}^{<\infty}$.

We are now going to show that $\mathcal{A}$ is not closed under direct limits. Let us start by collecting further information on the algebra $R$.

**Conventions:** We write $R$-modules by specifying the vector space $V_i$ corresponding to the vertex, $i = 1, 2$, together with three maps $f_\alpha, f_\beta, f_\gamma$ corresponding to the three arrows.

For each of the two vertices $i = 1, 2$, we denote by $P_i, I_i,$ and $S_i$ the corresponding projective, injective or simple module respectively, and set $I = I_1 \oplus I_2$.

$R$ has a factor algebra isomorphic to the Kronecker algebra which we will denote by $\Lambda$. The $R$-modules where the map $f_\gamma$ corresponding to the arrow $\gamma$ is zero are also $\Lambda$-modules and will be called Kronecker modules.

Note that every $R$-module $X$ admits a canonical short exact sequence $0 \to P_1^{(I)} \to X \to \tilde{X} \to 0$ where $I$ is a set and $\tilde{X}$ is a Kronecker module.

Recall the classification of the indecomposable finite length Kronecker modules:

i) The preprojectives $D_n, n \in \mathbb{N}_0$: Set $V_1 = k^{n+1}, V_2 = k^n, f_\beta = (E, 0)$ and $f_\alpha = (0, E)$ where $E$ denotes the unit in $K^{n \times n}$;

ii) The preinjectives $M_n, n \in \mathbb{N}_0$: Set $V_1 = k^n, V_2 = k^{n+1}, f_\beta = (E, 0)^t$ and $f_\alpha = (0, E)^t$;

iii) The simple regulars: A family $R_\lambda$ indexed by $\lambda \in k$ defined as $V_1 = V_2 = k, f_\beta$ the multiplication with $\lambda \in k, f_\alpha$ the identity, and further a module $R_\infty$ defined as $V_1 = V_2 = k, f_\beta$ the identity, and $f_\alpha = 0$;

iv) Every simple regular module $R_\lambda$ with $\lambda \in k \cup \{\infty\}$ moreover defines a tube $\mathcal{T}_\lambda$, that is a chain of indecomposable modules $X_i$ starting in the simple regular $X_0$ linked by non-split exact sequences $0 \to X_i \to X_{i-1} \oplus X_{i+1} \to X_i \to 0$ which are almost split in mod$\Lambda$. Any finite length indecomposable regular module occurs in this way.

The following is implicitly proven in [19].

**Proposition 5.1** An indecomposable finite length Kronecker module $X$ has finite projective dimension if and only if it lies in $\bigcup_{\lambda \in k} \mathcal{T}_\lambda$. In particular, $\mathcal{P}^{<\infty}$ consists of the finitely $\{P_1\} \cup \bigcup_{\lambda \in k} \mathcal{T}_\lambda$-filtered modules.

**Proof:** Since preprojectives and preinjectives have odd dimension, they are not in $\mathcal{P}^{<\infty}$, see [19]. So $X \in \mathcal{P}^{<\infty}$ must be regular, and taking into account the shape of the regular components (see iv), we see that all other modules in the tube $X$ belongs
to will be in $\mathcal{P}^{<\infty}$ too. But since the simple regular $R_{\infty}$ is isomorphic to $P_2/\text{Soc } P_2,$ where $\text{Soc } P_2 \notin \mathcal{P}^{<\infty},$ we have $R_{\infty} \notin \mathcal{P}^{<\infty},$ hence $X$ belongs to one of the other tubes. For the if-part, note that $R_{\lambda} \cong P_2/P_1$ for each $\lambda \in k.$ Again, we have that all modules in the corresponding tube are then in $\mathcal{P}^{<\infty}.$ □

**Lemma 5.2** [6] The finitely generated modules in $\mathcal{B}$ are precisely the finitely gen$I$-filtered modules. In particular, $S_2,$ and all the finite length indecomposable regular modules in $\mathcal{T}_\infty$ are in $\mathcal{B}.$

**Proof:** $S_2 \cong I_2/\text{Rad } I_2$ and $R_\infty \cong I_2/S_2$ belong to gen$I,$ hence to $\mathcal{B}.$ Since $\mathcal{B}$ is extension-closed, all the tube $\mathcal{T}_\infty$ lies in $\mathcal{B}.$ □

**Lemma 5.3** Let $X, Y$ be Kronecker modules. Assume that either $Y \in \mathcal{A},$ or $\text{Hom}_R(S_2, X) = 0.$ Then $\text{Ext}^1_\lambda(Y, X) = 0$ if and only if $\text{Ext}^1_R(Y, X) = 0.$

**Proof:** The if-part is always true. For the only-if-part, observe that $S_2 = M_0$ is a simple injective $\lambda$-module, and write $X = X' \oplus S_2^{(J)}$ with $\text{Hom}_R(S_2, X') = 0.$ If $Y \in \mathcal{A},$ then $\text{Ext}^1_R(Y, S_2^{(J)}) = 0$ by 5.2, so we can assume that we are in the second case, that is $X = X'.$ To verify $\text{Ext}^1_R(Y, X) = 0,$ we now show that every extension $E_R$ of $X$ by $Y$ is actually a Kronecker module. In fact, consider a short exact sequence $0 \rightarrow X_R \rightarrow E_R \rightarrow Y_R \rightarrow 0,$ and assume that $E_R$ contains a submodule isomorphic to $P_1.$ Since $P_1$ is uniserial and its socle $S_2$ is not contained in $\text{Im} f,$ we deduce that $g |_{P_1}$ is a monomorphism. But this is not possible because $Y$ is a Kronecker module. □

For each tube $\mathcal{T}_\lambda$ with $\lambda \in k \cup \{\infty\},$ we denote by $Y_\lambda$ the corresponding Prüfer module, that is, the direct limit of the chain of inclusions $X_0 \subset X_1 \ldots$ in the tube. It follows from 5.1 that the $Y_\lambda$ with $\lambda \in k$ are $\mathcal{P}^{<\infty}$-filtered and therefore belong to $\mathcal{A}.$

**Proposition 5.4** (1) Let $Y = \bigoplus_{\lambda \in k} Y_\lambda.$ Then $\mathcal{B} = Y \perp.$

(2) A Kronecker module $X$ belongs to $\mathcal{B}$ if and only if $\text{Ext}^1_\lambda(R_\lambda, X) = 0$ for all $\lambda \in k.$

**Proof:** (1) The inclusion $\subseteq$ is clear. For the other inclusion, let $X$ be a module in $Y \perp.$ For any $\lambda \in k$ and any $A \in \mathcal{T}_\lambda$ we have an exact sequence $0 \rightarrow A \rightarrow Y \rightarrow C \rightarrow 0$ where $C$ must belong to $\mathcal{P}$ and thus must have projective dimension at most one. This shows that $\text{Ext}^1_R(A, X) = 0.$ Then we infer from 5.1 that $\text{Ext}^1_R(A, X) = 0$ vanishes on all modules in $\mathcal{P}^{<\infty},$ that is $X \in \mathcal{B}.$

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(2) The only-if-part is clear since $R_\lambda \in \mathcal{P}^{<\infty} \subseteq \mathcal{A}$ for all $\lambda \in k$. For the if-part, we deduce from 5.3 that $\text{Ext}_R^1 (R_\lambda, X) = 0$. Then $\text{Ext}_R^1 (-, X) = 0$ vanishes also on all modules in $\mathcal{T}_\lambda$ and on the $\mathcal{T}_\lambda$-filtered module $Y_\lambda$, see [13, 7.3.4]. Hence $\text{Ext}_R^1 (Y, X) = 0$, and even $X \in Y^\perp$ since $Y \in \mathcal{P} = \mathcal{P}_1$. The claim now follows from (1). □

We will need some further notions from [25]. For a $\Lambda$-module $X$, we denote by $\tau X$ the sum of its finitely generated submodules without nonzero preprojective direct summand. If $X = \tau X$, then $X$ is said to be torsion, and if $\tau X = 0$, or equivalently, if $\text{Hom}_\Lambda (R_\lambda, X) = 0$ for all $\lambda \in k \cup \{\infty\}$, then $X$ is said to be torsionfree. Moreover, $X$ is called divisible if $\text{Ext}_\Lambda^1 (R_\lambda, X) = 0$ for all $\lambda \in k \cup \{\infty\}$. It is shown in [25] that the divisible modules are precisely the direct sums of preinjectives, Prüfer modules and copies of a module $Q$ which is the unique indecomposable torsionfree divisible module. Finally, $X$ is called regular if it does not have any preprojective, nor any preinjective summand.

We have the following immediate consequence of 5.4.

**Corollary 5.5** All Kronecker modules $X$ which are divisible over $\Lambda$ belong to $\mathcal{B}$. In particular, this shows that $Y$ belongs to $\mathcal{A} \cap \mathcal{B} = \text{Add} T$.

**Remark 5.6** (1) The finite length indecomposable Kronecker modules in $\mathcal{B}$ are precisely the preinjectives and the regular modules in $\mathcal{T}_\infty$. Indeed, if $X$ is a finite length indecomposable Kronecker modules in $\mathcal{B}$, then we know from 5.2 that there is a nonzero map $f : I_i \rightarrow X$ with $i = 1$ or $i = 2$. In case $i = 1$, the module $X$ is preinjective since $I_1 = M_1$. In case $i = 2$, the map $f$ cannot be a monomorphism and therefore factors through $I_2 / \text{Soc} I_2 \cong R_\infty$, which shows that $X$ is preinjective or regular in $\mathcal{T}_\infty$.

In particular, this shows that there are no non-zero modules in $\mathcal{P}^{<\infty} \cap \mathcal{B}$. For Kronecker modules, this follows immediately from 5.1. But then no other module $X$ can belong to $\mathcal{P}^{<\infty} \cap \mathcal{B}$, because otherwise we would obtain a contradiction from the canonical exact sequence $0 \rightarrow P_1^n \rightarrow X \rightarrow \tilde{X} \rightarrow 0$ where $n \geq 0$ and $\tilde{X}$ is a Kronecker module.

(2) It is well known that $Q$ is a direct summand of a product of copies of $Y$. Hence $Q$ belongs to $\mathcal{P}$ and is a direct limit of modules in $\mathcal{P}^{<\infty}$. Note however that $Q$ is not $\mathcal{P}^{<\infty}$-filtered. In fact, it follows from 5.1 that a Kronecker module which is $\mathcal{P}^{<\infty}$-filtered is always filtered by finite length regular modules and is therefore torsion.
Let us draw a further consequence from Proposition 5.4. For each tube \( T_\lambda \) with \( \lambda \in k \cup \{ \infty \} \), we denote by \( Z_\lambda \) the corresponding adic module, that is, the inverse limit of the chain of epimorphisms \( \ldots X_1 \to X_0 \) in the tube.

**Corollary 5.7** The adic module \( Z_\lambda \) belongs to \( \mathcal{B} \) if and only if \( \lambda = \infty \).

**Proof:** We know from [20, proof of 9.3] that for each \( \lambda \) there is a universal exact sequence \( 0 \to Z_\lambda \to Q_\lambda \to Y_\lambda \to 0 \) where \( Q_\lambda \) is a direct sum of copies of \( Q \). This shows that the \( Z_\lambda \) with \( \lambda \in k \) do not belong to \( Y^\perp = \mathcal{B} \). On the other hand, applying \( \text{Hom}_R(Y, \_ ) \) on the sequence for \( \lambda = \infty \), we deduce from long exact sequence \( \ldots = \text{Hom}_R(Y, Y_\infty) \to \text{Ext}^1_R(Y, Z_\infty) \to \text{Ext}^1_R(Y, Q_\infty) = 0 \ldots \) that \( Z_\infty \in \mathcal{B} \). \( \square \)

We are now in a position to decide whether \( \mathcal{A} \) is closed under direct limits.

**Theorem 5.8** \( T \) is not product-complete, hence \( \mathcal{A} \neq \mathcal{P} \), and \( \mathcal{A} \) is not closed under direct limits.

**Proof:** Assume that \( T \) is product-complete. Recall that \( Q \) is a direct summand of a product of copies of \( Y \), and \( Y \) lies in \( \text{Add} T \) by Corollary 5.5. So, we infer that \( Q \) must belong to \( \mathcal{A} \). On the other hand, we have just seen that the adic module \( Z_\infty \) is in \( \mathcal{B} \), and it is well-known that \( Z_\infty \) is pure-injective but not \( \Sigma \)-pure-injective. Then there is a cardinal \( \beta \) such that \( Z_\infty ^{(\beta)} \) is not pure-injective, but still is a torsionfree regular module in \( \mathcal{B} \). Now since \( Q \in \mathcal{A} \), we have \( \text{Ext}^1_\lambda (Q, Z_\infty ^{(\beta)}) = 0 \). But it was shown by Okoh in [26, Prop.1 and Remark on p.265] that a torsionfree regular module belongs to \( \text{Ker} \text{Ext}^1_\lambda (Q, -) \) if and only if it is pure-injective. So, we obtain a contradiction. Thus \( T \) is not product-complete. From Corollary 4.4 we then conclude that \( \mathcal{A} \) is not closed under direct limits, and in particular \( \mathcal{A} \neq \mathcal{P} \). \( \square \)

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**References**


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