Direct limits of modules of finite projective dimension

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Dedicated to Paul Eklof on his 60th birthday

Abstract

We describe in homological terms the direct limit closure of a class $\mathcal{C}$ of modules over a ring R. We also determine the closure of the cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ cogenerated by $\mathcal{C}$. As an application, we solve a problem of Fuchs and Salce on the structure of direct limits of modules of projective dimension at most one over commutative domains. Then we consider the case when R is a right coherent ring and $\mathcal{C} = \mathcal{P}^{<\infty}$, the class of all finitely presented modules of finite projective dimension. If findim $R < \infty$ then $\mathfrak{C}$ is a tilting cotorsion pair induced by a tilting module T. We characterize closure properties of $\mathcal{A}$ in terms of properties of T. Finally, we discuss an example where $\mathcal{A}$ is not closed under direct limits.

Let R be a ring. Denote by $\mathcal{P}$ the class of all modules of finite projective dimension, and by $\mathcal{P}^{<\infty}$ the class of all finitely presented modules in $\mathcal{P}$. For $n < \omega$ let $\mathcal{P}_n$ be the class of all modules of projective dimension at most n, and let $\mathcal{P}_n^{<\infty}$ be the corresponding subclass of $\mathcal{P}^{<\infty}$.

In this paper, we study the categories $\varinjlim \mathcal{P}_n$ and $\varinjlim \mathcal{P}_n^{<\infty}$ of all direct limits of modules in $\mathcal{P}_n$ and $\mathcal{P}_n^{<\infty}$, respectively. To this end, we consider the complete cotorsion pair $(\mathcal{A}_n, \mathcal{B}_n)$ cogenerated by $\mathcal{P}_n^{<\infty}$ and investigate the class $\mathcal{A}_n$.

Our main tool is a homological description of $\varinjlim \mathcal{A}_n$. We show that in many cases the limit closure $\varinjlim \mathcal{A}$ of the first component in a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ consists of all modules X satisfying $\operatorname{Ext}^1_R(X, B) = 0$ for each pure-injective module $B \in \mathcal{B}$; there is also a characterization of $\varinjlim \mathcal{A}$ in terms of vanishing of Tor (see Section 2).

This result allows us to discuss a more general question: What is the smallest complete cotorsion pair $\overline{\mathfrak{C}} = (\overline{\mathcal{A}}, \overline{\mathcal{B}})$ containing $\mathfrak{C}$ with $\overline{\mathcal{A}}$ being closed under direct limits? The question is of particular interest because of a classical result of Enochs saying that in this case $\overline{\mathcal{A}}$ is a covering class, and $\overline{\mathcal{B}}$ an enveloping class, in Mod$R$.

*Research of the first author supported by a HWP-grant of LMU Munich. Second author supported by grants GAČR 201-00-0766 and MSM 113200007.

In Corollary 2.4, we answer the question for cotorsion pairs cogenerated by classes of finitely presented modules over right coherent rings.

Section 3 deals with an application to the particular case when $n = 1$ and R is a commutative domain. In Theorem 3.5, we solve a problem of Fuchs and Salce ([15, Problem 22 on p.246]) by showing that a module belongs to $\varinjlim \mathcal{P}_1$ if and only if it has flat dimension at most one. Furthermore, we investigate the divisible modules of projective dimension at most one and answer a question related to the Fuchs' divisible module δ ([14, Problem 6 in Chapter VI]).

In Section 4, we consider right coherent rings and continue our investigation started in [4] of the complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ cogenerated by $\mathcal{P}^{<\infty}$. If findim$R < \infty$, then we know from [4] that there is a tilting module T such that $\mathcal{B} = T^{\perp}$. Furthermore, $\mathcal{A} = \mathcal{P}$ if and only if the category Add$T$ is closed under cokernels of monomorphisms, and in this case the little and the big finitistic dimensions of R coincide: $\operatorname{findim} R = \operatorname{Findim} R$.

Our focus here is on the category $\varinjlim \mathcal{P}^{<\infty}$. Note that $\mathcal{A}$ is always contained in $\varinjlim \mathcal{P}^{<\infty}$. Moreover, if $\mathcal{P}$ is closed under direct limits, e. g. if R is right perfect and Findim$R < \infty$, then $\mathcal{A} \subseteq \varinjlim \mathcal{P}^{<\infty} \subseteq \mathcal{P}$. Using the tilting module T from [4], we investigate closure properties of $\mathcal{A}$. We characterize the cases $\mathcal{A} = \varinjlim \mathcal{P}^{<\infty}$ and $\varinjlim \mathcal{P}^{<\infty} = \mathcal{P}$ in Theorems 4.2 and 4.7. In Theorem 4.3, we determine when $\mathcal{A}$ is a definable class.

Finally, in Section 5, we study an important example in detail, namely the artin algebra introduced by Igusa, Smalø, and Todorov [19]. In this case, we show that $\mathcal{A} = \varinjlim \mathcal{P}^{<\infty}$ fails while $\varinjlim \mathcal{P}^{<\infty} = \mathcal{P}$ holds true.

1 Preliminaries

First, we fix our terminology and notation.

Let R be an arbitrary ring, Mod$R$ be the category of all (right) R-modules, and mod$R$ the subcategory of all finitely presented modules. For a subcategory $\mathcal{M}$ of Mod$R$, we denote by Add$\mathcal{M}$ (respectively add$\mathcal{M}$) the subcategory of all modules isomorphic to a direct summand of a (finite) direct sum of modules of $\mathcal{M}$.

Following [24, p.210], we will say that a module M is FP_n provided that M has a projective resolution

$$\cdots \to P_{k+1} \to P_k \to \cdots \to P_1 \to P_0 \to M \to 0$$

where all P_i with $i \leq n$ are finitely generated. So FP_0 stands for finitely generated,

FP_1 for finitely presented, and a module M is FP_2 if and only if there are $n < \omega$, a finitely presented module K, and a short exact sequence $0 \to K \to R^n \to M \to 0$. Notice that $\mathcal{P}_1^{<\infty}$ coincides with the class of all FP_2 modules of projective dimension ≤ 1. If R is right coherent and M is finitely presented then M is FP_n for all $n \geq 1$.

A. PRECOVERS AND PREENVELOPES. Let $\mathcal{M}$ be a subcategory of $\mathrm{Mod}\,R$, and let A be a right R-module. A morphism $f \in \mathrm{Hom}_R(A, X)$ with $X \in \mathcal{M}$ is an *$\mathcal{M}$-preenvelope* (or a left $\mathcal{M}$-approximation) of A provided that the abelian group homomorphism $\mathrm{Hom}_R(f, M)\colon\ \mathrm{Hom}_R(X, M) \longrightarrow \mathrm{Hom}_R(A, M)$ is surjective for each $M \in \mathcal{M}$. An $\mathcal{M}$-preenvelope $f \in \mathrm{Hom}_R(A, X)$ of A is said to be *special* if f is a monomorphism and $\mathrm{Ext}^1_R(\mathrm{Coker}\, f, M) = 0$ for all $M \in \mathcal{M}$. An *$\mathcal{M}$-envelope* of A is an $\mathcal{M}$-preenvelope $f \in \mathrm{Hom}_R(A, X)$ which is left minimal, that is, h is an automorphism of X whenever $h \in \mathrm{End}_R(X)$ satisfies $hf = f$. If it exists, an $\mathcal{M}$-envelope is unique up to isomorphism.

The notions of an *$\mathcal{M}$-cover* and a (*special*) *$\mathcal{M}$-precover* are defined dually.

Finally, a subcategory $\mathcal{C}$ of $\mathrm{mod}R$ is said to be *covariantly* (respectively, *contravariantly*) *finite* in $\mathrm{mod}R$ if every finitely presented module has a $\mathcal{C}$-preenvelope (respectively, a $\mathcal{C}$-precover).

B. CLOSURE UNDER DIRECT LIMITS. Let $\mathcal{C}$ be a class of modules. Denote by $\varinjlim \mathcal{C}$ the class of all modules D such that $D = \varinjlim_{i \in I} C_i$ where $\{C_i \mid i \in I\}$ is a direct system of modules from $\mathcal{C}$.

In general, the class $\varinjlim \mathcal{C}$ is not closed under direct limits:

Example 1.1 Let $R = \mathbb{Z}$ and let $\mathcal{C} = \{A\}$ where A is an indecomposable torsion-free abelian group of rank $r \geq 2$ such that $\mathrm{End}(A) \cong \mathbb{Z}$. (There is a proper class of such groups: by [10, XII.3.5], there exist arbitrarily large indecomposable torsion-free abelian groups such that $\mathrm{End}(A) \cong \mathbb{Z}$.)

Consider the direct system $\{C_n \mid n < \infty\}$ where $C_n = A$ for all $n < \infty$ and $f_n : C_n \to C_{n+1}$ is the multiplication by n. Then $\varinjlim_{n<\infty} C_n$ coincides with the injective envelope $E(A) \cong \mathbb{Q}^{(r)}$ of A.

Now, consider the direct system $\{D_n \mid n < \infty\}$ where $D_n = E(A)$ for all $n < \infty$ and $g_n : D_n \to D_{n+1}$ is the projection on a fixed copy of $\mathbb{Q}$ in $E(A)$. Then $\varinjlim_{n<\infty} D_n \cong \mathbb{Q}$.

On the other hand, $\mathbb{Q} \notin \varinjlim \mathcal{C}$. Namely, let $(C_i \mid i \in I\}$ be a direct system with $C_i = A$ for all $i \in I$, and let $D = \varinjlim_{i \in I} C_i$. Since $\mathrm{End}(A) \cong \mathbb{Z}$ and A is torsion-free, all maps in the direct system are either monomorphisms or zero. It follows that either D contains a copy of A, or else $D = 0$. Anyway, D has rank $\neq 1$, so $D \ncong \mathbb{Q}$. $\square$

There is an important case when $\varinjlim \mathcal{C}$ is always closed under direct limits. The characterization goes back to Lenzing:

Lemma 1.2 [23] Let R be a ring, and let $\mathcal{C}$ be a full additive subcategory of mod R which is closed under isomorphisms and direct summands. The following statements are equivalent for a module A.
(1) $A \in \varinjlim \mathcal{C}$.
(2) There is a pure epimorphism $\coprod_{i\in I} X_i \to A$ for a sequence $(X_i \mid i \in I)$ of modules from $\mathcal{C}$.
(3) Every homomorphism $h : F \to A$ where F is finitely presented factors through a module in $\mathcal{C}$.
In particular, $\varinjlim \mathcal{C}$ is closed under direct limits, and $\varinjlim \mathcal{C} \cap \text{mod}R = \mathcal{C}$.

Crawley-Boevey and Krause observed that Lenzing's result implies a characterization of when $\varinjlim \mathcal{C}$ is a definable class of modules. Recall that a subcategory $\mathcal{M}$ of Mod R is *definable* provided it is closed under direct limits, direct products and pure submodules.

Theorem 1.3 [9, 4.2] [21, 3.11] Let R be a ring, and let $\mathcal{C}$ be a full additive subcategory of mod$R$ which is closed under isomorphisms and direct summands. The following statements are equivalent.
(1) $\mathcal{C}$ is covariantly finite in mod R.
(2) $\varinjlim \mathcal{C}$ is closed under products.
(3) Every right R-module has a $\varinjlim \mathcal{C}$-preenvelope.
(4) $\varinjlim \mathcal{C}$ is definable.

For example, if R is a left coherent and right perfect ring, then $\varinjlim \mathcal{P}_1^{<\infty} = \mathcal{P}_1$ is closed under products, so $\mathcal{P}_1^{<\infty}$ is covariantly finite in mod$R$, cf. [4], [18].

C. <u>Cotorsion pairs.</u> Next, we recall the notion of a cotorsion pair. This is the analog of the classical (non-hereditary) torsion pair where Hom is replaced by Ext^1.

For a class of modules $\mathcal{M} \subseteq \text{Mod}R$, we set $\mathcal{M}^{\perp_1} = \{X \in \text{Mod}R \mid \text{Ext}^1_R(M, X) = 0$ for all $M \in \mathcal{M}\}$ and ${}^{\perp_1}\mathcal{M} = \{X \in \text{Mod}R \mid \text{Ext}^1_R(X, M) = 0 \text{ for all } M \in \mathcal{M}\}$.

By the well-known properties of Ext collected below in Lemma 1.4, the class $\mathcal{M}^{\perp_1}$ is definable if $\mathcal{M}$ consists of finitely presented modules over a right coherent ring, and the class ${}^{\perp_1}\mathcal{M}$ is closed under direct limits if $\mathcal{M}$ consists of pure-injective modules.

Lemma 1.4 [13, Lemma 10.2.4], [5, Chap 1, Proposition 10.1] Let R be a ring, M an R-module, and $\{(N_\alpha, f_{\alpha\beta}) \mid \alpha \leq \beta \in I\}$ an arbitrary direct system of modules. Then the following hold true for each $n < \omega$.
(1) $\mathrm{Ext}^n_R(M, \varinjlim_{\alpha \in I} N_\alpha) \cong \varinjlim_{\alpha \in I} \mathrm{Ext}^n_R(M, N_\alpha)$ provided that M is FP_{n+1}.
(2) $\mathrm{Ext}^n_R(\varinjlim_{\alpha \in I} N_\alpha, M) \cong \varprojlim_{\alpha \in I} \mathrm{Ext}^n_R(N_\alpha, M)$ provided that M is pure-injective.

Let $\mathcal{A}, \mathcal{B} \subseteq \mathrm{Mod}R$ be classes of modules. Then $(\mathcal{A}, \mathcal{B})$ is said to be a *cotorsion pair* if $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$. The class $\mathcal{A} \cap \mathcal{B}$ is called the *kernel* of the cotorsion pair $(\mathcal{A}, \mathcal{B})$.

The basic relation between cotorsion pairs and approximations goes back to Salce [27]. It may be viewed as a substitute for the non-existence of a duality for arbitrary modules:

Lemma 1.5 [27, Corollary 2.4.] Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. The following are equivalent:
(1) Every module has a special $\mathcal{A}$-precover.
(2) Every module has a special $\mathcal{B}$-preenvelope.

In this case, the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called *complete*.

Moreover, we say that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *closed* if $\mathcal{A}$ is closed under direct limits. The importance of this notion comes from the following result of Enochs: If $(\mathcal{A}, \mathcal{B})$ is a complete and closed cotorsion pair, then every module has an $\mathcal{A}$-cover and a $\mathcal{B}$-envelope [13, 7.2.6].

Complete and/or closed cotorsion pairs occur quite frequently. For a class of modules $\mathcal{C}$, let $(\mathcal{A}, \mathcal{B})$ be the *cotorsion pair cogenerated by* $\mathcal{C}$, that is, let $\mathcal{B} = \mathcal{C}^{\perp_1}$ and $\mathcal{A} = {}^{\perp_1}(\mathcal{C}^{\perp_1})$. Then we know from [11] that $(\mathcal{A}, \mathcal{B})$ is complete provided that the isomorphism classes of modules in $\mathcal{C}$ form a set.
In this case, there is a useful description of the modules in $\mathcal{A}$. Recall that for an ordinal σ, a chain of modules $(M_\alpha \mid \alpha \leq \sigma)$ is said to be *continuous* provided that $M_\alpha \subseteq M_{\alpha+1}$ for all $\alpha < \sigma$ and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for all limit ordinals $\alpha \leq \sigma$. Moreover, if $\mathcal{S}$ is a class of modules, a module M is $\mathcal{S}$*-filtered* provided that there is a continuous chain $(M_\alpha \mid \alpha \leq \sigma)$ consisting of submodules of M such that $M = M_\sigma$, and each of the modules M_0, $M_{\alpha+1}/M_\alpha$ $(\alpha < \sigma)$, is isomorphic to an element of $\mathcal{S}$. Now, if the isomorphism classes of $\mathcal{C}$ form a set $\mathcal{S}$, then $\mathcal{A} = {}^{\perp_1}(\mathcal{C}^{\perp_1})$ consists of all direct summands of $\mathcal{S} \cup \{R\}$-filtered modules, [28, Theorem 2.2].

Dually, let $(\mathcal{A}, \mathcal{B})$ be the *cotorsion pair generated by* $\mathcal{C}$, that is, let $\mathcal{A} = {}^{\perp_1}\mathcal{C}$ and $\mathcal{B} = ({}^{\perp_1}\mathcal{C})^{\perp_1}$. Then we know from [12] that $(\mathcal{A}, \mathcal{B})$ is complete and closed provided that $\mathcal{C}$ consists of pure injective modules.

<u>D. Examples of complete cotorsion pairs.</u> Let $n < \omega$. Denote by $\mathcal{P}_n$ ($\mathcal{I}_n$) the class of all modules of projective (injective) dimension at most n, and by $\mathcal{F}_n$ the class of all modules of flat (= weak) dimension at most n. Then $(\mathcal{P}_n, (\mathcal{P}_n)^{\perp_1})$ and $({}^{\perp_1}\mathcal{I}_n, \mathcal{I}_n)$ are complete cotorsion pairs, cf. [13, 7.4.6] and [28, 2.1]. Moreover, $(\mathcal{F}_n, (\mathcal{F}_n)^{\perp_1})$ is complete and closed, [12].

For $N \in$ Mod-R, let $N^c = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$ be the *dual module* of N. Denote by $\mathcal{H}$ the class of all dual modules of all left R-modules. It is well-known that the class $\mathcal{PI}$ of all pure-injective modules consists of all direct summands of modules in $\mathcal{H}$, and that the cotorsion pair $(\mathcal{F}_0, (\mathcal{F}_0)^{\perp_1})$ is generated by $\mathcal{H}$ (and by $\mathcal{PI}$), cf. [13].

Next, we give a criterion for equality of two cotorsion pairs. Recall that a class $\mathcal{M} \subseteq \mathrm{Mod}R$ is *resolving* (*coresolving*) if it is closed under extensions, kernels of epimorphisms (cokernels of monomorphisms), and it contains all projective (injective) modules. For example, the class ${}^{\perp}\mathcal{M} = \{X \in \mathrm{Mod}R \mid \mathrm{Ext}^i_R(X, M) = 0$ for all $M \in \mathcal{M}$ and all $i > 0\}$ is resolving, while $\mathcal{M}^{\perp} = \{X \in \mathrm{Mod}R \mid \mathrm{Ext}^i_R(M, X) = 0$ for all $M \in \mathcal{M}$ and all $i > 0\}$ is coresolving.

Lemma 1.6 Let $(\mathcal{E}, \mathcal{D})$ be a complete cotorsion pair such that $\mathcal{E}$ is resolving. Let further $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair with $\mathcal{E} \subseteq \mathcal{X}$. Then the two cotorsion pairs coincide if and only if $\mathcal{X} \cap \mathcal{D} \subseteq \mathcal{E} \cap \mathcal{D}$.

Proof: It is enough to verify $\mathcal{X} \subseteq \mathcal{E}$ in case $\mathcal{X} \cap \mathcal{D} \subseteq \mathcal{E} \cap \mathcal{D}$. Take $X \in \mathcal{X}$ and consider a special $\mathcal{D}$-preenvelope $0 \longrightarrow X \longrightarrow D \longrightarrow E \longrightarrow 0$. Since $E \in \mathcal{E} \subseteq \mathcal{X}$ and $\mathcal{X}$ is closed under extensions, we have $D \in \mathcal{X} \cap \mathcal{D} \subseteq \mathcal{E} \cap \mathcal{D}$. Thus $X \in \mathcal{E}$ because $\mathcal{E}$ is resolving. □

Finally, we consider cotorsion pairs induced by a tilting module. Recall from [3] that a module T is a *tilting module* provided that

(T1) $\mathrm{pd}T < \infty$;

(T2) $\mathrm{Ext}^i_R(T, T^{(I)}) = 0$ for each $i > 0$ and all sets I;

(T3) There is $r \geq 0$ and a long exact sequence $0 \longrightarrow R_R \longrightarrow T_0 \longrightarrow \cdots \longrightarrow T_r \longrightarrow 0$ with $T_i \in \mathrm{Add}T$ for each $0 \leq i \leq r$.

In this case $({}^{\perp}(T^{\perp}), T^{\perp})$ is a complete cotorsion pair with the kernel $\mathrm{Add}T$, and ${}^{\perp}(T^{\perp}) \subseteq \mathcal{P}_n$ where $n = \mathrm{pd}T$, see [3, Section 2].

2 The closure of a cotorsion pair

We now consider the natural partial order $\leq$ on the class of all cotorsion pairs induced by inclusion of the first components. Observe that $\leq$ is a complete lattice order, the least element being $\mathfrak{S} = (\mathcal{P}_0, \text{Mod-}R)$, the largest $\mathfrak{L} = (\text{Mod-}R, \mathcal{I}_0)$, and the meet of the cotorsion pairs $\{(\mathcal{A}_\alpha, \mathcal{B}_\alpha) \mid \alpha \in I\}$ being $(\bigcap_{\alpha \in I} \mathcal{A}_\alpha, (\bigcap_{\alpha \in I} \mathcal{A}_\alpha)^{\perp_1})$.

Since $\mathfrak{L}$ is closed, and meets of closed cotorsion pairs are likewise closed, each cotorsion pair $\mathfrak{C}$ is contained in the smallest closed one, the *closure* of $\mathfrak{C}$.

The interesting case is when the closure is complete, hence provides for envelopes and covers of modules. We will show that this always occurs when $\mathfrak{C}$ is cogenerated by a class of finitely presented modules over a right coherent ring (see Corollary 2.4 below).

For a class of modules $\mathcal{C}$ we denote by $\widetilde{\mathcal{C}}$ the class of all pure epimorphic images of elements of $\mathcal{C}$. Clearly, $\mathcal{C} \cap \text{mod}R = \widetilde{\mathcal{C}} \cap \text{mod}R$ provided that $\mathcal{C}$ is closed under direct summands.

For example, if $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair, then the class $\widetilde{\mathcal{A}}$ is easily seen to coincide with the class of all modules M such that each (or some) special $\mathcal{A}$-precover of M is a pure epimorphism.

Define $\mathcal{C}^\intercal = \text{Ker Tor}_1^R(\mathcal{C}, -)$ for a class $\mathcal{C} \subseteq \text{Mod}R$, and ${}^\intercal\mathcal{D} = \text{Ker Tor}_1^R(-, \mathcal{D})$ for a class $\mathcal{D} \subseteq R\text{Mod}$. For a class $\mathcal{C} \subseteq \text{Mod}R$, we define $\widehat{\mathcal{C}} = {}^\intercal(\mathcal{C}^\intercal)$.

Note that $\varinjlim \mathcal{C} \subseteq \widehat{\mathcal{C}}$, and $\widetilde{\mathcal{C}} \subseteq \widehat{\mathcal{C}}$, since $\widehat{\mathcal{C}}$ is obviously closed under direct limits and pure epimorphic images. Moreover, we have

Lemma 2.1 Let R be a ring, $\mathcal{C}$ be a class of modules, and $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair cogenerated by $\mathcal{C}$.

1. $\widehat{\mathcal{C}} = {}^{\perp_1}(\mathcal{B} \cap \mathcal{H}) = \widehat{\mathcal{A}}$.
2. Assume that $\mathcal{C}$ is closed under arbitrary direct sums. Then $\varinjlim \mathcal{C} \subseteq \widetilde{\mathcal{C}} \subseteq \widehat{\mathcal{C}}$.
3. Assume that $\mathcal{C}$ consists of FP_2 modules. Then $M \in \mathcal{B}$ if and only if $M^{cc} \in \mathcal{B}$ for any module M. In particular, $\widehat{\mathcal{C}} = {}^{\perp_1}(\mathcal{B} \cap \mathcal{PI})$.

Proof: 1. Let $M \in \text{Mod}R$. The well-known Ext-Tor relations yield $M \in \widehat{\mathcal{C}}$ iff $M \in {}^{\perp_1}(N^c)$ for all $N \in \mathcal{C}^\intercal$. Moreover, $N \in \mathcal{C}^\intercal$ iff $N^c \in \mathcal{C}^{\perp_1} \cap \mathcal{H} = \mathcal{B} \cap \mathcal{H}$. Taking $\mathcal{C} = \mathcal{A}$, we get in particular $\widehat{\mathcal{A}} = {}^{\perp_1}(\mathcal{B} \cap \mathcal{H})$.
2. This is clear since $\widetilde{\mathcal{C}}$ is closed under direct limits in this case.
3. Let $M \in \text{Mod}R$. In this setting, the Ext-Tor relations yield $M \in \mathcal{B}$ iff $M^c \in \mathcal{C}^\intercal$

iff $M^{cc} \in \mathcal{B}$. Since each pure-injective module M is a direct summand in M^{cc}, ${}^{\perp_1}(\mathcal{B} \cap \mathcal{H}) = {}^{\perp_1}(\mathcal{B} \cap \mathcal{PI})$, and the assertion follows by part 1. $\square$

Lemma 2.2 Let R be a ring. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair such that $\mathcal{B}$ is closed under taking double dual modules. Then $\widetilde{\mathcal{A}} = \widehat{\mathcal{A}}$.

Proof: Let $M \in \widehat{\mathcal{A}}$. By Lemma 2.1.1, $M \in {}^{\perp_1}(\mathcal{B} \cap \mathcal{H})$. Let $0 \longrightarrow B \stackrel{\mu}{\longrightarrow} A \longrightarrow M \longrightarrow 0$ be an exact sequence with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Consider the canonical pure embedding $\nu : B \to B^{cc}$, and take the push-out of μ and ν:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \stackrel{\mu}{\longrightarrow} & A & \longrightarrow & M & \longrightarrow & 0 \\ & & \nu\downarrow & & \eta\downarrow & & \| & & \\ 0 & \longrightarrow & B^{cc} & \stackrel{\tau}{\longrightarrow} & N & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

By assumption, $B^{cc} \in \mathcal{B} \cap \mathcal{H}$, so the bottom row splits. It follows that ν factors through μ, hence μ is pure, and $M \in \widetilde{\mathcal{A}}$. $\square$

By Lemma 2.1, each cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is contained in the complete and closed cotorsion pair $(\widehat{\mathcal{A}}, \widehat{\mathcal{A}}^{\perp_1})$ generated by the class $\mathcal{B} \cap \mathcal{H}$. We now investigate whether the latter is the closure of $\mathfrak{C}$.

Theorem 2.3 Let R be a ring. Let $\mathcal{C}$ be a class consisting of FP_2 modules such that $\mathcal{C}$ is closed under extensions and direct summands and $R \in \mathcal{C}$. Then $\varinjlim \mathcal{C} = \widehat{\mathcal{C}}$.
If $(\mathcal{A}, \mathcal{B})$ denotes the cotorsion pair cogenerated by $\mathcal{C}$ then $\varinjlim \mathcal{C} = \varinjlim \mathcal{A} = \widetilde{\mathcal{A}} = \widehat{\mathcal{A}}$.

Proof: From Lemmas 2.1.1, 2.1.3 and 2.2 we get that $\widetilde{\mathcal{A}} = \widehat{\mathcal{A}} = \widehat{\mathcal{C}}$.

Next, we show that $\mathcal{A} \subseteq \varinjlim \mathcal{C}$. The proof is a generalization of a particular case considered in [4, 2.1]. First, the isomorphism classes of $\mathcal{C}$ form a set, so $\mathcal{A}$ consists of all direct summands of $\mathcal{C}$-filtered modules. By Lemma 1.2, $\varinjlim \mathcal{C}$ is closed under direct limits, hence under direct summands. So it suffices to prove that $\varinjlim \mathcal{C}$ contains all $\mathcal{C}$-filtered modules.

We proceed by induction on the length, δ, of the filtration. The cases when $\delta = 0$ and δ is a limit ordinal are clear (the latter by Lemma 1.2). Let δ be non-limit, so we have an exact sequence $0 \longrightarrow A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \longrightarrow 0$ with $A \in \varinjlim \mathcal{C}$ and $C \in \mathcal{C}$. We will apply Lemma 1.2 to prove that $B \in \varinjlim \mathcal{C}$.

Let $h : F \to B$ be a homomorphism with F finitely presented. Since C is FP_2, there is a presentation $0 \longrightarrow G \longrightarrow P \stackrel{p}{\longrightarrow} C \longrightarrow 0$ with P finitely generated projective and G finitely presented. There is also $q : P \to B$ such that $p = gq$. We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \xrightarrow{f'} & F \oplus P & \xrightarrow{(gh)\oplus p} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h\oplus q} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

Considering the pull-back of p and $(gh) \oplus p$, we see that the pull-back module U is an extension of G by $F \oplus P$, and F' is isomorphic to a direct summand in U. So U, and F', are finitely presented. Since $A \in \varinjlim \mathcal{C}$, Lemma 1.2 provides for a module $C' \in \mathcal{C}$ and maps $\sigma' : F' \to C'$, $\tau' : C' \to A$ such that $h' = \tau'\sigma'$. Consider the push-out of f' and σ':

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \xrightarrow{f'} & F \oplus P & \xrightarrow{gh\oplus p} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma'} & & \downarrow{\scriptstyle \sigma} & & \| & & \\ 0 & \longrightarrow & C' & \xrightarrow{\rho} & D & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

By assumption, $D \in \mathcal{C}$. By the push-out property, there is $\tau : D \to B$ such that $\tau\sigma = h \oplus q$, hence $\tau\sigma \restriction F = h$. So h factors through D, and $B \in \varinjlim \mathcal{C}$.

Now, since $\varinjlim \mathcal{C}$ is closed under pure epimorphic images by Lemma 1.2, we infer that $\widetilde{\mathcal{A}} \subseteq \varinjlim \mathcal{C}$. So $\varinjlim \mathcal{C} = \widetilde{\mathcal{A}} = \varinjlim \mathcal{A}$. $\square$

Corollary 2.4 Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair cogenerated by a class of FP_2 modules. (For example, let R be right coherent and $\mathfrak{C}$ be cogenerated by a subclass of mod$R$.) Then the closure $\overline{\mathfrak{C}} = (\overline{\mathcal{A}}, \overline{\mathcal{B}})$ of $\mathfrak{C}$ is generated by the class $\mathcal{B} \cap \mathcal{PI}$. In particular, $\overline{\mathfrak{C}}$ is complete, and $\overline{\mathcal{A}} = \varinjlim \mathcal{A} = \widehat{\mathcal{A}}$.

Proof: If $\mathfrak{C}$ is cogenerated by a class of FP_2 modules $\mathcal{D}$, we let $\mathcal{C}$ be the smallest class of modules closed under extensions and direct summands which contains $\mathcal{D} \cup \{R\}$. Then $\mathcal{C}$ also consists of FP_2 modules, and it cogenerates $\mathfrak{C}$, so Lemma 2.1.3 and Theorem 2.3 apply. $\square$

Corollary 2.5 Let R be a ring and $\mathcal{C}$ be a class consisting of FP_2 modules. Assume $R \in \mathcal{C}$. Then the class ${}^{\perp_1}(\mathcal{C}^{\perp_1})$ consists of all direct summands of $\mathcal{C}$-filtered modules while ${}^{\intercal}(\mathcal{C}^{\intercal})$ consists of all pure-epimorphic images of $\mathcal{C}$-filtered modules.

One of the ingredients in the proof of Theorem 2.3 was Lemma 1.4.1. Part 2 of that Lemma yields another case of coincidence of the classes $\varinjlim \mathcal{A}$ and $\widehat{\mathcal{A}}$:

Proposition 2.6 Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair generated by a class of dual modules. Then $\mathcal{A} = \varinjlim \mathcal{A} = \widehat{\mathcal{A}}$.

Proof: By Lemma 1.4.2, $\mathcal{A} = \varinjlim \mathcal{A}$. By Lemma 2.1.2, it suffices to prove that $\widehat{\mathcal{A}} \subseteq \mathcal{A}$. But $\widehat{\mathcal{A}} = {}^{\perp_1}(\mathcal{B} \cap \mathcal{H})$ by Lemma 2.1.1. Since $\mathcal{C} \subseteq \mathcal{B} \cap \mathcal{H}$, we have $\widehat{\mathcal{A}} \subseteq {}^{\perp_1}\mathcal{C} = \mathcal{A}$. □

As an application, we consider the classes of modules of bounded flat dimension:

Corollary 2.7 Let R be a ring and $n < \infty$. Let $\mathcal{H}_n = \{\Omega^{-n}(M) \mid M \in \mathcal{H}\}$ where $\Omega^{-n}(M)$ denotes an n-th cosyzygy of M. Then $\mathcal{F}_n = {}^{\perp_1}\mathcal{H}_n = \widehat{\mathcal{F}_n}$.

Proof: First, $\mathcal{F}_0 = {}^{\intercal}(R\mathrm{Mod}) = {}^{\perp_1}\mathcal{H}_0$. Let $n > 0$. For a module N, denote by $\Omega_n(N)$ the n-th syzygy (in a projective resolution) of N. We have $N \in \mathcal{F}_n$ iff $\Omega_n(N) \in \mathcal{F}_0$ iff $\mathrm{Ext}^1_R(\Omega_n(N), \mathcal{H}_0) = 0$ iff $\mathrm{Ext}^{n+1}_R(N, \mathcal{H}_0) = 0$ iff $\mathrm{Ext}^1_R(N, \mathcal{H}_n) = 0$. This proves that $\mathcal{F}_n = {}^{\perp_1}\mathcal{H}_n$. Finally, all cosyzygies of a dual module can be taken to be dual modules as well, so Proposition 2.6 applies. □

3 Modules of projective dimension at most one over commutative domains

In this section, R will denote a commutative domain and Q its quotient field.

We start by reviewing some properties of the class $\mathcal{DI}$ of all *divisible modules*. Recall that $\mathcal{DI} = \mathcal{CP}^{\perp_1}$ where $\mathcal{CP} = \{R/rR \mid r \in R\}$ denotes a set of representatives of all *cyclically presented modules*. It is well-known that the complete cotorsion pair $({}^{\perp_1}\mathcal{DI}, \mathcal{DI})$ is cogenerated by a tilting module of projective dimension one, namely the *Fuchs' divisible module* δ, cf. [15, §VII.1], [28].

Denote by $\mathcal{HD}$ the class of all *h-divisible modules*, that is, of all modules that are homomorphic images of direct sums of copies of Q. Clearly, $\mathcal{HD} \subseteq \mathcal{DI}$, and the equality holds true if and only if R is a *Matlis domain*, that is, $\mathrm{pd}Q = 1$, cf. [15, §VII.2].

For any domain R, we have $\mathcal{P}_1 = {}^{\perp_1}\mathcal{HD}$ by [15, VII.2.5], so the complete cotorsion pair $(\mathcal{P}_1, (\mathcal{P}_1)^{\perp_1})$ is generated by the class of all h-divisible modules.

Let us look more closely at the case when R is a Prüfer domain.

Example 3.1 Assume R is Prüfer. Then $({}^{\perp_1}\mathcal{DI}, \mathcal{DI})$ coincides with $(\mathcal{P}_1, (\mathcal{P}_1)^{\perp_1})$ and also with the cotorsion pair $(\mathcal{A}, \mathcal{B})$ cogenerated by $\mathcal{P}^{<\infty}$, so the Fuchs' divisible module δ is just the tilting module considered in [4].

Indeed, in this case we have $\mathcal{P}^{<\infty} = \mathcal{P}_1^{<\infty} = \operatorname{mod} R$, hence $\mathcal{A} \subseteq \mathcal{P}_1$, cf. [4, Section 2]. So, we have a chain of complete cotorsion pairs $({}^{\perp_1}\mathcal{DI}, \mathcal{DI}) \leq (\mathcal{A}, \mathcal{B}) \leq (\mathcal{P}_1, (\mathcal{P}_1)^{\perp_1})$. On the other hand, $\mathcal{P}_1 \subseteq {}^{\perp_1}\mathcal{DI}$ by [14, VI.3.9], and the claim follows.

Note that, in contrast to the artin algebra case [4, 4.2], the fact that $\mathcal{P}^{<\infty}$ is contravariantly finite in $\operatorname{mod} R$ does not force $\mathcal{A}$ to coincide with $\varinjlim \mathcal{P}^{<\infty}$. Indeed, $\mathcal{A}$ is not closed under direct limits unless R is a Dedekind domain. □

In particular, the above observations show that for all Matlis and all Prüfer domains, all divisible modules of projective dimension at most one belong to Addδ, cf. [14, VI.3.10]. Problem 6 of [14, Chapter VI] asks whether this is true for any domain. The following result provides for an answer.

Proposition 3.2 The following are equivalent for a commutative domain R.
(1) All divisible modules of projective dimension at most one belong to Addδ.
(2) The cotorsion pairs $({}^{\perp_1}\mathcal{DI}, \mathcal{DI})$ and $(\mathcal{P}_1, (\mathcal{P}_1)^{\perp_1})$ coincide.
(3) $\mathcal{DI} = ({}^{\perp_1}\mathcal{HD})^{\perp_1}$.
(4) A module has projective dimension at most one if and only if it is a direct summand of a $\mathcal{CP}$-filtered module.

Proof: We always have $\mathfrak{D} = ({}^{\perp_1}\mathcal{DI}, \mathcal{DI}) \leq (\mathcal{P}_1, (\mathcal{P}_1)^{\perp_1}) = \mathfrak{P}$. Condition (1) states that $\mathcal{P}_1 \cap \mathcal{DI} =Add\delta = {}^{\perp_1}\mathcal{DI} \cap \mathcal{DI}$. This is equivalent to condition (2) by Lemma 1.6. Further, (2) is equivalent to (3), since the cotorsion pair $\mathfrak{P}$ is generated by $\mathcal{HD}$. On the other hand, the cotorsion pair $\mathfrak{D}$ is cogenerated by the set $\mathcal{CP}$, hence ${}^{\perp_1}\mathcal{DI}$ consists of all direct summands of $\mathcal{CP}$-filtered modules, cf. section 1.C. So (2) is equivalent to (4). □

Next, we consider the class $\mathcal{TF}$ of all *torsion-free modules*, that is, $\mathcal{TF} = (\mathcal{CP})^{\intercal}$. There is a duality between torsion-free and divisible modules: a module N is torsion-free iff N^c is divisible, [28, §4]. By a result of Warfield, $(\mathcal{TF}, Q^{\perp_1} \cap \mathcal{I}_1)$ is a closed cotorsion pair, cf. [28, §2].

First, we show that $(\mathcal{TF}, Q^{\perp_1} \cap \mathcal{I}_1)$ is generated by the class of all pure-injective modules of injective dimension at most one:

Lemma 3.3 Let R be a commutative domain. Then $\mathcal{TF} = {}^{\perp_1}(\mathcal{I}_1 \cap \mathcal{PI}) = (\mathcal{F}_1)^{\intercal}$.

Proof: Since Q is a flat module, $Q^{\perp_1}$ contains all pure-injective modules, so $Q^{\perp_1} \cap \mathcal{I}_1 \cap \mathcal{PI} = \mathcal{I}_1 \cap \mathcal{PI}$ and $\mathcal{TF} \subseteq {}^{\perp_1}(\mathcal{I}_1 \cap \mathcal{PI})$. Consider $M \in {}^{\perp_1}(\mathcal{I}_1 \cap \mathcal{PI})$. Let

$N \in \mathcal{F}_1$. Then $N^c \in \mathcal{I}_1$, so $0 = \mathrm{Ext}^1_R(M, N^c) \cong (\mathrm{Tor}^R_1(M,N))^c$. It follows that ${}^{\perp_1}(\mathcal{I}_1 \cap \mathcal{PI}) \subseteq (\mathcal{F}_1)^\intercal$. Finally, since $R/rR \in \mathcal{F}_1$, we get $(\mathcal{F}_1)^\intercal \subseteq \mathcal{TF}$. □

Similarly, the cotorsion pair $(\mathcal{F}_1, (\mathcal{F}_1)^{\perp_1})$ is generated by the class of all divisible pure-injective modules. In fact, there is a more general result (where, for a class of modules $\mathcal{C}$, ${}^{\perp_n}\mathcal{C}$ denotes the class $\{M \in \mathrm{Mod}R \mid \mathrm{Ext}^n_R(M, C) = 0 \text{ for all } C \in \mathcal{C}\}$):

Lemma 3.4 Let R be a commutative domain and $n > 0$. Then $\mathcal{F}_n = {}^{\perp_n}(\mathcal{DI} \cap \mathcal{PI})$. In particular, $\mathcal{F}_1 = \widehat{\mathcal{CP}} = {}^{\perp_1}(\mathcal{DI} \cap \mathcal{PI})$.

Proof: First, $\mathcal{F}_1 = \widehat{\mathcal{CP}}$ by Corollary 2.7 and Lemma 3.3. Let $n > 0$. By Corollary 2.7, $\mathcal{F}_n = {}^{\perp_1}\mathcal{H}_n = {}^{\perp_n}\mathcal{H}_1$. Since $\mathcal{H}_1 \subseteq \mathcal{DI}$, we have $\mathcal{F}_n \supseteq {}^{\perp_n}(\mathcal{DI} \cap \mathcal{PI})$.

Conversely, let $M \in \mathcal{F}_n$, and let N be a module such that $N^c \in \mathcal{DI}$. Then $N \in \mathcal{TF} = (\mathcal{F}_1)^\intercal$ by Lemma 3.3. So $0 = \mathrm{Tor}^R_1(\Omega^{n-1}M, N) \cong \mathrm{Tor}^R_n(M, N)$. This shows that $M \in {}^{\perp_n}(\mathcal{DI} \cap \mathcal{H})$. Finally, by Lemma 2.1.3, ${}^{\perp_n}(\mathcal{DI} \cap \mathcal{H}) = {}^{\perp_n}(\mathcal{DI} \cap \mathcal{PI})$. □

Altogether, we have the following chain of complete cotorsion pairs

$$({}^{\perp_1}\mathcal{DI}, \mathcal{DI}) \leq (\mathcal{P}_1, (\mathcal{P}_1)^{\perp_1}) \leq (\mathcal{F}_1, (\mathcal{F}_1)^{\perp_1}) = (\widehat{\mathcal{CP}}, (\widehat{\mathcal{CP}})^{\perp_1}).$$

By Corollary 2.4 and Lemma 3.4, $(\mathcal{F}_1, (\mathcal{F}_1)^{\perp_1})$ is the closure of $({}^{\perp_1}\mathcal{DI}, \mathcal{DI})$ and hence of $(\mathcal{P}_1, (\mathcal{P}_1)^{\perp_1})$.

Problem 22 in [15, p.246] asks for the structure of the modules which are direct limits of modules in $\mathcal{P}_1$. The answer has already been known for Prüfer domains: $\varinjlim \mathcal{P}_1 = \mathrm{Mod}R$, cf. Example 3.1. Since Prüfer domains are characterized as the domains of weak global dimension at most one, $\varinjlim \mathcal{P}_1$ then coincides with the class of all modules of flat dimension at most one. The latter description extends to any commutative domain:

Theorem 3.5 Let R be a commutative domain. Then

$$\varinjlim \mathcal{P}_1^{<\infty} = \varinjlim \mathcal{P}_1 = \mathcal{F}_1.$$

Proof: Theorem 2.3 applied to the cotorsion pair $(\mathcal{A}_1, \mathcal{B}_1)$ cogenerated by $\mathcal{P}_1^{<\infty}$ and combined with Lemma 2.1.3 yields $\varinjlim \mathcal{P}_1^{<\infty} = \widehat{\mathcal{P}_1^{<\infty}} = {}^{\perp_1}(\mathcal{B}_1 \cap \mathcal{PI})$.
We now claim that ${}^{\perp_1}(\mathcal{B}_1 \cap \mathcal{PI}) = \mathcal{F}_1$. By Lemma 3.4, it suffices to show that $\mathcal{B}_1 \cap \mathcal{H} = \mathcal{DI} \cap \mathcal{H}$. Now, for a module $N \in R$-Mod, we have $N^c \in \mathcal{B}_1 = (\mathcal{P}_1^{<\infty})^{\perp_1}$ iff $N \in (\mathcal{P}_1^{<\infty})^\intercal$. By Lemma 3.3, the latter is equivalent to $N \in \mathcal{TF}$, and hence to $N^c \in \mathcal{DI}$.

So, the claim is proven, and we obtain $\mathcal{F}_1 = \varinjlim \mathcal{P}_1^{<\infty} \subseteq \varinjlim \mathcal{P}_1$. On the other hand, $\mathcal{P}_1 \subseteq \mathcal{F}_1$, and $\varinjlim \mathcal{P}_1^{<\infty}$ is closed under direct limits by Lemma 1.2. Thus $\varinjlim \mathcal{P}_1 = \mathcal{F}_1$. □

We don't know whether this result can be extended to higher dimensions, that is, whether the limit closure of $\mathcal{P}_n$ always coincides with $\mathcal{F}_n$.

4 Direct limits of finitely presented modules of finite projective dimension

Throughout this section, R will denote a right coherent ring, and $(\mathcal{A}, \mathcal{B})$ will be the (complete) cotorsion pair cogenerated by $\mathcal{P}^{<\infty}$. From Section 2 we obtain

$$(\mathcal{A}, \mathcal{B}) \leq (\varinjlim \mathcal{P}^{<\infty}, (\varinjlim \mathcal{P}^{<\infty})^{\perp})$$

where the right-hand term is the closure. In general, $(\mathcal{A}, \mathcal{B})$ is not closed.

Example 4.1 Let R be a von-Neumann-regular ring. Then $\mathcal{P}^{<\infty}$ consists of the finitely generated projective modules, so $\mathcal{A} = \mathcal{P}_0$, while $\varinjlim \mathcal{P}^{<\infty} = \mathrm{Mod}\, R$. □

In order to investigate when $(\mathcal{A}, \mathcal{B})$ is a closed cotorsion pair, we will assume findim$R < \infty$ and use the tilting module T from [4] satisfying $\mathcal{B} = T^{\perp}$.
Moreover, we will deal with the property that all pure submodules of a given module M are direct summands. Modules M with such property are called *pure-split* in [7]. We will say that M is $\sum$*-pure-split* if all modules in Add$M$ are pure-split. For example, every $\sum$-pure-injective module is $\sum$-pure-split.

Theorem 4.2 Let R be a right coherent ring with findim$R < \infty$, and let T be a tilting module with $\mathcal{B} = T^{\perp}$. Then the following statements are equivalent.
(1) $\mathcal{A}$ is closed under direct limits (that is, $\mathcal{A} = \varinjlim \mathcal{P}^{<\infty}$).
(2) T is $\sum$-pure-split.

Proof: (1)⇒(2): If $\mathcal{A}$ is closed under direct limits then it is closed under pure-epimorphic images by Theorem 2.3. Recall further that $\mathcal{B}$ is always closed under pure submodules as it is definable. So, if $0 \longrightarrow X \longrightarrow T' \longrightarrow Y \longrightarrow 0$ is a pure-exact sequence with $T' \in \mathrm{Add}\, T$, we have $Y \in \mathcal{A}$ and $X \in \mathcal{B}$, which shows that the sequence splits.
For the implication (2)⇒(1), we first observe that $(\varinjlim \mathcal{P}^{<\infty}, (\varinjlim \mathcal{P}^{<\infty})^{\perp_1})$ is a cotorsion pair such that $\varinjlim \mathcal{P}^{<\infty} = \widetilde{\mathcal{A}}$ contains the resolving class $\mathcal{A}$, cf. Theorem 2.3 and

Corollary 2.4. So, by Lemma 1.6, it suffices to prove that $\varinjlim \mathcal{P}^{<\infty} \cap \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B} = \operatorname{Add} T$. For this purpose, consider $M \in \varinjlim \mathcal{P}^{<\infty} \cap \mathcal{B}$ and let $0 \to B \to A \to M \to 0$ be an exact sequence with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then $A \in \operatorname{Add} T$. Further, since $M \in \widetilde{\mathcal{A}}$, the exact sequence above is pure-exact. Hence it splits by assumption, and $M \in \operatorname{Add} T$. □

Closure under direct limits of $\mathcal{A}$ can thus be interpreted as a pure-injectivity property of the module T. We now exhibit a further closure property of $\mathcal{A}$ that can be tested on T. Recall that a module M with $\operatorname{Add} M$ being closed under products is said to be *product-complete*, see [22].

Theorem 4.3 Let R be a right coherent ring with $\operatorname{findim} R < \infty$, and let T be a tilting module with $\mathcal{B} = T^{\perp}$. The following statements are equivalent.
(1) $\mathcal{A}$ is definable.
(2) $\mathcal{A}$ is closed under direct products.
(3) T is product-complete.
(4) $\mathcal{A}$ is closed under direct limits, and $\mathcal{P}^{<\infty}$ is covariantly finite in $\operatorname{mod} R$.

Proof: (1)⇒(2) is clear. Further, (2) implies (3) because $\operatorname{Add} T = \mathcal{A} \cap \mathcal{B}$, see Section 1.D. The implication (4)⇒(1) holds by Theorems 1.3 and 2.3.
(3)⇒(4): Observe that a module A belongs to $\mathcal{A}$ if and only if there is a long exact sequence $0 \to A \to T_0 \to T_1 \to \ldots \to T_n \to 0$ with $n = \operatorname{findim} R$ and $T_i \in \operatorname{Add} T$ for all $0 \leq i \leq n$. In fact, the if-part holds since $\mathcal{A}$ is resolving, and the only-if-part is shown as for the special case $A = R$ in the proof of [3, 4.1].
This shows that $\mathcal{A}$ is closed under direct products if so is $\operatorname{Add} T$. Moreover, T is Σ-pure-injective by [22] and thus Σ-pure-split, hence $\mathcal{A} = \varinjlim \mathcal{P}^{<\infty}$ by Theorem 4.2. From Theorem 1.3 it now follows that the category $\mathcal{P}^{<\infty}$ is covariantly finite in $\operatorname{mod} R$. □

Note that with the results above we can refine [4, 3.4 and 3.7]:

Corollary 4.4 Let R be a right coherent ring with $\operatorname{findim} R < \infty$, and let T be a tilting module with $\mathcal{B} = T^{\perp}$. Assume that $\mathcal{P}^{<\infty}$ is covariantly finite (this happens when R is a two-sided coherent and right perfect ring with $\operatorname{Findim} R = 1$, or more generally, with $\mathcal{P} = \varinjlim \mathcal{P}^{<\infty}$, see [4]). Then $\mathcal{A}$ is definable if and only if it is closed under direct limits. In other words, T is product-complete if and only if T is $\sum$-pure-split.

Recall that a family $(M_i)_{i\in I}$ is called *locally* (right) *t-nilpotent* if for each sequence of non-isomorphisms $M_{i_1} \xrightarrow{f_1} M_{i_2} \xrightarrow{f_2} M_{i_3}\dots$ with indices $(i_n)_{n\in IN}$ from I, and each element $x \in M_{i_1}$, there exists $m = m_x \in IN$ such that $f_m\, f_{m-1}\dots f_1(x) = 0$.

Corollary 4.5 Let R be a right coherent ring with $\operatorname{findim} R < \infty$, and assume that $\mathcal{A}$ is closed under direct limits. If T is a tilting module with $\mathcal{B} = T^{\perp}$, then there is a locally t-nilpotent family $(T_i)_{i\in I}$ of modules with local endomorphism ring such that $T = \bigoplus_{i\in I} T_i$. In particular, the direct sum M of a representative set of all indecomposable modules from $\mathcal{A}\cap\mathcal{B}$ is a tilting module with $\mathcal{B} = M^{\perp}$.

Proof: Recall that a submodule X of a module Y is called a local direct summand of Y if there is a decomposition $X = \coprod_{k\in K} X_k$ with the property that $\coprod_{k\in K_0} X_k$ is a direct summand of Y for every finite subset $K_0 \subset K$. Of course, X then is a pure submodule of Y. So, we can now conclude from Theorem 4.2 that all local direct summands of modules in $\operatorname{Add}T$ are direct summands. By [16, 2.3] this implies that T has a decomposition $T = \bigoplus_{i\in I} T_i$ in modules with local endomorphism ring. By [17, 7.3.5] and [1, 2.7] we further know that the family $(T_i)_{i\in I}$ is locally t-nilpotent. The statement on M then follows from Azumaya's Decomposition Theorem. □

The next result shows that we can test contravariant finiteness of $\mathcal{P}^{<\infty}$ on the indecomposable modules in $\mathcal{P}^{<\infty}\cap\mathcal{B}$. Note however that in general $\mathcal{P}^{<\infty}\cap\mathcal{B}$ can be zero, see Remark 5.6.

Corollary 4.6 The following statements are equivalent for an artin algebra R with $\operatorname{findim} R < \infty$.
(a) $\mathcal{P}^{<\infty}$ is contravariantly finite in $\operatorname{mod} R$.
(b) $\mathcal{P}^{<\infty}$ is covariantly finite in $\operatorname{mod} R$, and the direct sum U of a representative set of all indecomposable modules from $\mathcal{P}^{<\infty}\cap\mathcal{B}$ is a $\sum$-pure-split tilting module with $U^{\perp} = \mathcal{B}$.

Proof: Of course, if $\mathcal{P}^{<\infty}\cap\mathcal{B} \neq 0$, then $U \in \mathcal{A}\cap\mathcal{B} = \operatorname{Add}T$. Now, we know from [4, 4.2 and 4.3] that $\mathcal{P}^{<\infty}$ is contravariantly finite if and only if T is finitely presented, and that in this case $\mathcal{A} = \mathcal{P}$ is closed under direct limits. This proves that (a) implies the second statement in (b), while the first is shown in [18]. Conversely, if (b) is satisfied, then we know from Theorems 4.2 and 4.3 that U is product-complete, which implies by [2, 5.2] that U is finitely presented, hence (a) holds true. □

If $\operatorname{Findim} R < \infty$ and $\mathcal{P}$ is closed under direct limits (e. g. if R is right perfect), we also have

$$(\varinjlim \mathcal{P}^{<\infty}, (\varinjlim \mathcal{P}^{<\infty})^{\perp}) \leq (\mathcal{P}, \mathcal{P}^{\perp}).$$

Our next aim is a criterion for $\mathcal{P} = \varinjlim \mathcal{P}^{<\infty}$. More generally, for an integer $n \geq 0$, we will consider the cotorsion pair $(\mathcal{A}_n, \mathcal{B}_n)$ cogenerated by $\mathcal{P}_n^{<\infty}$ and describe when $\mathcal{P}_n = \varinjlim \mathcal{P}_n^{<\infty}$. Our criterion is in terms of properties of pure-injective modules:

Theorem 4.7 Let R be a right perfect and right coherent ring.

1. Let $n < \infty$. Then $\mathcal{P}_n = {}^{\perp_1}(\mathcal{P}_n^{\perp} \cap \mathcal{H})$.
   Moreover, $\mathcal{P}_n = \varinjlim \mathcal{P}_n^{<\infty}$ iff $\mathcal{B}_n \cap \mathcal{PI} = \mathcal{P}_n^{\perp} \cap \mathcal{PI}$ iff $\mathcal{P}_n{}^{\top} = (\mathcal{P}_n^{<\infty})^{\top}$.

2. Assume $\operatorname{Findim} R < \infty$. Then $\mathcal{P} = {}^{\perp_1}(\mathcal{P}^{\perp} \cap \mathcal{H})$.
   Moreover, $\mathcal{P} = \varinjlim \mathcal{P}^{<\infty}$ iff $\mathcal{B} \cap \mathcal{PI} = \mathcal{P}^{\perp} \cap \mathcal{PI}$ iff $\mathcal{P}^{\top} = (\mathcal{P}^{<\infty})^{\top}$.

Proof: 1. By assumption, $\mathcal{P}_n = \mathcal{F}_n$, so Corollary 2.7 and Lemma 2.1.1 give the first assertion.

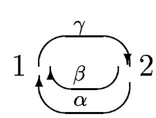
Assume $\mathcal{P}_n = \varinjlim \mathcal{P}_n^{<\infty}$. Then $\mathcal{B}_n \cap \mathcal{PI} = \mathcal{P}_n^{\perp} \cap \mathcal{PI}$ by Lemma 1.4.2. Furthermore, if $\mathcal{B}_n \cap \mathcal{PI} = \mathcal{P}_n^{\perp} \cap \mathcal{PI}$, then $\mathcal{B}_n \cap \mathcal{H} = \mathcal{P}_n^{\perp} \cap \mathcal{H}$, so $\mathcal{P}_n{}^{\top} = (\mathcal{P}_n^{<\infty})^{\top}$. The latter implies $\mathcal{P}_n = {}^{\top}(\mathcal{P}_n{}^{\top}) = {}^{\top}(\mathcal{P}_n^{<\infty})^{\top} = \varinjlim \mathcal{P}_n^{<\infty}$ by Corollary 2.7 and Theorem 2.3.

2. By assumption, $\mathcal{P} = \mathcal{P}_n$ and $\mathcal{P}^{<\infty} = \mathcal{P}_n^{<\infty}$ for some $n < \infty$, and part 1. applies. □

5 On an example of Igusa, Smalø, and Todorov

Examples 3.1 and 4.1 already yield cases where $\mathcal{P} = \varinjlim \mathcal{P}^{<\infty}$ and $\mathcal{A} = {}^{\perp_1}(\mathcal{P}^{<\infty\,\perp_1})$ is not closed under direct limits. We now show that the same can happen over an artin algebra. To this end, we study the following example from [19].

Let k be an algebraically closed field and R the finite-dimensional algebra given by the quiver



with the relations $\alpha\gamma = \beta\gamma = \gamma\alpha = 0$.

It was shown in [19] that $\operatorname{Findim} R = \operatorname{findim} R = 1$, but $\mathcal{P}^{<\infty}$ is not contravariantly finite in $\operatorname{mod} R$. We then know from [4] that there is a $\mathcal{P}^{<\infty}$-filtered tilting module T of projective dimension 1 which is not finitely generated and satisfies $T^{\perp} = \mathcal{B}$.

Moreover, each module of finite projective dimension is a direct limit of elements of $\mathcal{P}^{<\infty}$.

We are now going to show that $\mathcal{A}$ is not closed under direct limits. Let us start by collecting further information on the algebra R.

Conventions: We write R-modules by specifying the vector space V_i corresponding to the vertex , $i = 1, 2$, together with three maps f_α, f_β, f_γ corresponding to the three arrows.

For each of the two vertices $i = 1, 2$, we denote by P_i, I_i, and S_i the corresponding projective, injective or simple module respectively, and set $I = I_1 \oplus I_2$.

R has a factor algebra isomorphic to the Kronecker algebra which we will denote by Λ. The R-modules where the map f_γ corresponding to the arrow γ is zero are also Λ-modules and will be called Kronecker modules.

Note that every R-module X admits a canonical short exact sequence $0 \longrightarrow P_1^{(I)} \longrightarrow X \longrightarrow \bar{X} \longrightarrow 0$ where I is a set and $\bar{X}$ is a Kronecker module.

Recall the classification of *the indecomposable finite length Kronecker modules*:
i) The preprojectives D_n, $n \in I\!N_0$: Set $V_1 = k^{n+1}$, $V_2 = k^n$, $f_\beta = (E, 0)$ and $f_\alpha = (0, E)$ where E denotes the unit in $K^{n \times n}$;
ii) The preinjectives M_n, $n \in I\!N_0$: Set $V_1 = k^n$, $V_2 = k^{n+1}$, $f_\beta = (E, 0)^t$ and $f_\alpha = (0, E)^t$;
iii) The simple regulars: A family R_λ indexed by $\lambda \in k$ defined as $V_1 = V_2 = k$, f_β the multiplication with $\lambda \in k$, f_α the identity, and further a module R_∞ defined as $V_1 = V_2 = k$, f_β the identity, and $f_\alpha = 0$;
iv) Every simple regular module R_λ with $\lambda \in k \cup \{\infty\}$ moreover defines a tube $\mathcal{T}_\lambda$, that is a chain of indecomposable modules X_i starting in the simple regular X_0 linked by non-split exact sequences $0 \longrightarrow X_i \longrightarrow X_{i-1} \oplus X_{i+1} \longrightarrow X_i \longrightarrow 0$ which are almost split in modΛ. Any finite length indecomposable regular module occurs in this way.

The following is implicitly proven in [19].

Proposition 5.1 An indecomposable finite length Kronecker module X has finite projective dimension if and only if it lies in $\bigcup_{\lambda \in k} \mathcal{T}_\lambda$. In particular, $\mathcal{P}^{<\infty}$ consists of the finitely $\{P_1\} \cup \bigcup_{\lambda \in k} \mathcal{T}_\lambda$-filtered modules.

Proof: Since preprojectives and preinjectives have odd dimension, they are not in $\mathcal{P}^{<\infty}$, see [19]. So $X \in \mathcal{P}^{<\infty}$ must be regular, and taking into account the shape of the regular components (see iv), we see that all other modules in the tube X belongs to will be in $\mathcal{P}^{<\infty}$ too. But since the simple regular R_∞ is isomorphic to $P_2/\operatorname{Soc} P_2$, where $\operatorname{Soc} P_2 \notin \mathcal{P}^{<\infty}$, we have $R_\infty \notin \mathcal{P}^{<\infty}$, hence X belongs to one of the other tubes. For the if-part, note that $R_\lambda \cong P_2/P_1$ for each $\lambda \in k$. Again, we have that all modules in the correponding tube are then in $\mathcal{P}^{<\infty}$. □

Lemma 5.2 [6] The finitely generated modules in $\mathcal{B}$ are precisely the finitely gen$I$-filtered modules. In particular, S_2, and all the finite length indecomposable regular modules in $\mathcal{T}_\infty$ are in $\mathcal{B}$.

Proof: $S_2 \cong I_2/\operatorname{Rad} I_2$ and $R_\infty \cong I_2/S_2$ belong to gen$I$, hence to $\mathcal{B}$. Since $\mathcal{B}$ is extension-closed, all the tube $\mathcal{T}_\infty$ lies in $\mathcal{B}$. □

Lemma 5.3 Let X, Y be Kronecker modules. Assume that either $Y \in \mathcal{A}$, or $\operatorname{Hom}_R(S_2, X) = 0$. Then $\operatorname{Ext}^1_\Lambda(Y, X) = 0$ if and only if $\operatorname{Ext}^1_R(Y, X) = 0$.

Proof: The if-part is always true. For the only-if-part, observe that $S_2 = M_0$ is a simple injective Λ-module, and write $X = X' \oplus S_2^{(J)}$ with $\operatorname{Hom}_R(S_2, X') = 0$. If $Y \in \mathcal{A}$, then $\operatorname{Ext}^1_R(Y, S_2^{(J)}) = 0$ by 5.2, so we can assume that we are in the second case, that is $X = X'$. To verify $\operatorname{Ext}^1_R(Y, X) = 0$, we now show that every extension E_R of X by Y is actually a Kronecker module. In fact, consider a short exact sequence $0 \longrightarrow X_R \xrightarrow{f} E_R \xrightarrow{g} Y_R \longrightarrow 0$, and assume that E_R contains a submodule isomorphic to P_1. Since P_1 is uniserial and its socle S_2 is not contained in $\operatorname{Im} f$, we deduce that $g\,|_{P_1}$ is a monomorphism. But this is not possible because Y is a Kronecker module. □

For each tube $\mathcal{T}_\lambda$ with $\lambda \in k \cup \{\infty\}$, we denote by Y_λ the corresponding *Prüfer module*, that is, the direct limit of the chain of inclusions $X_0 \subset X_1 \ldots$ in the tube. It follows from 5.1 that the Y_λ with $\lambda \in k$ are $\mathcal{P}^{<\infty}$-filtered and therefore belong to $\mathcal{A}$.

Proposition 5.4 (1) Let $Y = \bigoplus_{\lambda \in k} Y_\lambda$. Then $\mathcal{B} = Y^{\perp}$.
(2) A Kronecker module X belongs to $\mathcal{B}$ if and only if $\operatorname{Ext}^1_\Lambda(R_\lambda, X) = 0$ for all $\lambda \in k$.

Proof: (1) The inclusion $\subseteq$ is clear. For the other inclusion, let X be a module in $Y^\perp$. For any $\lambda \in k$ and any $A \in \mathcal{T}_\lambda$ we have an exact sequence $0 \longrightarrow A \longrightarrow Y \longrightarrow C \longrightarrow 0$ where C must belong to $\mathcal{P}$ and thus must have projective dimension at most one. This shows that $\operatorname{Ext}^1_R(A, X) = 0$. Then we infer from 5.1 that $\operatorname{Ext}^1_R(\ , X) = 0$ vanishes on all modules in $\mathcal{P}^{<\infty}$, that is $X \in \mathcal{B}$.

(2) The only-if-part is clear since $R_\lambda \in \mathcal{P}^{<\infty} \subseteq \mathcal{A}$ for all $\lambda \in k$. For the if-part, we deduce from 5.3 that $\operatorname{Ext}^1_R(R_\lambda, X) = 0$. Then $\operatorname{Ext}^1_R(-, X) = 0$ vanishes also on all modules in $\mathcal{T}_\lambda$ and on the $\mathcal{T}_\lambda$-filtered module Y_λ, see [13, 7.3.4]. Hence $\operatorname{Ext}^1_R(Y, X) = 0$, and even $X \in Y^\perp$ since $Y \in \mathcal{P} = \mathcal{P}_1$. The claim now follows from (1). □

We will need some further notions from [25]. For a Λ-module X, we denote by τX the sum of its finitely generated submodules without nonzero preprojective direct summand. If $X = \tau X$, then X is said to be *torsion*, and if $\tau X = 0$, or equivalently, if $\operatorname{Hom}_\Lambda(R_\lambda, X) = 0$ for all $\lambda \in k \cup \{\infty\}$, then X is said to be *torsionfree*.
Moreover, X is called *divisible* if $\operatorname{Ext}^1_\Lambda(R_\lambda, X) = 0$ for all $\lambda \in k \cup \{\infty\}$. It is shown in [25] that the divisible modules are precisely the direct sums of preinjectives, Prüfer modules and copies of a module Q which is the unique indecomposable torsionfree divisible module.
Finally, X is called *regular* if it does not have any preprojective, nor any preinjective summand.

We have the following immediate consequence of 5.4.

Corollary 5.5 All Kronecker modules X which are divisible over Λ belong to $\mathcal{B}$. In particular, this shows that Y belongs to $\mathcal{A} \cap \mathcal{B} = \operatorname{Add} T$.

Remark 5.6 (1) The finite length indecomposable Kronecker modules in $\mathcal{B}$ are precisely the preinjectives and the regular modules in $\mathcal{T}_\infty$. Indeed, if X is a finite length indecomposable Kronecker modules in $\mathcal{B}$, then we know from 5.2 that there is a nonzero map $f : I_i \to X$ with $i = 1$ or $i = 2$. In case $i = 1$, the module X is preinjective since $I_1 = M_1$. In case $i = 2$, the map f cannot be a monomorphism and therefore factors through $I_2/\operatorname{Soc} I_2 \simeq R_\infty$, which shows that X is preinjective or regular in $\mathcal{T}_\infty$.
In particular, this shows that there are no non-zero modules in $\mathcal{P}^{<\infty} \cap \mathcal{B}$. For Kronecker modules, this follows immediately from 5.1. But then no other module X can belong to $\mathcal{P}^{<\infty} \cap \mathcal{B}$, because otherwise we would obtain a contradiction from the canonical exact sequence $0 \longrightarrow {P_1}^n \longrightarrow X \longrightarrow \bar{X} \longrightarrow 0$ where $n \geq 0$ and $\bar{X}$ is a Kronecker module.

(2) It is well known that Q is a direct summand of a product of copies of Y. Hence Q belongs to $\mathcal{P}$ and is a direct limit of modules in $\mathcal{P}^{<\infty}$. Note however that Q is not $\mathcal{P}^{<\infty}$-filtered. In fact, it follows from 5.1 that a Kronecker module which is $\mathcal{P}^{<\infty}$-filtered is always filtered by finite length regular modules and is therefore torsion.

Let us draw a further consequence from Proposition 5.4. For each tube $\mathcal{T}_\lambda$ with $\lambda \in k \cup \{\infty\}$, we denote by Z_λ the corresponding *adic module*, that is, the inverse limit of the chain of epimorphisms $\ldots X_1 \to X_0$ in the tube.

Corollary 5.7 The adic module Z_λ belongs to $\mathcal{B}$ if and only if $\lambda = \infty$.

Proof: We know from [20, proof of 9.3] that for each λ there is a universal exact sequence $0 \longrightarrow Z_\lambda \longrightarrow Q_\lambda \longrightarrow Y_\lambda \longrightarrow 0$ where Q_λ is a direct sum of copies of Q. This shows that the Z_λ with $\lambda \in k$ do not belong to $Y^\perp = \mathcal{B}$. On the other hand, applying $\operatorname{Hom}_R(Y,\ )$ on the sequence for $\lambda = \infty$, we deduce from long exact sequence $\ldots 0 = \operatorname{Hom}_R(Y, Y_\infty) \to \operatorname{Ext}^1_R(Y, Z_\infty) \to \operatorname{Ext}^1_R(Y, Q_\infty) = 0 \ldots$ that $Z_\infty \in \mathcal{B}$. □

We are now in a position to decide whether $\mathcal{A}$ is closed under direct limits.

Theorem 5.8 T is not product-complete, hence $\mathcal{A} \neq \mathcal{P}$, and $\mathcal{A}$ is not closed under direct limits.

Proof: Assume that T is product-complete. Recall that Q is a direct summand of a product of copies of Y, and Y lies in $\operatorname{Add}T$ by Corollary 5.5. So, we infer that Q must belong to $\mathcal{A}$. On the other hand, we have just seen that the adic module Z_∞ is in $\mathcal{B}$, and it is well-known that Z_∞ is pure-injective but not Σ-pure-injective. Then there is a cardinal β such that ${Z_\infty}^{(\beta)}$ is not pure-injective, but still is a torsionfree regular module in $\mathcal{B}$. Now since $Q \in \mathcal{A}$, we have $\operatorname{Ext}^1_\Lambda(Q, {Z_\infty}^{(\beta)}) = 0$. But it was shown by Okoh in [26, Prop.1 and Remark on p.265] that a torsionfree regular module belongs to $\operatorname{Ker}\operatorname{Ext}^1_\Lambda(Q, -)$ if and only if it is pure-injective. So, we obtain a contradiction. Thus T is not product-complete. From Corollary 4.4 we then conclude that $\mathcal{A}$ is not closed under direct limits, and in particular $\mathcal{A} \neq \mathcal{P}$. □

Acknowledgment We would like to thank Birge Huisgen-Zimmermann, Robert El Bashir, Mike Prest and Sverre Smalø for interesting comments.

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