

ON THE ABUNDANCE OF SILTING MODULES

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ABSTRACT. Silting modules are abundant. Indeed, they parametrise the definable torsion classes over a noetherian ring, and the hereditary torsion pairs of finite type over a commutative ring. Also the universal localisations of a hereditary ring, or of a finite dimensional algebra of finite representation type, can be parametrised by silting modules. In these notes, we give a brief introduction to the fairly recent concepts of silting and cosilting module, and we explain the classification results mentioned above.

1. INTRODUCTION

The notion of a (compact) silting complex was introduced by Keller and Vossieck [36], and it was later rediscovered in work of Aihara-Iyama, Keller-Nicolàs, König-Yang, Mendoza-Sáenz-Santiago-Souto Salorio, and Wei [2, 35, 45, 55, 37]. This renewed interest was motivated by cluster theory, and also by the interplay with certain torsion pairs in triangulated categories. In fact, silting complexes are closely related with t-structures and co-t-structures in the derived category. This connection was recently extended to non-compact silting complexes, and more generally, to silting objects in triangulated categories [5, 50, 48].

But silting is also closely related to torsion pairs and localisation of abelian categories, as well as to ring theoretic localisation. Aim of these notes is to review some of these connections. We will see that localisation techniques provide constructions of silting objects and lead to classification results over several important classes of rings. There is also an interesting interaction with combinatorial aspects of silting. Certain posets studied in cluster theory have a ring theoretic interpretation which sheds new light on their structure. On the other hand, silting theory yields a new approach to general questions on homological properties of ring epimorphisms.

We will consider silting modules, that is, the modules that arise as zero cohomologies of (not necessarily compact) 2-term silting complexes over an arbitrary ring. These modules were introduced in [5]. They provide a generalisation of (not necessarily finitely generated) tilting modules. Moreover, over a finite dimensional algebra, the finitely generated silting modules are precisely the support τ -tilting modules introduced in [1] and studied in cluster theory.

In representation theory, one usually studies finite dimensional tilting or support τ -tilting modules up to isomorphism and multiplicities. Similarly, in the infinite dimensional case, it is convenient to study silting classes rather than modules. The silting class $\text{Gen } T$ given by a silting module T consists of all T -generated modules, and it determines the additive closure $\text{Add } T$ of T . It has several useful closure properties, making it a definable torsion class. A silting class thus provides every module M with a (minimal) right $\text{Gen } T$ -approximation given by the trace of T in M , and with a left $\text{Gen } T$ -approximation. Furthermore, $\text{Gen } T$ - even when T is not finitely generated - satisfies an important finiteness condition: it is determined by a set Σ of morphisms between finitely generated projective modules [44]. This “finite type” result extends the analogous result for tilting modules from [13] stating that every tilting class is determined by a set of finitely presented modules of projective dimension at most one.

It turns out that in many respects the dual concept of a cosilting module studied in [16, 57, 17] is more accessible. Cosilting modules are pure-injective, and cosilting classes are definable torsionfree classes, in fact, they coincide with the definable torsionfree classes. Notice that the dual result is not true in general: not every definable torsion class is silting. Indeed, there are more cosilting than silting classes, because in general cosilting modules are not of “cofinite type”, that is, they need not be determined by a set Σ as above. Also this phenomenon was already known for cotilting modules [12], a cosilting example is exhibited in Example 4.10.

However, over a noetherian ring, every torsion pair with definable torsionfree class is generated by a set of finitely presented modules. This allows to show that all cosilting modules are of cofinite type. Then we obtain dually that the silting classes coincide with the definable torsion classes [4]. This can be regarded as a “large” analog of [1, Theorem 2.7] stating that over a finite dimensional algebra A , there is a bijection between isomorphism classes of basic support τ -tilting modules and functorially finite torsion classes in $\text{mod-}A$.

Similarly, one can show that over a commutative ring, silting modules correspond bijectively to hereditary torsion pairs of finite type, and if the ring is commutative noetherian, they are parametrised by subsets of the spectrum closed under specialisation [4].

Silting is thus often related to Gabriel localisation of module categories. But it is also related to ring theoretic localisation, in particular to universal localisation of rings in the sense of [52]. In fact, by [6] every partial silting module over a ring A induces a ring epimorphism $f : A \rightarrow B$ with nice homological properties, and it is proved in [44] that every universal localisation of A arises in this way from some partial silting module. Further, in certain cases, the universal localisations of A , or the homological ring epimorphisms starting in A , are parametrised by silting modules.

More precisely, one obtains a parametrisation by minimal silting modules, that is, silting modules for which the regular module A admits a minimal left $\text{Gen} T$ approximation $A \rightarrow T_0 \rightarrow T_1 \rightarrow 0$. For example, when A is a finite dimensional algebra, every finite dimensional silting module is minimal. Moreover, in this case, T_1 is a partial silting module uniquely determined by T , so one can associate to T a ring epimorphism $f : A \rightarrow B$, which turns out to be a universal localisation of A . If A is of finite representation type, or more generally, τ -tilting finite in the sense of [26], this assignment yields a bijection between silting modules and universal localisations of A , see [43].

A similar result holds true when A is a hereditary ring. As shown in [6], the minimal silting A -modules are in bijection with the universal localisations of A , which by [39] coincide with the homological ring epimorphisms. If A is a finite dimensional hereditary algebra, this correspondence yields a bijection between finite dimensional support tilting modules and functorially finite wide subcategories of $\text{mod-}A$, recovering results from [32, 41]. The combinatorial interpretation of finite dimensional support tilting modules in terms of noncrossing partitions or clusters can then be translated into ring theoretic terms by considering the poset of all universal localisations of A . Notice that universal localisations of A can have infinite dimension. Leaving the finite dimensional world, however, has the advantage that the poset becomes a lattice, while the same is not true if we restrict to ring epimorphisms with finite dimensional target, see e.g. Example 6.2(1).

The paper is organised as follows. Section 2 is devoted to the notion of a silting module. In Section 3, we collect some results on cosilting modules scattered in the literature. The interplay between silting and cosilting modules is discussed in Section 4, which also contains the classification results over noetherian or commutative rings mentioned above. In Section 5, we review the connections with ring epimorphisms, and we explain the parametrisation of all universal localisations of a hereditary ring. Finally, in Section 6, we focus on finite dimensional algebras and on the poset of their universal localisations.

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Notation. Throughout the paper, A will denote a ring, $\text{Mod-}A$ the category of all right A -modules, and $\text{mod-}A$ the category of all finitely presented right A -modules. The corresponding categories of left A -modules are denoted by $A\text{-Mod}$ and $A\text{-mod}$. The unbounded derived category of $\text{Mod-}A$ is denoted by $D(A)$. We further denote by $\text{Proj-}A$ and $\text{proj-}A$ the full subcategory of $\text{Mod-}A$ consisting of all projective and all finitely generated projective right A -modules, respectively, and we write $*$ = $\text{Hom}_A(-, A)$.

Given a subcategory \mathcal{C} of $\text{Mod-}A$, we denote

$$\begin{aligned}\mathcal{C}^\circ &= \{M \in \text{Mod-}A \mid \text{Hom}_A(\mathcal{C}, M) = 0\}, \\ \mathcal{C}^{\perp 1} &= \{M \in \text{Mod-}A \mid \text{Ext}_A^1(\mathcal{C}, M) = 0\}.\end{aligned}$$

The classes ${}^\circ\mathcal{C}$ and ${}^{\perp 1}\mathcal{C}$ are defined dually. Moreover, $\text{Add}\mathcal{C}$ and $\text{Prod}\mathcal{C}$ are the subcategories of $\text{Mod-}A$ formed by the modules that are isomorphic to a direct summand of a coproduct of modules in \mathcal{C} , or of a product of such modules, respectively. $\text{Gen}\mathcal{C}$ is the subcategory of the \mathcal{C} -generated modules, i. e. the epimorphic images of modules in $\text{Add}\mathcal{C}$, and $\text{Cogen}\mathcal{C}$ is defined dually. When \mathcal{C} just consists of a single module M , we write $\text{Add } M, \text{Prod } M, \text{Gen } M, \text{Cogen } M$.

Furthermore, we denote by $\text{Mor}(\mathcal{C})$ the class of all morphisms in $\text{Mod-}A$ between objects in \mathcal{C} . Given a set of morphisms $\Sigma \subset \text{Mor}(\mathcal{C})$, we consider the classes

$$\begin{aligned}\mathcal{D}_\Sigma &= \{X \in \text{Mod-}A \mid \text{Hom}_A(\sigma, X) \text{ is surjective for all } \sigma \in \Sigma\}; \\ \mathcal{X}_\Sigma &= \{X \in \text{Mod-}A \mid \text{Hom}_A(\sigma, X) \text{ is bijective for all } \sigma \in \Sigma\}; \\ \mathcal{C}_\Sigma &= \{X \in \text{Mod-}A \mid \text{Hom}_A(X, \sigma) \text{ is surjective for all } \sigma \in \Sigma\}; \\ \mathcal{F}_\Sigma &= \{X \in A\text{-Mod} \mid \sigma \otimes_A X \text{ is injective for all } \sigma \in \Sigma\}.\end{aligned}$$

Again, when $\Sigma = \{\sigma\}$, we write $\mathcal{D}_\sigma, \mathcal{X}_\sigma, \mathcal{C}_\sigma, \mathcal{F}_\sigma$.

2. SILTING MODULES

We start out by briefly reviewing the notion of a (not necessarily compact) silting complex.

Definition 2.1. [55] A bounded complex of projective A -modules σ is said to be *silting* if

- (1) $\text{Hom}_{D(A)}(\sigma, \sigma^{(I)}[i]) = 0$, for all sets I and $i > 0$.
- (2) the smallest triangulated subcategory of $D(A)$ containing $\text{Add } \sigma$ is $\text{K}^b(\text{Proj-}A)$.

A prominent role is played by the silting complexes of length two which were studied in [1] in connection with cluster mutation. Their endomorphism ring has interesting properties investigated in [18, 19, 20]. Another important feature is the fact that 2-term silting complexes (which we always assume concentrated in cohomological degrees -1 and 0) are determined by their zero cohomology. In fact, identifying a complex $\dots 0 \rightarrow P_{-1} \xrightarrow{\sigma} P_0 \rightarrow 0 \dots$ with the morphism σ in $\text{Mor}(\text{Proj-}A)$, one obtains the following result which goes back to work of Hoshino-Kato-Miyachi [29].

Proposition 2.2. *Let σ be 2-term complex in $\text{K}^b(\text{Proj-}A)$ and $T = H^0(\sigma)$. Then σ is a silting complex if and only if the class $\text{Gen } T$ of T -generated modules coincides with \mathcal{D}_σ .*

We are interested in the modules that occur in this way.

Definition 2.3. A right A -module T is *silting* if it admits a projective presentation $P_{-1} \xrightarrow{\sigma} P_0 \rightarrow T \rightarrow 0$ such that $\text{Gen } T = \mathcal{D}_\sigma$. The class $\text{Gen } T$ is then called a *silting class*.

We say that two silting modules T, T' are *equivalent* if they generate the same silting class. By [5, Section 3], this is equivalent to $\text{Add } T = \text{Add } T'$.

Examples 2.4. (1) A module T is silting with respect to a monomorphic projective presentation $\sigma : P_{-1} \rightarrow P_0$ if and only if $\text{Gen } T = \text{Ker Ext}_A^1(T, -)$, i. e. T is a *tilting* module (of projective dimension at most one, not necessarily finitely generated).

(2) If A is a finite dimensional algebra over a field, and $T \in \text{mod-}A$, then T is silting if and only if it is *support τ -tilting* in the sense of [1].

(3) Let A be the path algebra of the quiver Q having two vertices, 1 and 2, and countably many arrows from 1 to 2. Let $P_i = e_i A$ be the indecomposable projective A -module for $i = 1, 2$. Then $T := S_2$ with the projective presentation

$$0 \longrightarrow P_1^{(\mathbb{N})} \xrightarrow{\sigma} P_2 \longrightarrow T \longrightarrow 0,$$

is a silting module (of projective dimension one, not finitely presented) which is not tilting.

Indeed, it is not tilting as the class $\text{Gen } T$ consists precisely of the semisimple injective A -modules, so $\text{Gen } T = \text{Ker Hom}_A(P_1, -) \subsetneq \text{Ker Ext}_A^1(T, -)$. But T is silting with respect to the projective presentation γ of T obtained as the direct sum of σ with the trivial map $P_1 \rightarrow 0$, since $\mathcal{D}_\gamma = \text{Ker Ext}_A^1(T, -) \cap \text{Ker Hom}_A(P_1, -) = \text{Ker Hom}_A(P_1, -) = \text{Gen } T$.

Notice that every class of the form \mathcal{D}_σ for a morphism $\sigma \in \text{Mor}(\text{Proj-}A)$ is closed under extensions, epimorphic images, and direct products. In the silting case, $\mathcal{D}_\sigma = \text{Gen } T$ is also closed under coproducts, and it is therefore a torsion class. Moreover, even though σ need not belong to $\text{Mor}(\text{proj-}A)$, the silting class \mathcal{D}_σ is determined by a set Σ of morphisms in $\text{Mor}(\text{proj-}A)$, that is, $\mathcal{D}_\sigma = \mathcal{D}_\Sigma$. This property is obtained in [44] as a consequence of the analogous result for tilting modules proved in [13]; an alternate proof in [4] uses that \mathcal{D}_σ is definable (as shown in [5, 3.5 and 3.10]). Recall that a class of modules is said to be *definable* if it is closed under direct limits, direct products, and pure submodules.

In fact, these properties characterise silting classes.

Theorem 2.5. [44] *The following statements are equivalent for a full subcategory \mathcal{D} of $\text{Mod-}A$.*

- (i) \mathcal{D} is a silting class;
- (ii) there is $\sigma \in \text{Mor}(\text{Proj-}A)$ such that $\mathcal{D} = \mathcal{D}_\sigma$, and \mathcal{D} is closed for coproducts;
- (iii) there is a set $\Sigma \subset \text{Mor}(\text{proj-}A)$ such that $\mathcal{D} = \mathcal{D}_\Sigma$.

Moreover, \mathcal{D} is a tilting class if and only if $\mathcal{D} = \mathcal{D}_\Sigma$ for a set Σ of monomorphisms in $\text{Mor}(\text{proj-}A)$.

A silting class \mathcal{D} , being a torsion class, provides any module M with a minimal right \mathcal{D} -approximation $d(M) \hookrightarrow M$ where d denotes the torsion radical corresponding to \mathcal{D} . Moreover, being definable, \mathcal{D} also provides M with a left \mathcal{D} -approximation. In fact, the existence of special left approximations can be used to detect the torsion classes which are generated by a tilting module.

Theorem 2.6. [10] *A torsion class \mathcal{T} in $\text{Mod-}A$ is of the form $\mathcal{T} = \text{Gen } T$ for some tilting module T if and only if for every A -module M (or equivalently, for $M = A$) there is a short exact sequence $0 \rightarrow M \rightarrow B \rightarrow C \rightarrow 0$ with $B \in \mathcal{T}$ and $C \in {}^{\perp 1}\mathcal{T}$.*

At the end of the next section, we will discuss how this result extends to silting modules.

3. COSILTING MODULES

When dealing with classification results for silting modules, it turns out that the dual concept of a cosilting module, which was introduced by Breaz and Pop in [16], is often more accessible. In order to study the interplay between the two notions, it will be convenient to consider right silting and left cosilting modules.

Definition 3.1. A left A -module C is *cosilting* if it admits an injective copresentation $0 \rightarrow C \rightarrow E_0 \xrightarrow{\omega} E_1$ such that the class $\text{Cogen } C$ of C -cogenerated modules coincides with the class \mathcal{C}_ω . The class $\text{Cogen } C$ is then called a *cosilting class*.

Two cosilting modules C, C' are said to be *equivalent* if they cogenerate the same class $\text{Cogen } C = \text{Cogen } C'$. It will follow from Remark 3.4 below that this is equivalent to $\text{Prod } C = \text{Prod } C'$.

Dually to Example 2.4(1), we have that C is cosilting with respect to an epimorphic injective copresentation $E_0 \xrightarrow{\omega} E_1$ if and only if $\text{Cogen } C = \text{Ker Ext}_A^1(-, C)$, i. e. C is a *cotilting* module (of injective dimension at most one, not necessarily finitely generated).

Further, dually to the silting case, we see that every cosilting class $\mathcal{C}_\omega = \text{Cogen } C$ is a torsionfree class. How to describe the torsionfree classes that arise in this way? For cotilting modules there is the following characterisation in terms of the existence of approximations.

Theorem 3.2. [10] *A torsionfree class \mathcal{F} in $A\text{-Mod}$ is of the form $\mathcal{F} = \text{Cogen } C$ for some cotilting module C if and only if for every left A -module M (or equivalently, for an injective cogenerator E of $A\text{-Mod}$) there is a short exact sequence $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$ with $L \in \mathcal{F}$ and $K \in \mathcal{F}^\perp$.*

In order to extend this result to cosilting modules we need some preparation.

Lemma 3.3. *Let C be a left A -module with injective copresentation $0 \rightarrow C \rightarrow E_0 \xrightarrow{\omega} E_1$.*

- (1) *A module X belongs to the class \mathcal{C}_ω if and only if $\text{Hom}_{\text{D}(A)}(X, \omega[1]) = 0$, if and only if $\text{Hom}_{\text{D}(A)}(\sigma, \omega[1]) = 0$ for any injective copresentation $0 \rightarrow X \rightarrow I_0 \xrightarrow{\sigma} I_1$.*
- (2) *Assume that $0 \rightarrow C \rightarrow E_0 \xrightarrow{\omega} E_1$ is a minimal injective copresentation. Then $\text{Cogen } C \subset {}^\perp C$ if and only if $\text{Hom}_{\text{D}(A)}(\omega^I, \omega[1]) = 0$ for any set I , if and only if $\text{Hom}_{\text{D}(A)}(X, \omega[1]) = 0$ for all $X \in \text{Cogen } C$.*

Proof. (1) is shown by standard arguments. Moreover, using (1), the statement in (2) can be rephrased as follows: $\text{Cogen } C \subset {}^\perp C$ if and only if all products of copies of C are in \mathcal{C}_ω , if and only if $\text{Cogen } C \subset \mathcal{C}_\omega$. Now, since \mathcal{C}_ω is always closed under submodules, and $\mathcal{C}_\omega \subset {}^\perp C$, the second condition means $\text{Cogen } C \subset \mathcal{C}_\omega$, which in turn entails the first condition $\text{Cogen } C \subset {}^\perp C$. So it remains to show that the first condition implies the second. We sketch an argument from [57, Lemma 4.13]. Consider a cardinal κ and a map $f : C^\kappa \rightarrow E_1$. The map ω factors as $\omega = e \circ \omega'$ with $e : \text{Im } \omega \hookrightarrow E_1$ being an injective envelope. Setting $Z = \text{Coker } \omega$, we obtain the following commutative diagram with exact rows

$$(3.1) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i} & C^\kappa & \xrightarrow{gf} & Z & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow f & & \parallel & & \\ 0 & \longrightarrow & C & \longrightarrow & E_0 & \xrightarrow{\omega} & E_1 & \xrightarrow{g} & Z & \longrightarrow & 0 \end{array}$$

where h is constructed by first taking the map $f' : K \rightarrow \text{Im } \omega$ induced by f , and then lifting f' to h thanks to the fact that $K \in \text{Cogen } C \subset {}^\perp C$. Now the injectivity of E_0, E_1 yields maps $s_0 : C^\kappa \rightarrow E_0$ and $s_1 : Z \rightarrow E_1$ such that $f = \omega s_0 + s_1(gf)$, and one verifies $\text{Im}(f - \omega s_0) \cap \text{Im } \omega = 0$. Since $\text{Im } \omega$ is an essential submodule of E_1 , it follows $f = \omega s_0$, as required. \square

Remark 3.4. A module M in a torsionfree class \mathcal{F} is said to be *Ext-injective* in \mathcal{F} if $\text{Ext}_A^1(\mathcal{F}, M) = 0$. Notice that the condition $\text{Cogen } C \subset {}^{\perp_1}C$ implies that $({}^oC, \text{Cogen } C)$ is a torsion pair, and it can be rephrased by saying that C is an Ext-injective object in the torsionfree class $\text{Cogen } C$.

Now assume that C is a cosilting module with respect to an injective copresentation ω . Since $\mathcal{C}_\omega \subset {}^{\perp_1}C$, it follows that C is Ext-injective in $\text{Cogen } C$. More precisely, $\text{Prod } C$ is the class of all Ext-injective modules in $\text{Cogen } C$. Indeed, for every module $M \in \text{Cogen } C$ there is an embedding $f : M \rightarrow C'$ with $C' \in \text{Prod } C$ and $\text{Coker } f \in \text{Cogen } C$. If M is Ext-injective in $\text{Cogen } C$, then f is a split monomorphism and $M \in \text{Prod } C$. For details we refer to [5, Lemma 2.3 and 3.3, Proposition 3.10] where the dual statements are proved.

The following result is inspired by the proof of [57, Proposition 4.9].

Proposition 3.5. *Let C be a left A -module which is Ext-injective in $\text{Cogen } C$. Assume there are an injective cogenerator E of $A\text{-Mod}$, a cardinal κ , and a right $\text{Prod } C$ -approximation $g : C^\kappa \rightarrow E$. Then $\text{Ker } g \oplus C$ is a cosilting module with cosilting class $\text{Cogen } C$.*

Proof. Take a minimal injective copresentation $0 \rightarrow C \rightarrow E_0 \xrightarrow{\gamma} E_1$. Then γ^κ is an injective copresentation of C^κ . The map $g : C^\kappa \rightarrow E$ induces a map $\tilde{g} : E_0^\kappa \rightarrow E$, which can be viewed as a map of complexes $\tilde{g} : \gamma^\kappa \rightarrow E^\bullet$ where E^\bullet is the complex with E concentrated in degree 0. Note that \tilde{g} has zero cohomology g . Considering the mapping cone $K_{\tilde{g}} : E_0^\kappa \xrightarrow{(\tilde{g}, \gamma^\kappa)} E \oplus E_1^\kappa$ and setting $\omega = K_{\tilde{g}}[-1]$, we obtain a triangle in $D(A)$

$$\omega \rightarrow \gamma^\kappa \xrightarrow{\tilde{g}} E^\bullet \rightarrow$$

whose zero cohomologies give rise to the exact sequence

$$0 \rightarrow C_1 \rightarrow C^\kappa \xrightarrow{g} E$$

with $C_1 = \text{Ker } g = H^0(\omega)$.

We claim that $\text{Cogen } C = \mathcal{C}_\omega$. Indeed, for a module X , applying $\text{Hom}_{D(A)}(X, -)$ to the triangle above yields a long exact sequence

$$\text{Hom}_{D(A)}(X, \gamma^\kappa) \xrightarrow{\text{Hom}_{D(A)}(X, \tilde{g})} \text{Hom}_{D(A)}(X, E^\bullet) \rightarrow \text{Hom}_{D(A)}(X, \omega[1]) \rightarrow \text{Hom}_{D(A)}(X, \gamma^\kappa[1]) \rightarrow 0.$$

Now, since $g : C^\kappa \rightarrow E$ is a right $\text{Prod } C$ -approximation of an injective cogenerator, $X \in \text{Cogen } C$ if and only if $\text{Hom}_A(X, g)$ is surjective, which amounts to $\text{Hom}_{D(A)}(X, \tilde{g})$ being surjective. Further, recall from Lemma 3.3(1) that $X \in \mathcal{C}_\omega$ if and only if $\text{Hom}_{D(A)}(X, \omega[1]) = 0$. This immediately gives the inclusion $\text{Cogen } C \supset \mathcal{C}_\omega$. For the reverse inclusion, use that $\text{Hom}_{D(A)}(X, \gamma[1]) = 0$ for all $X \in \text{Cogen } C$ by assumption and Lemma 3.3(2).

In order to prove that $\tilde{C} = C_1 \oplus C$ is a cosilting module, we now consider its injective copresentation $\tilde{\omega} = \omega \oplus \gamma$. By construction $\mathcal{C}_\omega \subset \mathcal{C}_{\gamma^\kappa} = \mathcal{C}_\gamma$. Therefore $\mathcal{C}_{\tilde{\omega}} = \mathcal{C}_\omega \cap \mathcal{C}_\gamma = \mathcal{C}_\omega = \text{Cogen } C = \text{Cogen } \tilde{C}$. \square

We are now going to see that cosilting modules are precisely the ‘‘non-faithful cotilting’’ modules. This result is due to Zhang and Wei [57], who have compared several notions related to cotilting and cosilting modules. We collect the relevant arguments below for the reader’s convenience.

Theorem 3.6. [57] *The following statements are equivalent for an A -module C .*

- (1) C is a cosilting module.
- (2) $\text{Cogen } C$ is a torsionfree class, and C is cotilting over $\bar{A} = A/\text{Ann}(C)$.
- (3) C is Ext-injective in $\text{Cogen } C$, and there are an injective cogenerator E of $A\text{-Mod}$ and an exact sequence

$$0 \rightarrow C_1 \rightarrow C_0 \xrightarrow{g} E$$

such that $C_1, C_0 \in \text{Prod } C$ and g is a right $\text{Cogen } C$ -approximation.

Proof. First of all, notice that $\text{Ann}(C) = \text{Ann}(\text{Cogen } C)$ and $\text{Cogen}_{\bar{A}}C = \text{Cogen}_AC$. Moreover, if $\text{Cogen } C$ is extension closed (which holds true in all three statements), then $\text{Ext}_{\bar{A}}^1(\text{Cogen } C, C) = 0$ if and only if $\text{Ext}_A^1(\text{Cogen } C, C) = 0$. Finally, note also that $\text{Ann}(C) = \bigcap_{h \in \text{Hom}_A(A, C)} \text{Ker } h$, so the class $\text{Cogen}_{\bar{A}}C$ contains \bar{A} and all projective \bar{A} -modules.

(1) \Rightarrow (2): Every cosilting module satisfies $\text{Cogen } C \subset {}^{\perp}C$, thus $\text{Cogen}_{\bar{A}}C \subset \text{Ker Ext}_{\bar{A}}^1(-, C)$. To verify that $\text{Cogen}_{\bar{A}}C$ is cotilting, it remains to show the reverse inclusion. Pick a left \bar{A} -module X with $\text{Ext}_{\bar{A}}^1(X, C) = 0$. Since $\text{Cogen } C$ contains all projective \bar{A} -modules, there is a short exact sequence $0 \rightarrow K \rightarrow L \rightarrow X \rightarrow 0$ in $\bar{A}\text{-Mod}$ with $K, L \in \text{Cogen } C$. Now one proves (dually to the proof of [5, Proposition 3.10]) that the diagonal map $K \hookrightarrow C^I$ with $I = \text{Hom}_A(K, C)$ has cokernel $M \in \text{Cogen } C$. By assumption on X , the map $\text{Hom}_{\bar{A}}(L, C^I) \rightarrow \text{Hom}_{\bar{A}}(K, C^I)$ is surjective, yielding a map f and a commutative diagram

$$(3.2) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & X & \longrightarrow & 0 \\ & & \parallel & & \downarrow f & & \downarrow g & & \\ 0 & \longrightarrow & K & \longrightarrow & C^I & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Since $L, M \in \text{Cogen } C$, we infer $\text{Ker } g \cong \text{Ker } f \in \text{Cogen } C$, and $\text{Im } g \in \text{Cogen } C$. But $\text{Cogen } C$ is closed under extensions, so also $X \in \text{Cogen } C$.

(2) \Rightarrow (3): If E is an injective cogenerator of $A\text{-Mod}$, then $E' = \{x \in E \mid \text{Ann}(C) \cdot x = 0\}$ is an injective cogenerator of $\bar{A}\text{-Mod}$. By Theorem 3.2 there is a short exact sequence of \bar{A} -modules $0 \rightarrow C_1 \rightarrow C_0 \xrightarrow{g'} E' \rightarrow 0$ with $C_0 \in \text{Cogen } C$ and $C_1 \in \text{Ker Ext}_{\bar{A}}^1(\text{Cogen } C, -)$. Then g' is a right $\text{Cogen } C$ -approximation. Further, since $E' \in \text{Ker Ext}_{\bar{A}}^1(\text{Cogen } C, -)$, both C_1 and C_0 belong to $\text{Cogen } C \cap \text{Ker Ext}_{\bar{A}}^1(\text{Cogen } C, -) = \text{Prod } C$.

Keeping in mind that every module in $\text{Cogen } C$ is also a \bar{A} -module, we conclude that the map $g : C_0 \xrightarrow{g'} E' \subset E$ is a right $\text{Cogen } C$ -approximation with the stated properties. Finally, C is Ext-injective in $\text{Cogen } C$ by the first paragraph of the proof.

(3) \Rightarrow (1): We can assume w.l.o.g. that $C_0 = C^\kappa$ for some cardinal κ . By Proposition 3.5, there is an injective copresentation $\tilde{\omega}$ of the module $\tilde{C} = C_1 \oplus C$ such that $\mathcal{C}_{\tilde{\omega}} = \text{Cogen } \tilde{C} = \text{Cogen } C$. Take minimal injective copresentations α and γ of C_1 and C , respectively. Then $\alpha \oplus \gamma$ is a minimal injective copresentation of \tilde{C} , hence there are injective modules E, E' such that $\tilde{\omega} = (\alpha \oplus \gamma) \oplus (0 \rightarrow E) \oplus (E' \xrightarrow{\text{id}} E')$. Further, it is easy to see that $C_1 \in \text{Prod } C$ implies $\mathcal{C}_{C_1} \subset \mathcal{C}_\alpha$, so $\mathcal{C}_{\alpha \oplus \gamma} = \mathcal{C}_\gamma$. We infer $\mathcal{C}_{\tilde{\omega}} = \mathcal{C}_\gamma \cap \mathcal{C}_{(0 \rightarrow E)}$. So the injective copresentation $\omega = \gamma \oplus (0 \rightarrow E)$ of C satisfies $\mathcal{C}_\omega = \mathcal{C}_{\tilde{\omega}} = \text{Cogen } C$. This completes the proof. \square

It was shown in [11] that every cotilting module is pure-injective, and cotilting classes are definable. The characterisation given in Theorem 3.6(2) now allows to deduce the same properties for cosilting modules.

Corollary 3.7. [16, 57] *Every cosilting module is pure-injective, every cosilting class is definable.*

We are ready for a generalisation of Theorem 3.2 and [12, Theorem 6.1].

Theorem 3.8. [57, 17] *The following statements are equivalent for a torsionfree class \mathcal{F} in $A\text{-Mod}$.*

- (1) $\mathcal{F} = \text{Cogen } C$ for some cosilting module C .
- (2) For every left A -module M there is an exact sequence

$$0 \rightarrow K \rightarrow L \xrightarrow{g} M$$

such that g is a right \mathcal{F} -approximation and K is Ext-injective in \mathcal{F} .

- (3) Every left A -module admits a minimal right \mathcal{F} -approximation.

Let us sketch the proof. If \mathcal{F} is definable, then every module admits a minimal right \mathcal{F} -approximation, see [12, Corollary 2.6] and the references therein. Moreover, Wakamatsu's Lemma ensures that every minimal right \mathcal{F} -approximation has an Ext-injective kernel. Therefore (1) \Rightarrow (3) \Rightarrow (2). The implication (2) \Rightarrow (1) is shown by looking at the special case when $M = E$ is an injective cogenerator of $A\text{-Mod}$. In this case $\mathcal{F} = \text{Cogen } L = \text{Cogen } C$ for $C = K \oplus L$, and by Theorem 3.6, the module C is cosilting provided it is Ext-injective in \mathcal{F} . So it remains to verify Ext-injectivity of L . To this end, one works over $\bar{A} = \text{Ann}(C) = \text{Ann}(\mathcal{F})$, where the injective cogenerator $E' = \{x \in E \mid \text{Ann}(C) \cdot x = 0\}$ admits a short exact sequence $0 \rightarrow K \rightarrow L \rightarrow E' \rightarrow 0$, and L is in $\text{Ker Ext}_{\bar{A}}^1(\mathcal{F}, -)$ since so are K and E' (cf. the proof of Theorem 3.6).

Corollary 3.9. *The assignment $C \mapsto \text{Cogen } C$ defines a bijection between*

- (i) *equivalence classes of cosilting left A -modules;*
- (ii) *definable torsionfree classes in $A\text{-Mod}$.*

Notice that the statements above rely on the existence of *minimal* injective copresentations. It is shown in [17] that the dual version of Theorem 3.8 holds true over perfect or hereditary rings. In general, however, one only has that the conditions dual to statements (2) and (3) in Theorem 3.6 are equivalent [5, Proposition 3.2] and are satisfied by every silting module. In [5, Proposition 3.10] this is phrased by saying that every silting module is *finendo quasitilting*. But the converse is not true. A counterexample will be given in Example 4.10.

Moreover, there is a further asymmetry. Indeed, definable classes give rise to minimal right approximations, and also to left approximations, but not necessarily to *minimal* left approximations. So not all silting (nor tilting) classes satisfy the dual of condition (3) in Theorem 3.8. Moreover, not all definable torsion classes are generated by a finendo quasitilting module, an example is given in [12, Proposition 7.2]. However, we will see in the forthcoming section that definable torsion classes coincide with silting classes over noetherian rings.

4. DUALITY

For a better understanding of the asymmetries discussed above, we need to review the interplay between definable subcategories of $\text{Mod-}A$ and $A\text{-Mod}$. Let us briefly recall that a subcategory \mathcal{D} of $\text{Mod-}A$ is definable if and only if it is the intersection of the kernels of a set of coherent functors $\text{Mod-}A \rightarrow \text{Ab}$, that is, of additive functors commuting with direct limits and direct products. Since the coherent functors are precisely the cokernels of morphisms of functors $\text{Hom}_A(\sigma, -) : \text{Hom}_A(M, -) \rightarrow \text{Hom}_A(N, -)$ induced by morphisms $\sigma : N \rightarrow M$ in $\text{mod-}A$, one obtains that \mathcal{D} is definable if and only if it is of the form $\mathcal{D} = \mathcal{D}_\Sigma$ for a set Σ of morphisms between finitely presented right A -modules, see e.g. [25, Sections 2.1 and 2.3].

Further, denoting by $(\text{mod-}A, \text{Ab})$ and $(A\text{-mod}, \text{Ab})$ the categories of additive covariant functors on $\text{mod-}A$ and $A\text{-mod}$, respectively, one has an assignment

$$(\text{mod-}A, \text{Ab}) \rightarrow (A\text{-mod}, \text{Ab}), F \mapsto F^\vee$$

where F^\vee is defined on a module N in $A\text{-mod}$ by

$$F^\vee(N) = \text{Hom}(F, - \otimes N).$$

Notice that Hom-functors $F = \text{Hom}_A(M, -)$ given by $M \in \text{mod-}A$ are mapped to \otimes -functors $F^\vee = - \otimes_A M$, see e.g. [49, Example 10.3.1].

This assignment was first studied by Auslander and by Gruson and Jensen. It induces a duality between the finitely presented objects in $(\text{mod-}A, \text{Ab})$ and $(A\text{-mod}, \text{Ab})$, and it allows to associate to every definable category \mathcal{D} in $\text{Mod-}A$ a *dual definable category* \mathcal{D}^\vee in $A\text{-Mod}$. Indeed, $\mathcal{D}^\vee = \mathcal{F}_\Sigma$ where Σ is the maximal collection of morphisms in $\text{mod-}A$ such that $\mathcal{D} = \mathcal{D}_\Sigma$. In this way one obtains a bijection between definable subcategories of $\text{Mod-}A$ and $A\text{-Mod}$ interchanging definable

torsion classes with definable torsionfree classes. For details we refer to [12, Section 5] and the references therein.

We now review some results from [4] explaining how the assignment $\mathcal{D} \mapsto \mathcal{D}^\vee$ acts on the definable torsion or torsionfree classes given by silting and cosilting modules, respectively. We fix a commutative ring k such that A is a k -algebra, and given an A -module M , we denote by M^+ its dual with respect to an injective cogenerator of $\text{Mod-}k$. For example, take $k = \mathbb{Z}$ and M^+ the character dual of M .

Proposition 4.1. *Let $\sigma \in \text{Mor}(\text{Proj-}A)$.*

- (1) σ^+ is a morphism between injective left A -modules with $\mathcal{C}_{\sigma^+} = \mathcal{F}_\sigma$.
- (2) If \mathcal{D}_σ is a silting class, then $\mathcal{D}_\sigma^\vee = \mathcal{F}_\sigma$.
- (3) If T is a silting module with respect to σ , then T^+ is a cosilting module with respect to σ^+ .

Definition 4.2. A cosilting left A -module C (or the cosilting class $\text{Cogen } C$) is said to be of *cofinite type* if there is a set $\Sigma \subset \text{Mor}(\text{proj-}A)$ such that $\text{Cogen } C = \mathcal{F}_\Sigma$.

Corollary 4.3. *The assignment $\mathcal{D} \mapsto \mathcal{D}^\vee$ defines a bijection between silting classes in $\text{Mod-}A$ and cosilting classes of cofinite type in $A\text{-Mod}$.*

If we restrict to tilting classes, that is, to classes $\mathcal{D} = \mathcal{D}_\Sigma$ with Σ consisting of monomorphisms in $\text{Mor}(\text{proj-}A)$, then we recover the bijection between tilting classes and cotilting classes of cofinite type established in [12]. Indeed, in this case $\mathcal{F}_\Sigma = \text{Ker Tor}_1^A(\mathcal{S}, -)$ where \mathcal{S} is the set of finitely presented right A -modules of projective dimension at most one that arise as cokernels of the monomorphisms in Σ . So Definition 4.2 agrees with the definition of a cotilting class of cofinite type in [12].

Here is a useful criterion for a torsionfree class to be cosilting of cofinite type.

Lemma 4.4. *If a torsion pair $(\mathcal{T}, \mathcal{F})$ in $A\text{-Mod}$ is generated by a set of finitely presented left A -modules, then \mathcal{F} is a cosilting class of cofinite type.*

Proof. Let \mathcal{U} be a set in $A\text{-mod}$ such that $\mathcal{F} = \{M \in A\text{-Mod} \mid \text{Hom}_A(U, M) = 0 \text{ for all } U \in \mathcal{U}\}$. Choosing a projective presentation $\alpha_U \in \text{Mor}(A\text{-proj})$ for each $U \in \mathcal{U}$ and applying $*$ = $\text{Hom}_A(-, A)$ on it, we obtain a set $\Sigma = \{\alpha_U^* \mid U \in \mathcal{U}\} \subset \text{Mor}(\text{proj-}A)$ such that $\mathcal{F} = \mathcal{F}_\Sigma$. \square

Now recall that over a left noetherian ring every torsion pair $(\mathcal{T}, \mathcal{F})$ in $A\text{-Mod}$ restricts to a torsion pair $(\mathcal{U}, \mathcal{V})$ in $A\text{-mod}$ with $\mathcal{U} = \mathcal{T} \cap A\text{-mod}$ and $\mathcal{V} = \mathcal{F} \cap A\text{-mod}$, and moreover, taking direct limit closures, $(\mathcal{U}, \mathcal{V})$ extends to a torsion pair $(\varinjlim \mathcal{U}, \varinjlim \mathcal{V})$ in $A\text{-Mod}$, see [24, Lemma 4.4]. So, if \mathcal{F} definable, that is, closed under direct limits, we see that $(\mathcal{T}, \mathcal{F}) = (\varinjlim \mathcal{U}, \varinjlim \mathcal{V})$ is generated by \mathcal{U} . Using Lemma 4.4, we conclude that every definable torsionfree class in $A\text{-Mod}$ is a cosilting class of cofinite type. This also implies that every definable torsion class in $\text{Mod-}A$ is a silting class.

Theorem 4.5. [4, Corollary 3.8] *If A is a left noetherian ring, the assignment $\mathcal{D} \mapsto \mathcal{D}^\vee$ defines a bijection between silting classes in $\text{Mod-}A$ and cosilting classes in $A\text{-Mod}$, and there are bijections between*

- (i) equivalence classes of silting right A -modules;
- (ii) equivalence classes of cosilting left A -modules;
- (iii) definable torsion classes in $\text{Mod-}A$;
- (iv) definable torsionfree classes in $A\text{-Mod}$.

The bijection (i) \rightarrow (iii) is given by the assignment $T \mapsto \text{Gen } T$, the bijection (ii) \rightarrow (iv) is defined dually.

The theorem above can be regarded as a “large” version of the following result for finite dimensional algebras due to Adachi-Iyama-Reiten.

Theorem 4.6. [1, Theorem 2.7] *If A is a finite dimensional algebra over a field, there are bijections between*

- (i) *isomorphism classes of basic support τ -tilting (i.e. finite dimensional silting) right A -modules;*
- (ii) *isomorphism classes of basic finite dimensional cosilting left A -modules;*
- (iii) *functorially finite torsion classes in $\text{mod-}A$;*
- (iv) *functorially finite torsionfree classes in $A\text{-mod}$.*

The bijection (i) \rightarrow (iii) is given by the assignment $T \mapsto \text{gen } T = \text{Gen } T \cap \text{mod-}A$, the bijection (ii) \rightarrow (iv) is defined dually.

Next, we apply the criterion in Lemma 4.4 to hereditary torsion pairs. Recall that a torsion pair $(\mathcal{T}, \mathcal{F})$ is *hereditary* if the torsion class \mathcal{T} is closed under submodules, or equivalently, the torsionfree class \mathcal{F} is closed under injective envelopes. Moreover, $(\mathcal{T}, \mathcal{F})$ has *finite type* if \mathcal{F} is closed under direct limits.

Hereditary torsion pairs are in bijection with the Gabriel topologies on A . The *Gabriel filter* \mathcal{G} corresponding to a hereditary torsion pair $(\mathcal{T}, \mathcal{F})$ in $A\text{-Mod}$ is formed by the left ideals I of A such that $A/I \in \mathcal{T}$ (see e.g.[54, Ch.VI]). If the torsion pair is of finite type, then \mathcal{G} has a basis of finitely generated ideals [49, Section 11.1.2], and so the torsion pair $(\mathcal{T}, \mathcal{F})$ is generated by the set $\mathcal{U} = \{A/I \mid I \in \mathcal{G} \text{ finitely generated}\} \subset A\text{-mod}$. Thus \mathcal{F} is a cosilting class of cofinite type.

The dual definable category \mathcal{F}^\vee is then given by the right A -modules with $M \otimes A/I = 0$ for all finitely generated ideals $I \in \mathcal{G}$, and one easily checks that this amounts to M being *I -divisible*, i.e. $MI = M$, for all such I . We will say that \mathcal{F}^\vee is the *class of divisibility* by these ideals.

Corollary 4.7. *The assignment $\mathcal{D} \mapsto \mathcal{D}^\vee$ restricts to a bijection between the silting classes occurring as classes of divisibility by sets of finitely generated left ideals, and the torsionfree classes in hereditary torsion pairs of finite type.*

Over a commutative ring, every cosilting class of cofinite type arises from a hereditary torsion pair of finite type, because it turns out that every torsionfree class of the form \mathcal{F}_σ for some $\sigma \in \text{Mor}(\text{proj-}A)$ is closed under injective envelopes [4, Lemma 4.2]. This yields the following classification result.

Theorem 4.8. [4, Theorems 4.7 and 5.1] *If A is commutative ring, there is a bijection between*

- (i) *equivalence classes of silting A -modules,*
- (ii) *hereditary torsion pairs of finite type in $\text{Mod-}A$.*

If A is commutative noetherian, there are further bijections with

- (iii) *equivalence classes of cosilting A -modules,*
- (iv) *subsets $P \subseteq \text{Spec}(A)$ closed under specialisation.*

The bijection between (ii) and (iv) is well known, see [54, Chapter VI, §6.6]. An explicit construction of a silting module in (i) and a cosilting module in (iii) is provided in [4].

We close this section with an example of a cosilting module not of cofinite type.

Lemma 4.9. *Let A be a ring, and S a simple module such that S is a finitely generated module over $\text{End}_A S$ and $\text{Ext}_A^1(S, S) = 0$. Then S satisfies condition (3) in Theorem 3.6 and its dual. In particular, S is a cosilting module.*

Proof. Since $\text{End}_A S$ is a skew-field, the module S has finite length over $\text{End}_A S$, and therefore $\text{Add } S = \text{Prod } S$, see [38]. By assumption, $\text{Gen } S = \text{Add } S = \text{Cogen } S$ are contained both in $S^{\perp 1}$ and ${}^{\perp 1}S$. Moreover, if E is an injective cogenerator of $\text{Mod-}A$, the codiagonal map $S^{(I)} \rightarrow E$ given by $I = \text{Hom}_A(S, E)$ is a right $\text{Cogen } S$ -approximation with kernel in $\text{Prod } S$, yielding condition (3) in Theorem 3.6. For the dual condition take the diagonal map $A \rightarrow S^J$ given by $J = \text{Hom}_A(A, S)$. \square

Example 4.10. [4, Example 5.4] Let (A, \mathfrak{m}) be a valuation domain whose maximal ideal $\mathfrak{m} = \mathfrak{m}^2$ is idempotent and non-zero. Then $S = A/\mathfrak{m}$ satisfies the conditions of the Lemma above (indeed, we will see in Example 5.7(2) and Theorem 5.3(2) that the idempotency of \mathfrak{m} implies $\text{Ext}_A^1(S, S) = 0$), so it is a cosilting module with $\text{Cogen } S = \text{Gen } S = \text{Add } S = \{M \in \text{Mod-}A \mid M\mathfrak{m} = 0\}$.

Notice that $\text{Cogen } S$ does not contain the injective envelope of S , and so it does not arise from a hereditary torsion pair. It follows from Theorem 4.8 that S is not a cosilting module of cofinite type. Moreover, despite the fact that S satisfies the statement dual to condition (3) in Theorem 3.6, it is not a silting module. This follows again from Theorem 4.8, because if $\text{Gen } S$ were a silting class, it would be the class of divisibility by a set of finitely generated left ideals. But the only ideal I with $SI = S$ is $I = A$, which would entail $\text{Gen } S = \text{Mod-}A$, a contradiction.

5. RING EPIMORPHISMS

We have seen above that silting modules are closely related with localisation at Gabriel topologies, and silting classes are often given by divisibility conditions. In this section, we discuss the connections between silting and ring theoretic localisation. We first recall the relevant terminology.

Definition 5.1. [54, Ch.XI][28] A ring homomorphism $f : A \rightarrow B$ is a *ring epimorphism* if it is an epimorphism in the category of rings with unit, or equivalently, if the functor given by restriction of scalars $f_* : \text{Mod-}B \hookrightarrow \text{Mod-}A$ is full.

Further, f is a *homological ring epimorphism* if it is a ring epimorphism and $\text{Tor}_i^A(B, B) = 0$ for all $i > 0$, or equivalently, the functor given by restriction of scalars $f_* : \text{D}(B) \rightarrow \text{D}(A)$ is a full embedding.

Finally, f is a (*right*) *flat ring epimorphism* if it is a ring epimorphism and B is a flat right A -module.

Two ring epimorphisms $f_1 : A \rightarrow B_1$ and $f_2 : A \rightarrow B_2$ are *equivalent* if there is a ring isomorphism $h : B_1 \rightarrow B_2$ such that $f_2 = h \circ f_1$. We then say that they lie in the same *epiclass* of A .

Definition 5.2. [28],[33, Theorem 1.6.3][32] A full subcategory \mathcal{X} of $\text{Mod-}A$ is called *bireflective* if the inclusion functor $\mathcal{X} \hookrightarrow \text{Mod-}A$ admits both a left and right adjoint, or equivalently, if it is closed under products, coproducts, kernels and cokernels.

Moreover, a full subcategory \mathcal{W} of $\text{mod-}A$ is said to be *wide* if it is an abelian subcategory of $\text{mod-}A$ closed under extensions.

Theorem 5.3. (1) [27] *There is a bijection between*

- (i) *epiclasses of ring epimorphisms;*
- (ii) *bireflective subcategories of $\text{Mod-}A$.*

It assigns to a ring epimorphism $f : A \rightarrow B$ the essential image \mathcal{X}_B of the restriction functor $f_ : \text{Mod-}B \hookrightarrow \text{Mod-}A$.*

(2) [15][52, Theorem 4.8] *The following statements are equivalent for a ring epimorphism $f : A \rightarrow B$:*

- (1) *\mathcal{X}_B is closed under extensions in $\text{Mod-}A$;*
- (2) *$\text{Tor}_1^A(B, B) = 0$;*
- (3) *the functors Ext_A^1 and Ext_B^1 coincide on B -modules;*
- (4) *the functors Tor_1^A and Tor_1^B coincide on B -modules.*

(3) [27, 28], [33, Theorem 1.6.1] *If A is a finite dimensional algebra, the assignment $f \mapsto \mathcal{X}_B \cap \text{mod-}A$ defines a bijection between*

- (i) *epiclasses of ring epimorphisms $f : A \rightarrow B$ with B finite dimensional and $\text{Tor}_1^A(B, B) = 0$;*
- (ii) *functorially finite wide subcategories of $\text{mod-}A$.*

The following result provides a large supply of ring epimorphisms with the properties in Theorem 5.3(2).

Theorem 5.4. [52, Theorem 4.1] *Let A be a ring and Σ be a set of morphisms in $\text{Mor}(\text{proj-}A)$. Then there is a ring homomorphism $f : A \rightarrow A_\Sigma$, called universal localisation of A at Σ , such that*

- (i) *f is Σ -inverting, i.e. $\sigma \otimes_A A_\Sigma$ is an isomorphism for every σ in Σ , and*
- (ii) *f is universal Σ -inverting, i.e. for any Σ -inverting morphism $f' : A \rightarrow B$ there exists a unique ring homomorphism $g : A_\Sigma \rightarrow B$ such that $g \circ f = f'$.*

Moreover, $f : A \rightarrow A_\Sigma$ is a ring epimorphism with $\text{Tor}_1^A(A_\Sigma, A_\Sigma) = 0$.

Notice that A_Σ coincides with the universal localisation at $\Sigma^* = \{\sigma^* \mid \sigma \in \Sigma\} \subset \text{Mor}(A\text{-proj})$. Moreover, the essential image of the restriction functor along $f : A \rightarrow A_\Sigma$ consists of the right A -modules such that $X \otimes_A \sigma^*$ is an isomorphism for every $\sigma \in \Sigma$, and it equals

$$\mathcal{X}_\Sigma = \{X \in \text{Mod-}A \mid \text{Hom}_A(\sigma, X) \text{ is bijective for all } \sigma \in \Sigma\}.$$

We are going to discuss how classes of the latter shape are related with silting modules.

Definition 5.5. A module T_1 (the reason for this notation will become clear later) is called a *partial silting module* if it admits a projective presentation $P_{-1} \xrightarrow{\sigma} P_0 \rightarrow T_1 \rightarrow 0$ such that \mathcal{D}_σ is a torsion class containing T_1 .

By [5, Theorem 3.12] every partial silting module T_1 can be completed to a silting module T , called a *Bongartz completion* of T_1 , generating the same torsion class \mathcal{D}_σ . Moreover, $\text{Gen } T_1 \subset \mathcal{D}_\sigma \subset T_1^{\perp 1}$, and $(\text{Gen } T_1, T_1^\circ)$ is a torsion pair. Then

$$\mathcal{X}_\sigma = \mathcal{D}_\sigma \cap (\text{Coker } \sigma)^\circ = \text{Gen } T \cap T_1^\circ.$$

As shown in [6, Proposition 3.3], the class \mathcal{X}_σ is bireflective and extension closed. Thus it can be realised as \mathcal{X}_B for some ring epimorphism $f : A \rightarrow B$ as in Theorem 5.3(2), and by [6, Theorem 3.5] the ring B can be described as an idempotent quotient of $\text{End}_A T$.

A ring epimorphism arising from a partial silting module as above will be called *silting ring epimorphism*.

Theorem 5.6. [44] *Every universal localisation is a silting ring epimorphism.*

In order to prove the converse of the theorem above, one would need an analog of Theorem 2.5 for classes \mathcal{X}_σ . With the techniques employed in [44] (morphism categories and cotorsion pairs) one can only prove the existence of a set Σ of morphisms between *countably* generated projective modules such that $\mathcal{X}_\sigma = \mathcal{X}_\Sigma$. So the question whether silting ring epimorphisms coincide with universal localisations remains open.

Examples 5.7. (1) A module T_1 is partial silting with respect to an monomorphic projective presentation $\sigma : P_{-1} \hookrightarrow P_0$ if and only if it is partial tilting, and in this case $\mathcal{X}_\sigma = T_1^{\perp 1} \cap T_1^\circ$ is the perpendicular category of T_1 studied in [28, 22].

(2) For any ideal I of A , the canonical surjection $f : A \rightarrow \overline{A} = A/I$ is a ring epimorphism. Moreover, since $I/I^2 \cong \text{Tor}_1^A(\overline{A}, \overline{A})$, the ideal $I = I^2$ is idempotent if and only if $\mathcal{X}_{\overline{A}}$ is closed under extensions, in which case $\mathcal{X}_{\overline{A}} = I^\circ$.

An important example of an idempotent ideal is provided by the trace ideal $I = \tau_P(A)$ of a projective right A -module P . Notice that every idempotent ideal I has the form $I = AeA$ for some $e = e^2 \in A$ whenever A is a (one-sided) perfect ring [46, Proposition 2.1], or when A is commutative and I is finitely generated, in which case f is even a split epimorphism [40, Lemma 2.43]. Moreover, every idempotent ideal I with ${}_A I$ being finitely generated is the trace ideal of a countably generated projective right A -module, see [56].

(3) Let us focus on the case when $I = \tau_P(A)$ is the trace ideal of a projective right A -module P , and $f : A \rightarrow \overline{A} = A/I$. In this case $\mathcal{X}_{\overline{A}}$ consists of the modules M with $\tau_P(M) = MI = 0$, so

it is the perpendicular category of the partial tilting module P , and f is therefore a silting ring epimorphism. In fact, f is even a universal localisation. This is clear when P is finitely generated (then f is the universal localisation at $\Sigma = \{0 \rightarrow P\}$). For the general case, one uses the following argument due to Pavel Příhoda. First of all, keeping in mind that every projective module is a direct sum of countably generated projectives by a celebrated result of Kaplansky, we can assume w.l.o.g. that P is countably generated. Then P can be written as a direct limit of a direct system of finitely generated free modules $F_1 \xrightarrow{\alpha_1} F_2 \xrightarrow{\alpha_2} F_3 \xrightarrow{\alpha_3} \dots$ where each map α_i is given by multiplication with a matrix X_i having its entries in the trace ideal I , and moreover, for each $i > 1$ there is a map $\beta_i : F_{i+1} \rightarrow F_i$ such that $\beta_i \alpha_i \alpha_{i-1} = \alpha_{i-1}$, see [56, Theorem 1.9], [34, Proposition 1.4]. Now we set $\Sigma = \{1_{F_i} - \beta_i \alpha_i \mid i > 1\}$. Since f is Σ -invertig, the universal property in Theorem 5.4 implies that $\mathcal{X}_{\bar{A}} \subset \mathcal{X}_{\Sigma}$. Conversely, the fact that $A \rightarrow A_{\Sigma}$ is Σ -invertig entails $\alpha_i \otimes_A A_{\Sigma} = 0$ for all $i \geq 1$, hence $P \otimes_A A_{\Sigma} = 0$. But then $\text{Hom}_A(P, A_{\Sigma}) \cong \text{Hom}_{A_{\Sigma}}(P \otimes_A A_{\Sigma}, A_{\Sigma}) = 0$, that is, A_{Σ} belongs to $\mathcal{X}_{\bar{A}}$, and so do all A_{Σ} -modules.

(4) If A is a semihereditary ring, then all flat ring epimorphisms $A \rightarrow B$ are universal localisations [3, Proposition 5.3], and the converse is true if A is also commutative [14, Theorem 7.8]. Moreover, Tor_i^A vanishes for $i \geq 2$ (see e.g. [40, Theorem 4.67]), hence every epimorphism as in Theorem 5.3(2) is homological, and its kernel is an idempotent ideal by [14, Lemma 4.5]. On the other hand, for a semihereditary commutative ring, the fact that every homological epimorphism is a universal localisation amounts to the validity of the Telescope Conjecture for $D(A)$, and it fails in general, see [14, Section 8] and the example below.

(5) Let (A, \mathfrak{m}) be as in Example 4.10. Then $f : A \rightarrow \bar{A} = A/\mathfrak{m}$ is a homological ring epimorphism. But f is not a silting ring epimorphism, and thus not a universal localisation. Indeed, if there were a partial silting module T_1 with Bongartz completion T such $\mathcal{X}_{\bar{A}} = \text{Gen } T \cap T_1^\circ$, then by Theorem 4.8 the silting class $\text{Gen } T$ would be the class of divisibility by a set of finitely generated ideals of A , and then also by all ideals in the corresponding Gabriel filter \mathcal{G} . Since \mathfrak{m} is the unique maximal ideal and $\bar{A} \in \text{Gen } T$ is certainly not \mathfrak{m} -divisible, we infer that \mathcal{G} can only contain the ideal A . But then $\text{Gen } T = \text{Mod-}A$, and T and T_1 are projective, hence free, thus $\text{Gen } T \cap T_1^\circ = 0$ cannot coincide with $\mathcal{X}_{\bar{A}}$.

(6) If A is a hereditary ring, then homological ring epimorphisms and universal localisations of A coincide [39]. Moreover, by [53, Theorem 2.3], there is a bijection between wide subcategories of $\text{mod-}A$ and universal localisations of A , which maps a wide subcategory \mathcal{W} to the universal localisation at (projective resolutions of the modules in) \mathcal{W} . Conversely, every $A \rightarrow A_{\Sigma}$ is associated to the wide subcategory \mathcal{W} formed by the Σ -trivial modules, that is, the modules $M \in \text{mod-}A$ admitting a projective resolution $\sigma : P_{-1} \hookrightarrow P_0$ with $\sigma \otimes_A A_{\Sigma}$ being an isomorphism.

Our next aim is to present an important case where universal localisations are parametrised by silting modules. Recall that every silting module T satisfies the statement dual to Theorem 3.6(3), so there is a left $\text{Gen } T$ -approximation sequence for A with $T_0, T_1 \in \text{Add } T$

$$(*) \quad A \xrightarrow{f} T_0 \rightarrow T_1 \rightarrow 0.$$

Definition 5.8. If in the sequence $(*)$ the left approximation f can be chosen left minimal (i. e. such that any $h \in \text{End}_A T_0$ with $hf = f$ is an isomorphism), then T is said to be a *minimal* silting module.

For example, every finite dimensional silting module over a finite dimensional algebra is minimal. Moreover, ring epimorphisms with nice homological properties give rise to minimal silting modules. This is due to the following construction of silting modules which is dual to Proposition 3.5.

Proposition 5.9. [8] *Let T be a module with a projective presentation $P_{-1} \xrightarrow{\sigma} P_0$ such that $\text{Gen } T \subset \mathcal{D}_\sigma$. Assume there are a cardinal κ and a left $\text{Add } T$ -approximation $f : A \rightarrow T^{(\kappa)}$. Then $T \oplus \text{Coker } f$ is a silting A -module.*

Furthermore, by arguments dual to the proof of Proposition 3.5, there is a projective presentation ω of $\text{Coker } f$ such that $\text{Gen } T = \mathcal{D}_\omega$, which in particular shows that $\text{Coker } f$ is partial silting. Now we apply this construction to a ring epimorphism $f : A \rightarrow B$. Recall that f can always be regarded as a minimal left $\text{Add } B$ -approximation of A , in fact, it is the \mathcal{X}_B -reflection of A .

Corollary 5.10. [8] *Let $f : A \rightarrow B$ be a ring epimorphism such that B admits a projective presentation $P_{-1} \xrightarrow{\sigma} P_0$ with $\text{Gen } B \subset \mathcal{D}_\sigma$. Then $B \oplus \text{Coker } f$ is a minimal silting A -module, $\text{Coker } f$ is partial silting and f is the corresponding silting ring epimorphism.*

In particular, the assumptions of the Corollary are satisfied by any homological ring epimorphism such that B is an A -module of projective dimension at most one, recovering a construction of tilting modules from [9].

The importance of minimal silting modules is due to the fact that an approximation sequence $(*)$ with f being left minimal is unique up to isomorphism, and so is the module T_1 . If T_1 is partial silting, we can thus associate to T a uniquely determined silting ring epimorphism f .

Let us now assume that A is a hereditary ring. In this case $I = \text{Ann}(T)$ is an idempotent (projective) ideal [6, Lemma 5.5]. Since T is tilting over $\overline{A} = A/I$, the exact sequence $0 \rightarrow \overline{A} \rightarrow T_0 \rightarrow T_1 \rightarrow 0$ induced by a minimal approximation sequence $(*)$ gives $\text{Gen } T = T_1^{\perp 1} \cap \mathcal{X}_{\overline{A}} = T_1^{\perp 1} \cap I^\circ$, and one verifies that T_1 is partial silting with respect to the projective presentation σ given as the direct sum of a monomorphic presentation with the trivial map $I \rightarrow 0$.

So we can associate to T the silting ring epimorphism $f : A \rightarrow B$ with $\mathcal{X}_B = \mathcal{X}_\sigma$, which is a homological epimorphism. Notice that the class \mathcal{X}_σ can also be described in a different way. For any torsion class \mathcal{T} in an abelian category \mathcal{A} , we consider the subcategory of \mathcal{A}

$$(5.1) \quad \mathfrak{a}(\mathcal{T}) := \{X \in \mathcal{T} : \text{if } (g : Y \rightarrow X) \in \text{Mor}(\mathcal{T}), \text{ then } \text{Ker}(g) \in \mathcal{T}\}$$

studied in [32]. It turns out that in our situation $\mathcal{X}_\sigma = \mathfrak{a}(\text{Gen } T)$, see [6, Proposition 5.6]. Furthermore, the category of projective B -modules is equivalent to $\text{Add } T_0$. This shows that the epiclass of f determines $\text{Add } T_0$, thus $\text{Gen } T_0 = \text{Gen } T$. So, \mathfrak{a} defines an injective map $T \mapsto f$ up to equivalence. On the other hand, we also have a reverse assignment. According to Corollary 5.10, to every homological ring epimorphism $f : A \rightarrow B$ we can associate the minimal silting module $T = B \oplus \text{Coker } f$, and taking the silting ring epimorphism corresponding to T we recover f . Combined with the results from [53, 39] explained in Example 5.7(6), we obtain

Theorem 5.11. [6] *Let A be a hereditary ring. There is a bijection between*

- (i) *equivalence classes of minimal silting A -modules;*
- (ii) *epiclasses of homological ring epimorphisms of A ;*
- (ii') *epiclasses of universal localisations of A ;*
- (iii) *wide subcategories of $\text{mod-}A$.*

Under this bijection, epiclasses of injective homological ring epimorphisms of A correspond to minimal tilting A -modules.

Example 5.12. If A is a commutative hereditary ring, then every silting module is minimal. Indeed, by Theorem 4.8 every silting module T corresponds to a hereditary torsion pair of finite type, that is, to a Gabriel topology \mathcal{G} with a basis of finitely generated ideals. Since all ideals are projective, \mathcal{G} is a *perfect* Gabriel topology according to [54, Chapter XI, Proposition 3.3 and Theorem 3.4], and it induces a flat ring epimorphism $f : A \rightarrow B$ satisfying the assumptions of

Corollary 5.10. Then one can show as in [30, Theorem 5.4] that T is equivalent to the minimal silting module $B \oplus \text{Coker } f$.

In particular, if A is a Dedekind domain, there is a bijection between equivalence classes of tilting modules and sets of maximal ideals, and the trivial module 0 is the only silting module that is not tilting, see [9, Corollary 6.12].

6. COMBINATORICS OF UNIVERSAL LOCALISATIONS

The partial order on bireflective subcategories given by inclusion corresponds under the bijection in Theorem 5.3(1) to the partial order on the epiclasses of A defined by setting

$$f_1 \geq f_2$$

whenever there is a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{f_2} & B_2 \\ & \searrow f_1 & \nearrow g \\ & & B_1 \end{array}$$

Since bireflective subcategories are determined by closure properties, the poset induced by \geq is a lattice, and the ring epimorphisms $A \rightarrow B$ with $\text{Tor}_1^A(B, B) = 0$ form a sublattice in it by Theorem 5.3(2). We now compare these posets with other posets recently studied in cluster theory.

First of all, observe that over a finite dimensional algebra, Theorem 5.11 has also a “small” version.

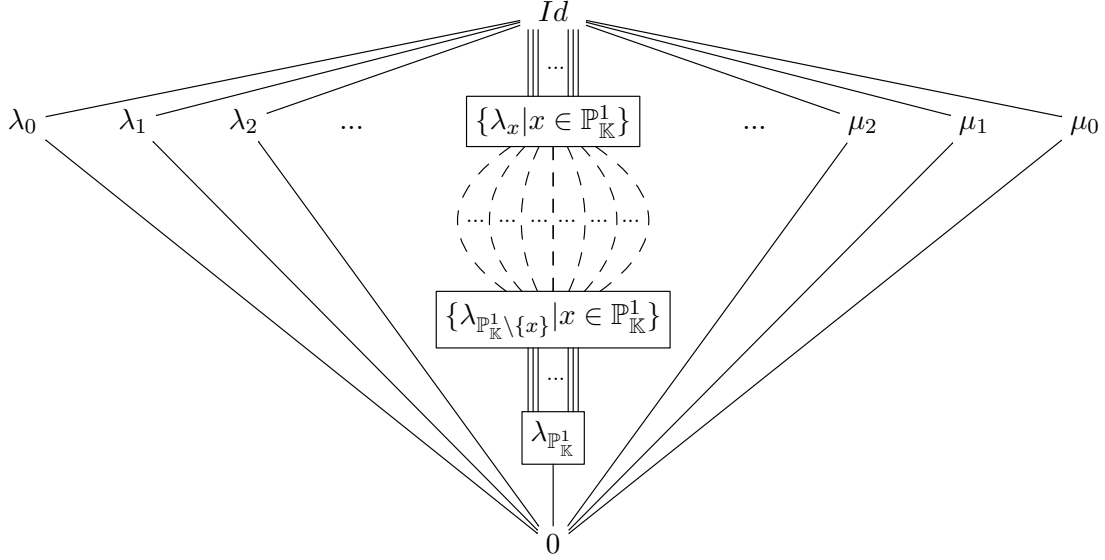
Corollary 6.1. [32, Section 2],[41, Theorem 4.2] *If A is a finite dimensional hereditary algebra, there are bijections between*

- (i) *isomorphism classes of basic support tilting (i.e. finite dimensional silting) modules;*
- (ii) *epiclasses of homological ring epimorphisms $A \rightarrow B$ with B finite dimensional;*
- (iii) *functorially finite wide subcategories of $\text{mod-}A$.*

In [32, 31, 51] further bijections are established, providing a combinatorial interpretation of finitely generated silting modules in terms of noncrossing partitions, clusters, or antichains. Observe that the poset of noncrossing partitions corresponds to the poset given by \geq on the epiclasses in condition (ii), and it does not form a lattice in general (unless we relax the condition that B is finite dimensional, as discussed above).

Examples 6.2. (1) Let A be a tame hereditary finite dimensional algebra with a tube \mathcal{U} of rank 2, and let S_1, S_2 be the simple regular modules in \mathcal{U} . The universal localisation $f_i : A \rightarrow A_{\{S_i\}}$ of A at (a projective resolution of) S_i has a finite dimensional codomain $A_{\{S_i\}}$ for $i = 1, 2$. The meet of f_1 and f_2 , however, is the universal localisation $A \rightarrow A_{\{S_1, S_2\}}$ of A at \mathcal{U} and $A_{\{S_1, S_2\}}$ is an infinite dimensional algebra by [23, Theorem 4.2].

(2) Let A be the Kronecker algebra, i.e. the path algebra of the quiver $\bullet \rightrightarrows \bullet$ over an algebraically closed field \mathbb{K} . The lattice of universal localisations has the following shape



Here the λ_i are the homological ring epimorphisms corresponding to preprojective silting modules, the μ_i correspond to preinjective silting modules, and the ring epimorphisms in frames are those with infinite dimensional codomain, that is, those given by universal localisation at (projective resolutions of) simple regular modules. The interval between Id and $\lambda_{\mathbb{P}^1_{\mathbb{K}}}$ represents the dual poset of subsets of the projective line $\mathbb{P}^1_{\mathbb{K}}$ over \mathbb{K} .

Up to equivalence, there is just one additional silting module which is not minimal and thus does not appear in the lattice above. It is called Lukas tilting module and it generates the class of all modules without preprojective summands. More details are given in [6, Examples 5.10 and 5.18].

We have seen in Section 5 that over a hereditary ring one can associate to any minimal silting module T a well-defined silting ring epimorphism f which turns out to be a universal localisation. Another important case where the assignment $T \mapsto f$ is well-defined is established by Marks and Stovicek in [43]. Let A be a finite dimensional algebra and T a finite dimensional silting module. Recall from Theorem 4.6 that $\text{gen } T = \text{Gen } T \cap \text{mod-}A$ is a functorially finite torsion class in $\text{mod-}A$, so the minimal approximation sequence

$$(*) \quad A \xrightarrow{f} T_0 \rightarrow T_1 \rightarrow 0$$

has all terms in $\text{mod-}A$. Dually to the proof of Proposition 3.5, we find a triangle in $\mathcal{D}(A)$

$$A^\bullet \xrightarrow{\tilde{f}} \gamma \rightarrow \omega \rightarrow$$

whose zero cohomologies give rise to (*). In particular, ω is a projective presentation of T_1 . It is shown in [43, Lemma 4.3] that T_1 is partial silting with respect to the projective presentation $\sigma = \omega \oplus (eA \rightarrow 0) \in \text{Mor}(\text{proj-}A)$, where $e \in A$ is a maximal idempotent element such that $\text{Hom}_A(eA, T) = 0$.

Proposition 6.3. *If A is a finite dimensional algebra, there is a commutative triangle of injections*

$$\begin{array}{ccc}
 \left\{ \begin{array}{l} \text{equivalence classes of} \\ \text{finite dimensional} \\ \text{silting } A\text{-modules} \end{array} \right\} & \xrightarrow{\alpha} & \left\{ \begin{array}{l} \text{epiclasses of} \\ \text{universal localisations} \\ f : A \rightarrow B \text{ with } \dim B < \infty \end{array} \right\} \\
 \searrow \mathfrak{a} & & \swarrow \epsilon \\
 & & \left\{ \begin{array}{l} \text{functorially finite wide} \\ \text{subcategories of } \text{mod-}A \end{array} \right\}
 \end{array}$$

Here $\alpha : T \mapsto f$ associates to a silting module T the silting ring epimorphism $f : A \rightarrow B$ induced by the partial silting module T_1 , and ϵ is the assignment $f \mapsto \mathcal{X}_B \cap \text{mod-}A$. Furthermore, \mathfrak{a} is the map defined in (5.1) for the abelian category $\text{mod-}A$.

Proof. We combine results from [43]. First of all, if T is a finite dimensional silting module, and T_1 and σ are as above, then by [43, Lemma 3.8] the functorially finite torsion class $\text{gen } T$ gives rise to the functorially finite wide subcategory $\mathfrak{a}(\text{gen } T) = \text{gen } T \cap T_1^\circ = \mathcal{X}_\sigma \cap \text{mod-}A$. But this is precisely the wide subcategory corresponding to the silting ring epimorphism associated to T_1 . It follows from Theorem 5.3(3) that f has a finite dimensional codomain, and it is the universal localisation of A at σ by [43, Proposition 4.4]. We conclude that α is well-defined, and the triangle commutes. Finally, ϵ is obviously injective, and \mathfrak{a} is injective by [43, Proposition 3.9]. \square

The image of the map \mathfrak{a} in the proposition above is determined in [43, Theorem 3.10]. It consists of the functorially finite wide subcategories $\mathcal{W} \subset \text{mod-}A$ for which the smallest torsion class in $\text{mod-}A$ containing \mathcal{W} is functorially finite. In particular, \mathfrak{a} is bijective whenever all torsion classes are functorially finite. Let us turn to algebras with such property.

Definition 6.4. [1, 26] Let A be a finite dimensional algebra, and denote by τ the Auslander-Reiten transpose. A module $M \in \text{mod-}A$ is τ -rigid if $\text{Hom}_A(M, \tau M) = 0$, and it is τ -tilting if, in addition, the number of non-isomorphic indecomposable direct summands of M coincides with the number of isomorphism classes of simple A -modules. The algebra A is τ -tilting finite if there are only finitely many isomorphism classes of basic τ -tilting A -modules.

We list some characterisations of τ -tilting finite algebras from [26], and we add two new conditions [7].

Theorem 6.5. *The following statements are equivalent for a finite dimensional algebra A .*

- (1) A is τ -tilting finite.
- (2) There are only finitely many finite dimensional silting A -modules up to equivalence.
- (3) Every torsion class in $\text{mod-}A$ is functorially finite.
- (4) Every silting A -module is finite dimensional up to equivalence.
- (5) There are only finitely many epiclasses of ring epimorphisms $A \rightarrow B$ with $\text{Tor}_1^A(B, B) = 0$.

In particular, if A is τ -tilting finite, all ring epimorphisms $A \rightarrow B$ with $\text{Tor}_1^A(B, B) = 0$ are universal localisations with a finite dimensional codomain.

Theorem 6.6. [43] *Let A be a finite dimensional τ -tilting finite algebra. Then there are bijections between*

- (i) isomorphism classes of basic support τ -tilting (i.e. silting) modules;
- (ii) epiclasses of ring epimorphisms $A \rightarrow B$ with $\text{Tor}_1^A(B, B) = 0$;
- (ii') epiclasses of universal localisations of A ;
- (iii) wide subcategories of $\text{mod-}A$.

The theorem above applies in particular to all algebras of finite representation type, but also to many representation-infinite algebras.

Examples 6.7. (1) [41, Lemma 3.5] If A is a local finite dimensional algebra, then A and 0 are the only silting modules, up to equivalence.

(2) Let A be a preprojective algebra of Dynkin type. Then the collections in Theorem 6.6 are further in bijection with the elements of the Weyl group of the underlying Dynkin quiver [47]. The interplay between combinatorial and ring theoretic data is used in [42] to describe the algebras arising as universal localisations of A and to determine the homological ring epimorphisms.

(3) The combinatorics of universal localisations is also used in [41] to classify silting modules over Nakayama algebras.

(4) Finally, let us remark that the maps α and \mathbf{a} in Proposition 6.3 do not preserve the poset structure, see e.g. [41, Example 4.8] or [6, Example 5.18].

We close this survey with some open questions.

Question 1. What is the image of the map α in Proposition 6.3? Recall that all universal localisations are silting ring epimorphisms by Theorem 5.6, but in general we do not know how finite dimensional silting modules interact with silting ring epimorphisms having a finite dimensional codomain.

Question 2. Is there a “large” version of Proposition 6.3 relating (minimal) silting modules with universal localisations and bireflective subcategories of $\text{Mod-}A$? Notice that it is open whether the module T_1 in a minimal approximation sequence $(*) A \rightarrow T_0 \rightarrow T_1 \rightarrow 0$ is always partial silting.

Question 3. In Theorems 4.8, 5.11, 6.6, we have seen several instances of the interplay between silting modules and localisations. Is there a suitable notion of localisation with a general statement encompassing all these cases?

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