Existence, Multiplicity and Perturbation Results for Quasilinear Elliptic Problems via Nonsmooth Critical Point Theory

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TO MY PARENTS
AND TO
MARIA MALANDRINO
FOR THEIR PATIENCE AND UNDERSTANDING

“A mathematician is a machine for turning coffee into theorems”

“If I were a medical man, I should prescribe a holiday to
any patient who considered his work important”

“All’inizio e alla fine abbiamo il mistero.
Potremmo dire che abbiamo il disegno di Dio.
A questo mistero la matematica ci avvicina, senza penetrarlo”
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Introduction

Let $\Omega$ be an open bounded subset in $\mathbb{R}^n$ ($n \geq 2$) and $f : H^1_0(\Omega) \to \mathbb{R}$ a functional of the form

$$f(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x) D_x_i u D_x_j u \, dx - \int_{\Omega} G(x, u) \, dx.$$ 

Since the pioneering paper of Ambrosetti–Rabinowitz [6], critical point theory has been successfully applied to the functional $f$, yielding several important results (see e.g. [41, 84, 93, 113]).

However, the assumption that $f : H^1_0(\Omega) \to \mathbb{R}$ is of class $C^1$ turns out to be very restrictive for more general functionals of calculus of variations, like

$$f(u) = \int_{\Omega} \mathcal{L}(x, u, \nabla u) \, dx - \int_{\Omega} G(x, u) \, dx,$$

(see e.g. [50]). In particular, if $f$ has the form

$$f(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x, u) D_x_i u D_x_j u \, dx - \int_{\Omega} G(x, u) \, dx,$$

we may expect $f$ to be of class $C^1$ only when the $a_{ij}$'s are independent of $u$ (see [35]). On the other hand, since the papers of Chang [42] and Marino–Scolozzi [82], techniques of critical point theory have been extended to some classes on nonsmooth functionals. In our setting, in which $f$ is naturally continuous but not locally Lipschitz, it turns out to be convenient to apply the theory developed in [48, 54, 68, 69] according to the approach started by Canino [35].

Let us point out that a different approach has been also considered in the literature. If we consider the space $H^1_0(\Omega) \cap \mathbb{L}^\infty(\Omega)$ endowed with the family of norms

$$\|u\|_\varepsilon := \|u\|_{H^1_0} + \varepsilon \|u\|_{\mathbb{L}^\infty}, \quad \varepsilon > 0,$$

then, under suitable assumptions, $f$ is of class $C^1$ in $(H^1_0(\Omega) \cap \mathbb{L}^\infty(\Omega), \| \cdot \|_\varepsilon)$ for each $\varepsilon > 0$. This allows an approximation procedure by smooth problems (the original one is obtained as a limit when $\varepsilon \to 0$). The papers of Struwe [114] and Arcaya–Boccardo [7, 8] follow, with some variants, this kind of approach. However, in view of multiplicity results, it is hard to keep the multiplicity of solutions at the limit. In particular, when $f$ is even and satisfies assumptions of Ambrosetti–Rabinowitz type, the existence of infinitely many solutions has been so far proved only by the former approach.
The aim of this Ph.D. thesis is to extend some results concerning existence, nonexistence, multiplicity and perturbation from symmetry for quasilinear problems like

\[
\begin{aligned}
- \sum_{i,j=1}^{n} D_{x_i} (a_{ij}(x,u) D_{x_j} u) + \frac{1}{2} \sum_{i,j=1}^{n} D_{s} a_{ij}(x,u) D_{x_i} u D_{x_j} u &= g(x,u) \quad \text{in } \Omega \\
\text{with } u = 0 \quad \text{on } \partial \Omega 
\end{aligned}
\]

and even for the more general class of elliptic problems

\[
\begin{aligned}
- \text{div} (\nabla \xi \mathcal{L}(x,u,\nabla u)) + D_{s} \mathcal{L}(x,u,\nabla u) &= g(x,u) \quad \text{in } \Omega \\
\text{with } u = 0 \quad \text{on } \partial \Omega 
\end{aligned}
\]

including the case when \( g \) has a critical growth in the sense of the Sobolev embedding

\[
W^{1,p}_0(\Omega) \hookrightarrow L^{np/(n-p)}(\Omega), \quad 1 < p < n.
\]

New results for (*) and (**) have been obtained in the following situations:

- **Chapter II**
  Fully nonlinear eigenvalue problems for elliptic systems; infinitely many solutions for quasilinear elliptic systems; existence of a weak solution for a general class of Euler’s equations of multiple integrals of calculus of variations (see [100, 101, 102]).

- **Chapter III**
  Multiplicity of solutions for perturbed symmetric \((g(x,-s) \neq -g(x,s))\) quasilinear elliptic systems; multiplicity of solutions for quasilinear elliptic equations at exponential growth and broken symmetry; multiplicity results for semilinear elliptic systems with broken symmetry and non–homogeneous boundary data (double loss of symmetry) (see [97, 98, 104, 111, 112]).

- **Chapter IV**
  Problems of jumping type for a general class of Euler’s equations of multiple integrals of calculus of variations; problems of jumping type for a general class of nonlinear variational inequalities (see [107, 108]).

- **Chapter V**
  Positive entire solutions for fully nonlinear elliptic equations; existence of two solutions for fully nonlinear problems at critical growth with perturbations of lower order; asymptotics of solutions for a class of nonlinear problems at nearly critical growth (see [96, 103, 99, 110]).

- **Chapter VI**
  Pucci–Serrin identities type for solutions of class \(C^1(\Omega)\) of Euler’s equations and related non–existence results (see [105]).

Work is in progress on higher order elliptic problems at critical growth (see [106, 109]).
Chapter 1

Recalls of non–smooth critical point theory

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1.1 Notions of nonsmooth analysis

In this section, we shall recall some results of abstract critical point theory [37, 48, 54, 68, 69]. For the proofs, we refer to [37] or [48]. Let $X$ be a metric space endowed with the metric $d$ and let $f : X \to \mathbb{R}$ be a function. We denote by $B_r(u)$ the open ball of center $u$ and radius $r$ and we set

$$ epi(f) = \{(u, \lambda) \in X \times \mathbb{R} : f(u) \leq \lambda\}. $$

In the following, $X \times \mathbb{R}$ will be endowed with the metric

$$ d((u, \lambda), (v, \mu)) = (d(u, v))^2 + (\lambda - \mu)^2 \frac{1}{2} $$

and $epi(f)$ with the induced metric.

**Definition 1.1.1.** For every $u \in X$ with $f(u) \in \mathbb{R}$, we denote by $|df|(u)$ the supremum of the $\sigma$‘s in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map

$$ \mathcal{H} : (B_{\delta}(u, f(u)) \cap epi(f)) \times [0, \delta] \to X $$

satisfying

$$ d(\mathcal{H}((v, \mu), t), v) \leq t, \quad f(\mathcal{H}((v, \mu), t)) \leq \mu - \sigma t, $$

whenever $(v, \mu) \in B_{\delta}(u, f(u)) \cap epi(f)$ and $t \in [0, \delta]$. The extended real number $|df|(u)$ is called the weak slope of $f$ at $u$. 
**Proposition 1.1.2.** Let \( u \in X \) with \( f(u) \in \mathbb{R} \). If \((u_h)\) is a sequence in \( X \) with \( u_h \to u \) and \( f(u_h) \to f(u) \), then we have \( |df|(u) \leq \lim\inf_h |df|(u_h) \).

**Remark 1.1.3.** If the restriction of \( f \) to \( \{u \in X : f(u) \in \mathbb{R}\} \) is continuous, then
\[
|df| : \{u \in X : f(u) \in \mathbb{R}\} \to [0, +\infty]
\]
is lower semicontinuous.

**Proposition 1.1.4.** Let \( f : X \to \mathbb{R} \cup \{+\infty\} \) be a function. Set
\[
\mathcal{D}(f) := \{u \in X : f(u) < +\infty\}
\]
and assume that \( f|_{\mathcal{D}(f)} \) is continuous. Then for every \( u \in \mathcal{D}(f) \) we have
\[
|df|(u) = |df|_{\mathcal{D}(f)}|(u)
\]
and this value is in turn equal to the supremum of the \( \sigma \)'s in \([0, +\infty]\) such that there exist \( \delta > 0 \) and a continuous map
\[
\mathcal{H} : (B_\delta(u) \cap \mathcal{D}(f)) \times [0, \delta] \to X
\]
satisfying
\[
d(\mathcal{H}(v, t), v) \leq t, \quad f(\mathcal{H}(v, t)) \leq f(v) - \sigma t,
\]
whenever \( v \in B_\delta(u) \cap \mathcal{D}(f) \) and \( t \in [0, \delta] \).

**Definition 1.1.5.** An element \( u \in X \) is said to be a (lower) critical point of \( f \) if \( |df|(u) = 0 \). A real number \( c \) is said to be a (lower) critical value of \( f \) if there exists a critical point \( u \in X \) of \( f \) such that \( f(u) = c \). Otherwise \( c \) is said to be a regular value of \( f \).

**Definition 1.1.6.** Let \( c \) be a real number. The function \( f \) is said to satisfy the Palais–Smale condition at level \( c \) ((PS)\(_c\) for short), if every sequence \((u_h)\) in \( X \) with \( |df|(u_h) \to 0 \) and \( f(u_h) \to c \) admits a subsequence \((u_{h_k})\) converging in \( X \) to some \( u \).

Let us also introduce some usual notations. For every \( b \in \mathbb{R} \cup \{+\infty\} \) and \( c \in \mathbb{R} \) we set
\[
f^b = \{u \in X : f(u) \leq b\}, \quad K_c = \{u \in X : |df|(u) = 0, f(u) = c\}.
\]

**Theorem 1.1.7.** (Deformation Theorem) Let \( c \in \mathbb{R} \). Assume that \( X \) is complete, \( f : X \to \mathbb{R} \) is a continuous function which satisfies (PS)\(_c\). Then, given \( \varepsilon > 0 \), a neighborhood \( U \) of \( K_c \) (if \( K_c = \emptyset \), we allow \( U = \emptyset \)) and \( \lambda > 0 \), there exist \( \varepsilon > 0 \) and a continuous map
\[
\eta : X \times [0, 1] \to X
\]
such that for every \( u \in X \) and \( t \in [0, 1] \) we have:
\begin{enumerate}[(a)]
\item \( d(\eta(u, t), u) \leq \lambda t; \)
\item \( f(\eta(u, t)) \leq f(u); \)
\item \( f(u) \notin [c - \varepsilon, c + \varepsilon] \implies \eta(u, t) = u; \)
\item \( \eta(f^{c+\varepsilon} \setminus U, 1) \subset f^{c-\varepsilon}. \)
\end{enumerate}
1.1. Notions of nonsmooth analysis

Theorem 1.1.8. (Noncritical Interval Theorem) Let \( a \in \mathbb{R} \) and \( b \in \mathbb{R} \cup \{+\infty\} \) \((a < b)\). Assume that \( f : X \to \mathbb{R} \) is a continuous function which has no critical points \( u \) with \( a \leq f(u) \leq b \), that \((PS)_c\) holds and \( f^- \) is complete whenever \( c \in [a, b] \). Then there exists a continuous map \( \eta : X \times [0, 1] \to X \) such that for every \( u \in X \) and \( t \in [0, 1] \) we have:

(a) \( \eta(u, 0) = u \);
(b) \( f(\eta(u, t)) \leq f(u) \);
(c) \( f(u) \leq a \implies \eta(u, t) = u \);
(d) \( f(u) \leq b \implies f(\eta(u, 1)) \leq a \).

Theorem 1.1.9. Let \( X \) be a complete metric space and \( f : X \to \mathbb{R} \cup \{+\infty\} \) a function such that \( \mathbb{D}(f) \) is closed in \( X \) and \( f|_{\mathbb{D}(f)} \) is continuous. Let \( u_0, v_0, v_1 \) be in \( X \) and suppose that there exists \( r > 0 \) such that \( \|v_0 - u_0\| X < r \), \( \|v_1 - u_0\| X > r \), \( \inf f(B_r(u_0)) > -\infty \), and

\[
a' = \inf \{ f(u) : u \in X, \|u - u_0\| X = r \} > \max \{ f(v_0), f(v_1) \}.
\]

Let

\[ \Gamma = \{ \gamma : [0, 1] \to \mathbb{D}(f) \text{ continuous with } \gamma(0) = v_0, \gamma(1) = v_1 \} \]

and assume that \( \Gamma \neq \emptyset \) and that \( f \) satisfies the Palais–Smale condition at the two levels

\[
c_1 = \inf f(B_r(u_0)), \quad c_2 = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} f(\gamma(t)).
\]

Then \(-\infty < c_1 < c_2 < +\infty\) and there exist at least two critical points \( u_1, u_2 \) of \( f \) such that \( f(u_i) = c_i \) \((i = 1, 2)\).

We now recall the mountain pass theorem without Palais–Smale.

Theorem 1.1.10. Let \( X \) be a Banach space and \( f : X \to \mathbb{R} \) is a continuous functional. Assume that the following facts hold:

(a) there exist \( \eta > 0 \) and \( \varrho > 0 \) such that:

\[
\forall u \in X : \|u\| X = \varrho \implies f(u) > \eta ;
\]

(b) \( f(0) = 0 \) and there exists \( w \in X \) such that:

\[
f(w) < \eta \text{ and } \|w\| X > \varrho .
\]

Moreover, let us set:

\[
\Phi = \{ \gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = w \}
\]

and

\[
\eta \leq \beta = \inf_{\gamma \in \Phi, t \in [0, 1]} f(\gamma(t)).
\]

Then there exists a Palais–Smale sequence for \( f \) at level \( \beta \).
In the next theorem, we recall a generalization of the classical perturbation argument of Bahri, Berestycki, Rabinowitz and Struwe devised around 1980 for dealing with problems with broken symmetry adapted to our non-smooth framework (See [92]).

In the next theorem, we recall a generalization due to Struwe [113] of the classical perturbation argument for dealing with problems with broken symmetry, here adapted to our nonsmooth framework.

**Theorem 1.1.11.** Let $X$ be a Hilbert space endowed with a norm $\| \cdot \|_X$ and let $f : X \to \mathbb{R}$ be a continuous functional. Assume that there exists $M > 0$ such that $f$ satisfies the concrete Palais–Smale condition at each level $c \geq M$. Let $Y$ be a finite dimensional subspace of $X$ and $u^* \in X \setminus Y$ and set

$$Y^* = Y \oplus \langle u^* \rangle, \quad Y^*_+ = \{ u + \lambda u^* : u \in Y, \lambda \geq 0 \}.$$

Assume now that $f(0) \leq 0$ and that:

(a) there exists $R > 0$ such that:

$$\forall u \in Y : \|u\|_X \geq R \implies f(u) \leq f(0);$$

(b) there exists $R^* \geq R$ such that:

$$\forall u \in Y^* : \|u\|_X \geq R^* \implies f(u) \leq f(0).$$

Let us set

$$\mathcal{P} = \left\{ \gamma \in C(X, X) : \gamma \text{ odd, } \gamma(u) = u \text{ if } \max\{f(u), f(-u)\} \leq 0 \right\}.$$

Then, if

$$c^* = \inf_{\gamma \in \mathcal{P}} \sup_{u \in Y^*_+} f(\gamma(u)) > c = \inf_{\gamma \in \mathcal{P}} \sup_{u \in Y} f(\gamma(u)) \geq M,$$

$f$ admits at least one critical value $\bar{c} \geq c^*$.

### 1.2 Functionals of the calculus of variations

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$ and let $f : W^{1,p}_0(\Omega; \mathbb{R}^N) \to \mathbb{R}$ $(N \geq 1)$ be a functional of the form

$$f(u) = \int_{\Omega} L(x, u, \nabla u) \, dx. \quad (1.1)$$

The associated Euler’s equation is formally given by the quasilinear problem

$$\begin{cases}
- \text{div} \left( \nabla_x L(x, u, \nabla u) \right) + \nabla_s L(x, u, \nabla u) = 0 & \text{in } \Omega \\
\quad u = 0 & \text{on } \partial \Omega. \quad (1.2)
\end{cases}$$

Assume that $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is measurable in $x$ for all $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$ and of class $C^1$ in $(s, \xi)$ for a.e. $x \in \Omega$. Moreover, assume that there exist $a_0 \in L^1(\Omega)$, $b_0 \in \mathbb{R}$, $a_1 \in L^1_{\text{loc}}(\Omega)$ and $b_1 \in L^\infty_{\text{loc}}(\Omega)$ such that for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$ we have

$$|L(x, s, \xi)| \leq a_0(x) + b_0|s|^{np/(n-p)} + b_0|\xi|^p, \quad (1.3)$$
Let $c \in \mathbb{R}$. A sequence $(u_h)$ in $W^{1,p}_0(\Omega, \mathbb{R}^N)$ is said to be a concrete Palais–Smale sequence at level $c$ ((CPS)$_c$–sequence, in short) for $f$, if $f(u_h) \to c$,

$$-\text{div} \left( \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \right) + \nabla_s \mathcal{L}(x, u_h, \nabla u_h) \in W^{-1,p'}(\Omega, \mathbb{R}^N)$$

eventually as $h \to \infty$ and

$$-\text{div} \left( \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \right) + \nabla_s \mathcal{L}(x, u_h, \nabla u_h) \to 0$$

strongly in $W^{-1,p'}(\Omega, \mathbb{R}^N)$.

We say that $f$ satisfies the concrete Palais–Smale condition at level $c$ ((CPS)$_c$ in short), if every (CPS)$_c$–sequence for $f$ admits a strongly convergent subsequence in $W^{1,p}_0(\Omega, \mathbb{R}^N)$.

The next result allow us to connect these “concrete” notions with the abstract critical point theory.
Chapter 1. Recalls of non-smooth critical point theory

Theorem 1.2.3. The functional $f$ is continuous and
\[ \forall u \in W_0^{1,p}(\Omega, \mathbb{R}^N) : |df|(u) \geq \sup \left\{ \int_{\Omega} \left( \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla v + \nabla_x \mathcal{L}(x, u, \nabla u) \cdot v \right) \, dx : v \in C_c^\infty(\Omega; \mathbb{R}^N), \|v\|_{1,p} \leq 1 \right\}. \]
Therefore, if $|df|(u) < +\infty$ it follows
\[ -\text{div} \left( \nabla_\xi \mathcal{L}(x, u, \nabla u) \right) + \nabla_x \mathcal{L}(x, u, \nabla u) \in W^{-1,p'}(\Omega, \mathbb{R}^N) \]
and
\[ \| -\text{div} \left( \nabla_\xi \mathcal{L}(x, u, \nabla u) \right) + \nabla_x \mathcal{L}(x, u, \nabla u) \|_{1,p'} \leq |df|(u). \]

Corollary 1.2.4. Let $u \in W_0^{1,p}(\Omega, \mathbb{R}^N)$, $c \in \mathbb{R}$ and let $(u_h)$ be a sequence in $W_0^{1,p}(\Omega, \mathbb{R}^N)$.
Then the following facts hold:
(a) if $u$ is a (lower) critical point of $f$, then $u$ is a weak solution of (1.2);
(b) if $(u_h)$ is a $(PS)_c$-sequence for $f$, then $(u_h)$ is a $(CPS)_c$-sequence for $f$;
(c) if $f$ satisfies $(CPS)_c$, then $f$ satisfies $(PS)_c$.

By means of the previous result, it is easy to deduce some versions of the Mountain Pass Theorem adapted to the functional $f$.

Theorem 1.2.5. Let $(D, S)$ be a compact pair, let $\psi : S \to W_0^{1,p}(\Omega, \mathbb{R}^N)$ be a continuous map and let
\[ \Phi = \left\{ \varphi \in C(D, W_0^{1,p}(\Omega, \mathbb{R}^N)) : \varphi|_S = \psi \right\}. \]
Assume that there exists a closed subset $A$ of $W_0^{1,p}(\Omega, \mathbb{R}^N)$ such that
\[ \inf_A f \geq \max_{\psi(S)} f, \]
\[ A \cap \psi(S) = \emptyset \text{ and } A \cap \varphi(D) \neq \emptyset \text{ for all } \varphi \in \Phi. \]
If $f$ satisfies the concrete Palais–Smale condition at level
\[ c = \inf_{\varphi \in \Phi} \max_{\varphi(D)} f, \]
then there exists a weak solution $u$ of (1.2) with $f(u) = c$. Furthermore, if $\inf_A f \geq c$, then there exists a weak solution $u$ of (1.2) with $f(u) = c$ and $u \in A$.

Theorem 1.2.6. Suppose that
\[ \mathcal{L}(x, -s, -\xi) = \mathcal{L}(x, s, \xi) \]
for a.e. $x \in \Omega$ and every $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$. Assume also that
(a) there exist $\rho > 0$, $\alpha > f(0)$ and a subspace $V \subset W_0^{1,p}(\Omega, \mathbb{R}^N)$ of finite codimension with
\[ \forall u \in V : \|u\| = \rho \implies f(u) \geq \alpha; \]
1.3 Non-smooth Lagrange multipliers

Let us now recall two definitions from [30], where a new notion of subdifferential for continuous functionals on normed spaces has been recently introduced.

**Definition 1.3.1.** Let $X$ be a real normed space and $C \subseteq X$. For each $u \in C$, we denote with $T_C(u)$ the set of all $v \in X$ such that for each $\varepsilon > 0$, there exist $\delta > 0$ and

$$\nu : (B(u, \delta) \cap C) \times [0, \delta] \to B(v, \varepsilon)$$

continuous with

$$\xi + t\nu(\xi, t) \in C,$$

when $\xi \in B(u, \delta) \cap C$ and $t \in [0, \delta]$. $T_C(u)$ is said the cone tangent to $C$ at $u$.

**Definition 1.3.2.** For each $u \in X$, set

$$\partial f(u) := \{\alpha \in X^* : (\alpha, -1) \in N_{\text{epi} f}(u, f(u))\},$$

where

$$N_{\text{epi} f}(u) := \{\nu \in X^* : \langle\nu, v\rangle \leq 0 \text{ for all } v \in T_{\text{epi} f}(u)\}.$$

$\partial f(u)$ is said to be the subdifferential of $f$ at $u$.

Via $\partial f(u)$ we shall connect critical points of $f$ constrained to $M$, with weak solutions to the related eigenvalue problem, where $M$ is the submanifold of $W_0^{1,p}(\Omega, \mathbb{R}^N)$

$$M = \left\{ u \in W_0^{1,p}(\Omega, \mathbb{R}^N) : \Phi(u) = 1 \right\},$$

$\Phi : W_0^{1,p}(\Omega, \mathbb{R}^N) \to W^{-1,p'}(\Omega, \mathbb{R}^N)$ is of class $C^1$, $M \neq \emptyset$, $0 \notin M$ and moreover $\nabla \Phi(u) \neq 0$ for each $u \in M$. Since $M$ is metric space endowed with the metric of $W_0^{1,p}(\Omega, \mathbb{R}^N)$, the weak slope $|df_M|(u)$ and the $(PS)_c$-condition for $f_M$ may of course be defined.

**Theorem 1.3.3.** For every $u \in W_0^{1,p}(\Omega, \mathbb{R}^N)$ there exists $\lambda \in \mathbb{R}$ such that

$$|df_M|(u) \geq \sup \left\{ \nabla f(u)(v) - \lambda \nabla \Phi(u)(v) : v \in C_c^\infty(\Omega, \mathbb{R}^N), \|v\|_{1,p} \leq 1 \right\}.$$

In particular, for each $(PS)_c$-sequence $(u_h)$ for $f_M$ there exists $(\lambda_h) \subseteq \mathbb{R}$ such that

$$\lim_{h} \sup \left\{ \nabla f(u_h)(v) - \lambda_h \nabla \Phi(u_h)(v) : v \in C_c^\infty(\Omega, \mathbb{R}^N), \|v\|_{1,p} \leq 1 \right\} = 0.$$
Chapter 1. Recalls of non-smooth critical point theory

Proof. By conditions (1.4) and (1.5), for every $u \in M$ and $v \in C^\infty_c(\Omega, \mathbb{R}^N)$ there exists

$$f'(u)(v) = \int_\Omega \nabla \xi L(x, u, \nabla u) \cdot \nabla v \, dx + \int_\Omega \nabla_s L(x, u, \nabla u) \cdot v \, dx$$

and the function $\{u \mapsto f'(u)(v)\}$ is continuous from $M$ into $\mathbb{R}$. Now, let us extend $f|_M$ to the functional $f^*: W^{1,p}_0(\Omega, \mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ given by

$$f^*(u) = \begin{cases} f(u) & \text{if } u \in M \\ +\infty & \text{if } u \notin M. \end{cases}$$ (1.6)

We may assume that $|df|_M(u) < +\infty$. As it is known it results $|df^*|(u) = |df|_M(u)$, so that by [30, Theorem 4.13] there exists $\omega \in \partial f^*(u)$ with $|df^*|(u) \geq \|\omega\|_{-1,p'}$. Moreover, by [30, Corollary 5.10] we have

$$\partial f^*(u) \subseteq \partial f(u) + \mathbb{R} \nabla \Phi(u).$$

Finally, by [30, Theorem 6.1], we get $\partial f(u) = \{\eta\}$ where

$$\langle \eta, v \rangle = \int_\Omega \nabla \xi L(x, u, \nabla u) \cdot \nabla v \, dx + \int_\Omega \nabla_s L(x, u, \nabla u) \cdot v \, dx = \nabla f(u)(v)$$

for each $v \in C^\infty_c(\Omega, \mathbb{R}^N)$ and the proof is complete. \qed
Chapter 2

Superlinear elliptic problems

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We refer the reader to [100, 101, 102]. Some parts of these publications have been slightly modified to give this collection a more uniform appearance.

2.1 Eigenvalue problems for elliptic systems

2.1.1 Introduction

Let $\Omega$ be a bounded and open subset of $\mathbb{R}^n$, $n \geq 3$ and $N \geq 1$. Existence and multiplicity results for quasilinear eigenvalue problems of the type:

\[
\left\{ - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left( a_{ij}(x,u) \frac{\partial u_k}{\partial x_i} \right) + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{h=1}^{N} \frac{\partial a_{ij}(x,u)}{\partial x_i} \frac{\partial u_h}{\partial x_j} \frac{\partial u_h}{\partial x_h} = \lambda \frac{\partial G}{\partial u_k}(x,u) \right. \\
\left. \begin{array}{c}
\text{in } \Omega \\
k = 1, \ldots, N \end{array} \right. \quad (u, \lambda) \in M \times \mathbb{R}
\]

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on the submanifold of $H^1_0(\Omega, \mathbb{R}^N)$

$$M = \left\{ u \in H^1_0(\Omega, \mathbb{R}^N) : \int_{\Omega} G(x, u) \, dx = 1 \right\},$$

have been firstly studied in 1983 by M. Struwe [114] and recently by G. Arioli [9] via techniques of non-smooth critical point theory.

The goal of this section is to study the following more general eigenvalue problem:

$$-\text{div} (\nabla \mathcal{L}(x, u, \nabla u)) + \nabla_s \mathcal{L}(x, u, \nabla u) = \lambda \nabla_s G(x, u) \quad (u, \lambda) \in M \times \mathbb{R}, \quad (2.1)$$

on the submanifold of $W^{1,p}_0(\Omega, \mathbb{R}^N)$

$$M = \left\{ u \in W^{1,p}_0(\Omega, \mathbb{R}^N) : \int_{\Omega} G(x, u) \, dx = 1 \right\}.$$

We shall consider functionals $f : W^{1,p}_0(\Omega, \mathbb{R}^N) \to \mathbb{R}$ defined by

$$f(u) = \int_{\Omega} \mathcal{L}(x, u, \nabla u) \, dx, \quad (2.2)$$

with $1 < p < n$. As it is known, in general $f$ is not even locally Lipschitzian unless $\mathcal{L}$ does not depend on $u$ or $n = 1$, so that classical critical point theory fails. Therefore, we refer to non-smooth critical point theory.

We shall prove that problem (2.1) admits a nontrivial weak solution in $M \times \mathbb{R}$ by restricting $f$ to $M$ and looking for constrained critical points.

We assume that $M \neq \emptyset$, that $\mathcal{L} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is measurable in $x$ for all $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$, of class $C^1$ in $(s, \xi)$ for a.e. $x \in \Omega$ and $\mathcal{L}(x, s, \cdot)$ is strictly convex. Moreover, we shall assume that:

- there exist $\nu > 0$ such that for each $\varepsilon > 0$ there is $a_\varepsilon \in L^1(\Omega)$ and $b_\varepsilon \in \mathbb{R}$ with

$$\nu|\xi|^p \leq \mathcal{L}(x, s, \xi) \leq a_\varepsilon(x) + \varepsilon|s|^p + b_\varepsilon|\xi|^p, \quad (2.3)$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$, where $p^*$ denotes the critical Sobolev’s exponent.

- there exists $b \in \mathbb{R}$ such that for each $\varepsilon > 0$ there exists $a_\varepsilon \in L^1(\Omega)$ with

$$|\nabla_s \mathcal{L}(x, s, \xi)| \leq a_\varepsilon(x) + \varepsilon|s|^p + b|\xi|^p, \quad (2.4)$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$. Moreover there is $a_1 \in L^{p'}(\Omega)$ with

$$|\nabla_s \mathcal{L}(x, s, \xi)| \leq a_1(x) + b|s|^\frac{p^*}{p} + b|\xi|^{p-1}, \quad (2.5)$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$.

- for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$

$$\nabla_s \mathcal{L}(x, s, \xi) \cdot s \geq 0. \quad (2.6)$$
2.1. Eigenvalue problems for elliptic systems

- (if $N > 1$) there exists a bounded Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$ such that
  \[
  \nabla_s L(x, s, \xi) \cdot \exp_\sigma(r, s) + \nabla_\xi L(x, s, \xi) \cdot \nabla \exp_\sigma(r, s, \xi) \leq 0
  \]
  for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^{nN}$, $\sigma \in \{-1, 1\}^N$ and $r, s \in \mathbb{R}^N$ where
  \[
  (\exp_\sigma(r, s))_h := \sigma_h \exp[\sigma_h(\psi(r_h) - \psi(s_h))]
  \]
  and
  \[
  [\nabla \exp_\sigma(r, s, \xi)]_{hi} := -\exp[\sigma_h(\psi(r_h) - \psi(s_h))]|\psi'(s_h)|_{ii} \xi_i^h
  \]
  for each $h = 1, \ldots, N$ and $i = 1, \ldots, n$.
- $G(x, s)$ is measurable in $x$ and of class $C^1$ in $s$ with $G(x, 0) = 0$ a.e. in $\Omega$. If $g(x, s)$ denotes $\nabla_s G(x, s)$, for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^{\frac{np}{np - 1 + p}}(\Omega)$ such that
  \[
  |g(x, s)| \leq a_\varepsilon(x) + \varepsilon|s|^{p-1}
  \]
  for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^N$.
- we have
  \[
  g(x, s) \cdot s > 0
  \]
  for a.e. $x \in \Omega$ and for each $s \neq 0$.

Under the previous assumptions the following is our main result.

**Theorem 2.1.1.** The eigenvalue problem:

\[
-\text{div} (\nabla_\xi L(x, u, \nabla u)) + \nabla_s L(x, u, \nabla u) = \lambda g(x, u) \quad (u, \lambda) \in M \times \mathbb{R},
\]

has at least one nontrivial weak solution $(u, \lambda) \in M \times \mathbb{R}$.

In the vectorial case ($N > 1$), to the author’s knowledge, problem (2.1) has only been considered in [114] and in [9] in the particular case

\[
L(x, s, \xi) = \frac{1}{2} \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x, s) \xi_i^h \xi_j^k,
\]

for coefficients $a_{ij}^{hk} : \Omega \times \mathbb{R}^N \to \mathbb{R}^{n^2}$ of the type $a_{ij}^{hk}(x, s) = \delta^{hk} \alpha_{ij}(x, s)$.

In [114, Theorem 3.2] the statement is essentially of perturbative nature, since it says that if for each $k \in \mathbb{N}$ there exists a $g_k > 0$ with

\[
|\nabla_s \alpha_{ij}(x, s)| < g_k \quad \text{for a.e. } x \in \Omega, \text{ for all } s \in \mathbb{R}^N,
\]

then the problem has at least $k$ distinct weak solutions:

\[
(u_\ell, \lambda_\ell) \in H^1_0(\Omega, \mathbb{R}^n) \times \mathbb{R}, \quad \ell = 1, \ldots, k.
\]

In other words, the less the coefficients $\alpha_{ij}(x, s)$ vary in $s$, the more solutions one gets.
In [9] a new technical condition is introduced to be compared with (2.7). It is assumed that there exist $K > 0$ and an increasing bounded Lipschitz function $\psi$ from $[0, +\infty]$ to $[0, +\infty]$ with $\psi(0) = 0$, $\psi'$ non-increasing, $\psi(s) \to K$ as $s \to +\infty$ and such that

$$\sum_{i,j=1}^{n} \sum_{k=1}^{N} \left| \frac{\partial \alpha_{ij}}{\partial s_k} (x, s) \xi_i \xi_j \right| \leq 2e^{-4K} \psi'(|s|) \sum_{i,j=1}^{n} \alpha_{ij} (x, s) \xi_i \xi_j$$

(2.13)

for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^n$ and for all $r, s \in \mathbb{R}^N$.

The proof itself of [11, Lemma 6.1] shows that this condition implies assumption (2.7) in the case of integrands $\mathcal{L}$ like (2.11). On the other hand, if $N > 2$, the two conditions look quite similar. However, our condition (2.7) seems to be preferable, because when $N = 1$ and $\mathcal{L}$ is given by (2.11), it reduces to the inequality

$$\left| \sum_{i,j=1}^{n} \frac{\partial \alpha_{ij}}{\partial s} (x, s) \xi_i \xi_j \right| \leq 2\psi'(s) \sum_{i,j=1}^{n} \alpha_{ij} (x, s) \xi_i \xi_j,$$

which is not so restrictive in view of the ellipticity of $\alpha_{ij}$, while (2.13) is in this case much stronger. For a general Lagrangian $\mathcal{L}$, in the case $N = 1$, condition (2.7) reduces to

$$|D_s \mathcal{L}(x, s, \xi)| \leq \psi'(s) \nabla_\xi \mathcal{L}(x, s, \xi) \cdot \xi$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. This assumption has already been considered in literature in jumping problems (see e.g. [33]).

In remark (2.1.7) we will show an example of $\mathcal{L}$ not of the form (2.11) and satisfying (2.7). Finally, we point out that (2.13) and (2.7) are not easily comparable to (2.12).

### 2.1.2 The concrete Palais–Smale condition

Recall first a very useful consequence of the Brezis-Browder’s Theorem [27, 28] in the vectorial case.

**Proposition 2.1.2.** Let $T \in L^1_{\text{loc}}(\Omega, \mathbb{R}^N) \cap W^{-1, p'}(\Omega, \mathbb{R}^N)$, $v \in W^{1, p}_0(\Omega, \mathbb{R}^N)$ and $\eta \in L^1(\Omega)$ with $T \cdot v \geq \eta$. Then $T \cdot v \in L^1(\Omega)$ and

$$\langle T, v \rangle = \int_{\Omega} T \cdot v \, dx$$

**Proof.** Let $(v_h) \subseteq C_c^\infty(\Omega, \mathbb{R}^N)$ with $v_h \to v$. Define $\Theta_h(v) \in W^{1, p}_0 \cap L^\infty$ with compact support in $\Omega$ by setting

$$\Theta_h(v) = \min\{|v|, |v_h|\} \frac{v}{\sqrt{|v|^2 + \frac{1}{h}}}.$$

Since

$$\min\{|v|, |v_h|\} \frac{T \cdot v}{\sqrt{|v|^2 + \frac{1}{h}}} \geq -\eta^- \in L^1(\Omega),$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. This assumption has already been considered in literature in jumping problems (see e.g. [33]).
and

\[ \left\langle T, \min\{|v|, |v_h|\} \frac{v}{\sqrt{|v|^2 + \frac{1}{h}}} \right\rangle = \int_\Omega \min\{|v|, |v_h|\} \frac{T \cdot v}{\sqrt{|v|^2 + \frac{1}{h}}} dx, \]

a variant of Fatou’s Lemma implies \( \int_\Omega T \cdot v dx \leq \langle T, v \rangle \), so that \( T \cdot v \in L^1(\Omega) \). Finally, since

\[
\left| \min\{|v|, |v_h|\} \frac{T \cdot v}{\sqrt{|v|^2 + \frac{1}{h}}} \right| \leq |T \cdot v|,
\]

Lebesgue’s Theorem yields

\[ \langle T, v \rangle = \int_\Omega T \cdot v dx, \]

and the proof is complete. \( \square \)

As a consequence of assumption (2.3) and convexity of \( L(x, s, \cdot) \), for each \( \varepsilon > 0 \) there exists \( a_\varepsilon \in L^1(\Omega) \) such that

\[
\nabla_\xi L(x, s, \xi) \cdot \xi \geq \nu|\xi|^p - a_\varepsilon(x) - \varepsilon|s|^{p^*}
\]

for a.e. \( x \in \Omega \) and for all \((s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN} \).

We now come to one of the main result of this section, i.e. the local compactness property for \((PS)_c\)-sequences.

**Theorem 2.1.3.** Let \((u_h)\) be a bounded sequence in \( W^{-1,p}_0(\Omega, \mathbb{R}^N) \) and set

\[
\langle w_h, v \rangle = \int_\Omega \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla v dx + \int_\Omega \nabla_s L(x, u_h, \nabla u_h) \cdot v dx,
\]

for all \( v \in C_c^\infty(\Omega, \mathbb{R}^N) \). If \((w_h)\) strongly converges to some \( w \) in \( W^{-1,p'}(\Omega, \mathbb{R}^N) \), then \((u_h)\) admits a strongly convergent subsequence in \( W^{1,p}_0(\Omega, \mathbb{R}^N) \).

**Proof.** Since \((u_h)\) is bounded in \( W^{1,p}_0(\Omega, \mathbb{R}^N) \), we find a \( u \) in \( W^{1,p}_0(\Omega, \mathbb{R}^N) \) such that, up to subsequences,

\[
\nabla u_h \rightharpoonup \nabla u \text{ in } L^p(\Omega, \mathbb{R}^N), \quad u_h \rightharpoonup u \text{ in } L^p(\Omega, \mathbb{R}^N), \quad u_h(x) \rightarrow u(x) \text{ a.e. } x \in \Omega.
\]

By [23, Theorem 2.1], up to a subsequence, we have

\[
\nabla u_h(x) \rightarrow \nabla u(x) \text{ a.e. } x \in \Omega.
\]

Therefore, by (2.5) we get

\[
\nabla \xi L(x, u_h, \nabla u_h) \rightharpoonup \nabla \xi L(x, u, \nabla u) \text{ in } L^{p'}(\Omega, \mathbb{R}^{nN}).
\]

We now want to prove that we have

\[
\langle w, u \rangle = \int_\Omega \nabla \xi L(x, u, \nabla u) \cdot \nabla u dx + \int_\Omega \nabla_s L(x, u, \nabla u) \cdot u dx.
\]
Let $\psi$ be as in (2.7) and test equation (2.15) with the following functions

$$v_h = \varphi(\sigma_1 \exp\{\sigma_1 (\psi(u^1) - \psi(u_h^1))\}, \ldots, \sigma_N \exp\{\sigma_N (\psi(u^N) - \psi(u_h^N))\}),$$

where $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$ and $\sigma_i = \pm 1$ for all $i = 1, \ldots, N$. By direct computation we obtain for a.e. $x \in \Omega$

$$D_j v_{hi} = (\sigma_i D_j \varphi + (\psi'(u_i)D_j u_i - \psi'(u_{hi})D_j u_{hi}) \exp[\sigma_i (\psi(u_i) - \psi(u_{hi}))])$$

for each $i = 1, \ldots, N$ and $j = 1, \ldots, n$. Therefore, with the notation

$$[\exp\{\sigma(\psi(u) - \psi(u_h))\} \psi'(u_h) \nabla u_h]_{ij} := \exp\{\sigma_i (\psi(u_i) - \psi(u_{hi}))\} \psi'(u_{hi}) D_j u_{hi}$$

for each $i = 1, \ldots, N$ and $j = 1, \ldots, n$, it results

$$\int_\Omega \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot [\varphi \nabla \varphi + \psi'(u) \nabla u \varphi] \exp\{\sigma(\psi(u) - \psi(u_h))\} \, dx +$$

$$- \langle w_h, \varphi \exp[\sigma(\psi(u) - \psi(u_h))]) \rangle +$$

$$+ \int_\Omega \left\{ \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \sigma \exp\{\sigma(\psi(u) - \psi(u_h))\} \right\} \, dx +$$

$$- \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \exp\{\sigma(\psi(u) - \psi(u_h))\} \psi'(u_h) \nabla u_h \right\} \varphi \, dx = 0.$$

Observe that if $v = (\sigma_1 \varphi, \ldots, \sigma_N \varphi)$ we have

$$\lim_h \langle w_h, \varphi \exp[\sigma(\psi(u) - \psi(u_h))]) \rangle = \langle w, v \rangle.$$

Since $u_h \rightharpoonup u$ in $W^{1,p}_0(\Omega, \mathbb{R}^N)$, we have

$$\lim_h \int_\Omega \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot [\varphi \nabla \varphi + \psi'(u) \nabla u \varphi] \exp\{\sigma(\psi(u) - \psi(u_h))\} \, dx =$$

$$= \int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla v \, dx + \int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \psi'(u) \nabla u \varphi \, dx.$$

Note now that by assumption (2.7) for each $h \in \mathbb{N}$ we have

$$\nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \sigma \exp\{\sigma(\psi(u) - \psi(u_h))\} +$$

$$- \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \exp\{\sigma(\psi(u) - \psi(u_h))\} \psi'(u_h) \nabla u_h \leq 0.$$

Therefore, Fatou’s Lemma implies that

$$\limsup_h \left\{ \int_\Omega \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \sigma \exp\{\sigma(\psi(u) - \psi(u_h))\} \varphi \, dx +$$

$$- \int_\Omega \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \exp\{\sigma(\psi(u) - \psi(u_h))\} \psi'(u_h) \nabla u_h \varphi \, dx \right\} \leq$$

$$\leq \int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot v \, dx - \int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \psi'(u) \nabla u \varphi \, dx.$$
Combining the previous inequalities we get

$$
\int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla v \, dx + \int_{\Omega} \nabla_s \mathcal{L}(x, u, \nabla u) \cdot v \, dx \geq \langle w, v \rangle.
$$

for each \( v = (\sigma_1 \varphi, \ldots, \sigma_N \varphi) \), with \( \varphi \in C_c^\infty(\Omega) \), \( \varphi \geq 0 \). Since we may exchange \( v \) with \( -v \), we obtain

$$
\int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla v \, dx + \int_{\Omega} \nabla_s \mathcal{L}(x, u, \nabla u) \cdot v \, dx = \langle w, v \rangle.
$$

(2.17)

for each \( v = (\sigma_1 \varphi, \ldots, \sigma_N \varphi) \) with \( \varphi \in C_c^\infty(\Omega) \) and \( \varphi \geq 0 \). Since each \( v \in C_c^\infty(\Omega, \mathbb{R}^N) \) is a linear combination of such functions, taking into account Proposition (2.1.2), we obtain relation (2.16). The final step is to prove that \( (u_h) \) goes to \( u \) in \( W^{1,p}_0(\Omega, \mathbb{R}^N) \). To this aim, let us first get the following inequality

$$
\limsup_h \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leq \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla u \, dx.
$$

(2.18)

Because of (2.6) Fatou’s Lemma yields

$$
\int_{\Omega} \nabla_s \mathcal{L}(x, u, \nabla u) \cdot u \, dx \leq \liminf_h \int_{\Omega} \nabla_s \mathcal{L}(x, u_h, \nabla u_h) \cdot u_h \, dx.
$$

(2.19)

Combining this fact with (2.16) and taking into account that

$$
\langle w_h, u_h \rangle \to \langle w, u \rangle \quad \text{as} \quad h \to +\infty,
$$

we deduce

$$
\limsup_h \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx =
$$

$$
= \limsup_h \left[ - \int_{\Omega} \nabla_s \mathcal{L}(x, u_h, \nabla u_h) \cdot u_h \, dx + \langle w_h, u_h \rangle \right] \leq
$$

$$
\leq \left[ - \int_{\Omega} \nabla_s \mathcal{L}(x, u, \nabla u) \cdot u \, dx + \langle w, u \rangle \right] =
$$

$$
= \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla u \, dx.
$$

In particular, again by Fatou’s Lemma, we have

$$
\int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla u \, dx \leq
$$

$$
\leq \liminf_h \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leq
$$

$$
\leq \limsup_h \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leq
$$

$$
\leq \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla u \, dx,
$$
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that is
\[ \lim_{h} \int_{\Omega} \nabla \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx = \int_{\Omega} \nabla \mathcal{L}(x, u, \nabla u) \cdot \nabla u \, dx, \]
which gives convergence in \( L^1(\Omega) \). Therefore, by (2.14), we conclude that:
\[ \lim_{h} \int_{\Omega} |\nabla u_h|^p \, dx = \int_{\Omega} |\nabla u|^p \, dx, \]
which gives convergence of \((u_h)\) to \(u\) in \( W^{1,p}_0(\Omega, \mathbb{R}^N) \).

**Corollary 2.1.4.** Let \((u_h)\) be a bounded sequence in \( W^{1,p}_0(\Omega, \mathbb{R}^N) \), \((\lambda_h)\) a sequence in \( \mathbb{R} \) and set for all \( v \in C_c^{\infty}(\Omega, \mathbb{R}^N) \)
\[ \langle \lambda_h w_h, v \rangle = \int_{\Omega} \nabla \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla v \, dx + \int_{\Omega} \nabla \mathcal{L}(x, u_h, \nabla u_h) \cdot v \, dx \]
If \((w_h)\) converges to some \( w \neq 0 \) in \( W^{-1,p'}(\Omega, \mathbb{R}^N) \) then \((u_h, \lambda_h)\) admits a strongly convergent subsequence in \( W^{1,p}_0(\Omega, \mathbb{R}^N) \times \mathbb{R} \).

**Proof.** By density, we can find \( \eta \in C_c^{\infty}(\Omega, \mathbb{R}^N) \) such that
\[ \lim_{h} \langle w_h, \eta \rangle = \langle w, \eta \rangle > 0. \]
Since of course the sequence
\[ \left\{ \int_{\Omega} \nabla \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \eta \, dx + \int_{\Omega} \nabla \mathcal{L}(x, u_h, \nabla u_h) \cdot \eta \, dx \right\} \]
is bounded, \((\lambda_h)\) is also bounded and the assertion follows by Theorem (2.1.3). \( \square \)

In the next result we prove that \( f \) satisfies \((PS)_c\)–condition.

**Lemma 2.1.5.** Let \( c \in \mathbb{R} \). Then, for each \((PS)_c\) sequence \((u_h)\) for \( f|_M \) there exists \( u \in M \) and \( \lambda \in \mathbb{R} \) such that, up to subsequences, \( u_h \to u \) in \( W^{1,p}_0(\Omega, \mathbb{R}^N) \) and \( \lambda_h \to \lambda \) in \( \mathbb{R} \). In particular, we have
\[ \int_{\Omega} \nabla \mathcal{L}(x, u, \nabla u) \cdot \nabla v \, dx + \int_{\Omega} \nabla \mathcal{L}(x, u, \nabla u) \cdot v \, dx = \lambda \int_{\Omega} g(x, u) \cdot v \, dx \]
for each \( v \in C_c^{\infty}(\Omega, \mathbb{R}^N) \).

**Proof.** Let \((u_h)\) be a \((PS)_c\) sequence for \( f|_M \). Since by (2.3) \((u_h)\) is bounded in \( W^{1,p}_0(\Omega, \mathbb{R}^N) \), \((u_h)\) weakly goes to a \( u \in M \) and since by (2.8) \( g \) is completely continuous from \( W^{1,p}_0(\Omega, \mathbb{R}^N) \) to \( W^{-1,p'}(\Omega, \mathbb{R}^N) \), up to subsequences it is
\[ g(x, u_h) \to g(x, u) \ \text{in} \ \ W^{-1,p'}(\Omega). \]
Now, by Theorem (1.3.3), there exists a sequence \((\lambda_h) \subset \mathbb{R} \) with
\[ \sup \left\{ \int_{\Omega} \nabla \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla v \, dx + \int_{\Omega} \nabla \mathcal{L}(x, u_h, \nabla u_h) \cdot v \, dx + \right\} \]

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$$-\lambda_h \int_{\Omega} g(x, u_h) \cdot v \, dx : v \in C_c^\infty(\Omega, \mathbb{R}^N), \|v\|_{1,p} \leq 1 \} \to 0,$$

as $h \to +\infty$. Hence, by applying Corollary (2.1.4) to

$$w_h = g(x, u_h) + A_h, \quad A_h \to 0 \text{ in } W^{-1,p'}(\Omega, \mathbb{R}^N),$$

up to subsequences $(u_h, \lambda_h)$ converges to $(u, \lambda)$ in $W_0^{1,p}(\Omega, \mathbb{R}^N) \times \mathbb{R}$.

We may now prove the main result of the section.

Proof of Theorem (2.1.1). By assumption (2.9) and $G(x,0) = 0$ we easily see that $0 \not\in M$ and $g(x,u) \neq 0$ for each $u \in M$. Since $f$ is bounded from below, there exists a $(PS)_c$-sequence $(u_h)$ for $f_{|\Omega}$ at the level:

$$c = \inf_{u \in M} f(u).$$

Indeed, let $(u_h)$ a sequence of minimizers for $f$ in $M$. Of course we have $f(u_h) \to c$. Moreover, if it was $|df|(u_h) \not\to 0$, we would find a $\sigma > 0$ such that $|df|(u_h) \geq \sigma$. Then by [37, Theorem 1.1.11] there exists a continuous deformation

$$\eta : M \times [0, \delta] \to M$$

for some $\delta > 0$ such that for all $t \in [0, \delta]$ and $h \in \mathbb{N}$

$$f(\eta(u_h, t)) \leq f(u_h) - \sigma t.$$

This easily yields the contradiction $f(\eta(u_h, t)) < c$ for sufficiently large values of $h \in \mathbb{N}$. Thus $(u_h)$ is a $(PS)_c$-sequence for $f_{|\Omega}$. The previous Lemma now provides a weak solution $(u, \lambda) \in M \times \mathbb{R}$ to (2.1). Of course $u \neq 0$.

2.1.3 Final remarks

We refer the reader to [9, section 6] for some concrete examples where the condition (2.7) is fulfilled for an integrand $L$ like (2.11).

Remark 2.1.6. Assume that there exists $R > 0$ such that

$$|s| \geq R \implies \nabla s L(x, s, \xi) = 0,$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$ and

$$|s| \leq R \implies \sum_{k=1}^N |D_{s_k} L(x, s, \xi)| \leq \frac{1}{4cR} \nabla \xi L(x, s, \xi) \cdot \xi$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$. Then (2.7) holds for a $\psi$ defined by

$$\psi(s) = \begin{cases} \frac{s}{4R} & \text{if } 0 \leq s \leq R \\ \frac{1}{4} & \text{if } s \geq R. \end{cases}$$
Remark 2.1.7. Let \( 2 < p < n \) and \( \mathcal{L} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{n^2} \to \mathbb{R} \) be defined by
\[
\mathcal{L}(x,s,\xi) = \frac{1}{p}(\nu + \arctan |s|^2)|\xi|^p \quad \nu \geq e\sqrt{N} (\sqrt{3} + \pi),
\]
for a.e. \( x \in \Omega \) and for all \((s,\xi) \in \mathbb{R}^N \times \mathbb{R}^{n^2}\). By following [11, Example 9.2] it is possible to show that there exist \( K > 0 \) and an increasing bounded Lipschitz function \( \psi : [0, +\infty[ \to [0, +\infty[ \) with \( \psi(0) = 0 \), \( \psi' \) non-increasing, \( \psi(s) \to K \) as \( s \to +\infty \) given by
\[
\psi(s) = \frac{\sqrt{N} e^{4K}}{\nu} \begin{cases} 
\frac{3^{3/4}s}{4} & \text{if } s \in [0,3^{-1/4}] \\
\sqrt{3} + \int_3^{3^{-1/4}} \frac{\tau}{\tau + 4} \, d\tau & \text{if } s \in [3^{-1/4},+\infty[ 
\end{cases}
\]
such that \( \mathcal{L} \) satisfies (2.20) and therefore (2.7).

Remark 2.1.8. Of course (2.7) does not look very nice and it is not clear how to describe the class of systems that satisfy this assumption. Therefore, it seems natural to look for some classes of quasilinear systems with a more particular structure but no need of this complicated hypothesis. Let us now consider the eigenvalue problem:
\[
\begin{cases}
-\text{div} (A_1(u_1)|\nabla u_1|^{p-2}\nabla u_1) + \frac{1}{p}A'_1(u_1)|\nabla u_1|^p = \lambda g_1(x,u) & \text{in } \Omega \\
\vdots & \vdots \\
-\text{div} (A_N(u_N)|\nabla u_N|^{p-2}\nabla u_N) + \frac{1}{p}A'_N(u_N)|\nabla u_N|^p = \lambda g_N(x,u) & \text{in } \Omega.
\end{cases}
\tag{2.21}
\]

In a variational setting, the weak solutions of (2.21) are the critical points of the functional \( f|_M : M \to \mathbb{R} \) defined by
\[
f(u) = \frac{1}{p} \sum_{k=1}^N \int_{\Omega} A_k(u_k)|\nabla u_k|^p \, dx
\]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( N \geq 1 \) and \( 1 < p < n \).

Consider now the following assumptions \((k = 1, \ldots, N)\):
- \( A_k \in C^1(\mathbb{R}) \) with \( a_k \leq A_k \leq a_k \) for some \( a_k, \bar{a}_k > 0 \).
- \( A'_k(s)s \geq 0 \) for each \( s \in \mathbb{R} \).
- there exists a bounded Lipschitz function \( \psi : \mathbb{R} \to \mathbb{R} \) such that
\[
A_k(s)e^{-p(\psi(t)-\psi(s))} \leq A_k(t) \leq A_k(s)e^{p(\psi(t)-\psi(s))}
\tag{2.22}
\]
for each \( s, t \in \mathbb{R} \) with \( s \leq t \).

Under the previous assumptions, by exploiting the proof of Theorem (2.1.3) it is possible to see that for system (2.21) assumption (2.7) may be replaced by (2.22) that looks much simpler and understandable. In some sense, this condition says that for each \( s \in \mathbb{R} \) fixed, \( A_k(t) \) must remain within the “exponential cone” determined by

\[
A_k(s)e^{-p(t)-p(s)}, \quad A_k(s)e^{p(t)-p(s)}
\]

and starting from \( A_k(s) \).

**Remark 2.1.9.** Condition (2.7) in only needed when \( N > 1 \), since for \( N = 1 \) Theorem (2.1.3) may be substituted by [100, Theorem 3.4] where no condition like (2.7) is requested in order to get the compactness property of \((PS)_c\) sequences.

Secondly, we remark that in the case \( N = 1 \) condition (2.6) can be assumed only for large values of \( |s| \), that is, there exists \( R > 0 \) such that

\[
|s| \geq R \implies D_s \mathcal{L}(x, s, \xi)s \geq 0
\]

for a.e. \( x \in \Omega \) and for all \((s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}\) (see again [100, Theorem 3.4]).

## 2.2 Quasilinear elliptic systems

### 2.2.1 Introduction

Many papers have been published on the study of multiplicity of solutions for quasilinear elliptic equations via non-smooth critical point theory; see e.g. [7, 10, 11, 35, 34, 37, 47, 88, 114]. However, for the vectorial case only a few multiplicity results have been proven: [114, Theorem 3.2] and recently [11, Theorem 3.2], where systems with multiple identity coefficients are treated. In this section, we consider the following diagonal quasilinear elliptic system, in an open bounded set \( \Omega \subset \mathbb{R}^n \) with \( n \geq 3 \),

\[
- \sum_{i,j=1}^{n} D_j(a^k_{ij}(x, u) D_i u_k) + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_s a^h_{ij}(x, u) D_i u_h D_j u_h = D_s G(x, u) \quad \text{in} \ \Omega,
\]

for \( k = 1, \ldots, N \), where \( u : \Omega \to \mathbb{R}^N \) and \( u = 0 \) on \( \partial \Omega \). To prove the existence of weak solutions, we look for critical points of the functional \( f : H^1_0(\Omega, \mathbb{R}^N) \to \mathbb{R} \),

\[
f(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} a^h_{ij}(x, u) D_i u_h D_j u_h \, dx - \int_{\Omega} G(x, u) \, dx.
\]

This functional is not locally Lipschitz if the coefficients \( a^h_{ij} \) depend on \( u \); however, as pointed out in [7, 35], it is possible to evaluate \( f' \),

\[
f'(u)(v) = \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} a^h_{ij}(x, u) D_i u_h D_j v_h \, dx + \\
\frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_s a^h_{ij}(x, u) \cdot v D_i u_h D_j u_h \, dx - \int_{\Omega} D_s G(x, u) \cdot v \, dx
\]
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for all $v \in H^1_0(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$.

To prove our main result and to provide some regularity of solutions, we consider the following assumptions.

- $(a_{ij}^h(\cdot, s))$ is measurable in $x$ for every $s \in \mathbb{R}^N$, and of class $C^1$ in $s$ for a.e. $x \in \Omega$ with

$$a_{ij}^h = a_{ji}^h.$$  

Furthermore, we assume that there exist $\nu > 0$ and $C > 0$ such that for a.e. $x \in \Omega$, all $s \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{nN}$

$$\sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x, s) \xi_i^h \xi_j^h \geq \nu |\xi|^2, \quad |a_{ij}^h(x, s)| \leq C, \quad |D_s a_{ij}^h(x, s)| \leq C$$  \hspace{1cm} (2.26)

and

$$\sum_{i,j=1}^{n} \sum_{h=1}^{N} s \cdot D_s a_{ij}^h(x, s) \xi_i^h \xi_j^h \geq 0.$$  \hspace{1cm} (2.27)

- there exists a bounded Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$, such that for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^{nN}$, $\sigma \in \{-1, 1\}^N$ and $r, s \in \mathbb{R}^N$

$$\sum_{i,j=1}^{n} \sum_{h=1}^{N} \left( \frac{1}{2} D_s a_{ij}^h(x, s) \cdot \exp(\sigma(r, s) + s \cdot D_s \exp(\sigma(r, s))_h) \right) \xi_i^h \xi_j^h \leq 0,$$  \hspace{1cm} (2.28)

where $(\exp(\sigma(r, s)))_i := \sigma_i \exp[\sigma_i(\psi(r_i) - \psi(s_i))]$ for each $i = 1, \ldots, N$.

- The function $G(x, s)$ is measurable in $x$ for all $s \in \mathbb{R}^N$ and of class $C^1$ in $s$ for a.e. $x \in \Omega$, with $G(x, 0) = 0$. Moreover for a.e. $x \in \Omega$ we will denote with $g(x, \cdot)$ the gradient of $G$ with respect to $s$.

- for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^{2n/(n+2)}(\Omega)$ such that

$$|g(x, s)| \leq a_\varepsilon(x) + \varepsilon |s|^{(n+2)/(n-2)}$$  \hspace{1cm} (2.29)

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^N$ and that there exist $q > 2$, $R > 0$ such that for all $s \in \mathbb{R}^N$ and for a.e. $x \in \Omega$

$$|s| \geq R \implies 0 < qG(x, s) \leq s \cdot g(x, s).$$  \hspace{1cm} (2.30)

- there exists $\gamma \in (0, q - 2)$ such that for all $\xi \in \mathbb{R}^{nN}$, $s \in \mathbb{R}^N$ and a.e. in $\Omega$

$$\sum_{i,j=1}^{n} \sum_{h=1}^{N} s \cdot D_s a_{ij}^h(x, s) \xi_i^h \xi_j^h \leq \gamma \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x, s) \xi_i^h \xi_j^h.$$  \hspace{1cm} (2.31)

Under these assumptions we will prove the following.
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Theorem 2.2.1. Assume that for a.e. \( x \in \Omega \) and for each \( s \in \mathbb{R}^N \)

\[
a_{ij}^h(x, -s) = a_{ij}^h(x, s), \quad g(x, -s) = -g(x, s).
\]

Then there exists a sequence \((u^m) \subseteq H_0^1(\Omega, \mathbb{R}^N)\) of weak solutions to (2.24) such that

\[
\lim_{m} f(u^m) = +\infty.
\]

The above result is well known for the semilinear scalar problem

\[
- \sum_{i,j=1}^{n} D_j(a_{ij}(x) D_i u) = g(x, u) \quad \text{in} \ \Omega
\]

\[
u = 0 \quad \text{on} \ \partial \Omega.
\]

A. Ambrosetti and P. H. Rabinowitz in [6, 93] studied this problem using techniques of classical critical point theory. The quasilinear scalar problem

\[
- \sum_{i,j=1}^{n} D_j(a_{ij}(x, u) D_i u) + \frac{1}{2} \sum_{i,j=1}^{n} D_s a_{ij}(x, u) D_i u D_j u = g(x, u) \quad \text{in} \ \Omega
\]

\[
u = 0 \quad \text{on} \ \partial \Omega,
\]

was studied in [34, 35, 37] and in [88] in a more general setting. In this case the functional

\[
f(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x, u) D_i u D_j u \, dx - \int_{\Omega} G(x, u) \, dx
\]

is continuous under appropriate conditions, but it is not locally Lipschitz. Consequently, techniques of non–smooth critical point theory have to be applied. In the vectorial case, to my knowledge, problem (2.24) has only been considered in [114, Theorem 3.2] and recently in [11, Theorem 3.2] for coefficients of the type \( a_{ij}^{hk}(x, s) = \delta_{hk} \alpha_{ij}(x, s) \).

2.2.2 The concrete Palais–Smale condition

The first step for the \((CPS)_c\) to hold is the boundedness of \((CPS)_c\) sequences.

Lemma 2.2.2. For all \( c \in \mathbb{R} \) each \((CPS)_c\) sequence of \( f \) is bounded in \( H_0^1(\Omega, \mathbb{R}^N) \).

Proof. Let \( a_0 \in L^1(\Omega) \) be such that for a.e. \( x \in \Omega \) and all \( s \in \mathbb{R}^N \)

\[
qG(x, s) \leq s \cdot g(x, s) + a_0(x).
\]

Now let \((u^m)\) be a \((CPS)_c\) sequence for \( f \) and let \( u^m \to 0 \) in \( H^{-1}(\Omega, \mathbb{R}^N) \) such that for all \( v \in C_0^\infty(\Omega, \mathbb{R}^N) \),

\[
\langle w^m, v \rangle = \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x, u^m) D_i u^m_h D_j v_h \, dx + \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_s a_{ij}^h(x, u^m) \cdot v D_i u^m_h D_j u^m_h \, dx - \int_{\Omega} g(x, u^m) \cdot v.
\]
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Taking into account the previous Lemma, for every \( m \in \mathbb{N} \) we obtain

\[
-\|w^m\|_{H^{-1}(\Omega, \mathbb{R}^N)} \|u^m\|_{H^1_0(\Omega, \mathbb{R}^N)} \leq \quad \leq \int_{\Omega} \frac{n}{2} \sum_{i,j=1}^{N} a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx + \\
+ \int_{\Omega} \sum_{i,j=1}^{N} D_s a_{ij}^h(x, u^m) \cdot u^m D_i u_h^m D_j u_h^m \, dx - \int_{\Omega} g(x, u^m) \cdot u^m \, dx \leq \\
\leq \int_{\Omega} \frac{n}{2} \sum_{i,j=1}^{N} a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx + \\
+ \int_{\Omega} \sum_{i,j=1}^{N} D_s a_{ij}^h(x, u^m) \cdot u^m D_i u_h^m D_j u_h^m \, dx + \\
- q \int_{\Omega} G(x, u^m) \, dx + \int_{\Omega} a_0 \, dx.
\]

Taking into account the expression of \( f \) and assumption (2.31), we have that for each \( m \in \mathbb{N} \),

\[
-\|w^m\|_{H^{-1}(\Omega, \mathbb{R}^N)} \|u^m\|_{H^1_0(\Omega, \mathbb{R}^N)} \leq \quad \leq - \left( \frac{q}{2} - 1 \right) \int_{\Omega} \frac{n}{2} \sum_{i,j=1}^{N} a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx + \\
+ \int_{\Omega} \sum_{i,j=1}^{N} D_s a_{ij}^h(x, u^m) \cdot u^m D_i u_h^m D_j u_h^m \, dx + qf(u^m) + \int_{\Omega} a_0 \, dx \leq \\
\leq - \left( \frac{q}{2} - 1 - \frac{\gamma}{2} \right) \int_{\Omega} \frac{n}{2} \sum_{i,j=1}^{N} a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx + \\
+ qf(u^m) + \int_{\Omega} a_0 \, dx.
\]

Because of (2.26), for each \( m \in \mathbb{N} \),

\[
\nu(q - 2 - \gamma) \|Du^m\|^2 \leq (q - 2 - \gamma) \int_{\Omega} \frac{n}{2} \sum_{i,j=1}^{N} a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx \\
\leq 2 \|w^m\|_{H^{-1}(\Omega, \mathbb{R}^N)} \|u^m\|_{H^1_0(\Omega, \mathbb{R}^N)} + 2qf(u^m) + 2 \int_{\Omega} a_0 \, dx.
\]

Since \( w^m \to 0 \) in \( H^{-1}(\Omega, \mathbb{R}^N) \), we conclude that \( (u^m) \) is a bounded sequence in \( H^1_0(\Omega, \mathbb{R}^N) \).

Lemma 2.2.3. If condition (2.29) holds, then the map

\[
H^1_0(\Omega, \mathbb{R}^N) \rightarrow L^{2n/(n+2)}(\Omega, \mathbb{R}^N) \quad u \mapsto g(x, u)
\]

is completely continuous.
Proof. This is a direct consequence of [37, Theorem 2.2.7]. □

The next result is crucial for the \((CPS)_c\) condition to hold for our elliptic system.

**Lemma 2.2.4.** Let \((u^m)\) be a bounded sequence in \(H^1_0(\Omega, \mathbb{R}^N)\), and set

\[
\langle w^m, v \rangle = \int_{\Omega} \sum_{i=1}^{n} \sum_{j=1}^{N} a^h_{ij}(x, u^m) D_i u^m_h D_j v_h \, dx + \\
+ \frac{1}{2} \int_{\Omega} \sum_{i=1}^{n} \sum_{j=1}^{N} D_s a^h_{ij}(x, u^m) \cdot v D_i u^m_h D_j u^m_h \, dx
\]

for all \(v \in C^\infty_c(\Omega, \mathbb{R}^N)\). If \((w^m)\) is strongly convergent to some \(w\) in \(H^{-1}(\Omega, \mathbb{R}^N)\), then \((u^m)\) admits a strongly convergent subsequence in \(H^1_0(\Omega, \mathbb{R}^N)\).

Proof. Since \((u^m)\) is bounded, we have \(u^m \rightharpoonup u\) for some \(u\) up to a subsequence. Each component \(u^m_k\) satisfies (2.5) in [23], so we may suppose that \(D_s u^m_k \rightharpoonup D_s u_k\) a.e. in \(\Omega\) for all \(k = 1, \ldots, N\) (see also [51]). We first prove that

\[
\int_{\Omega} \sum_{i=1}^{n} \sum_{j=1}^{N} a^h_{ij}(x, u) D_i u_h D_j u_h \, dx + \\
+ \frac{1}{2} \int_{\Omega} \sum_{i=1}^{n} \sum_{j=1}^{N} D_s a^h_{ij}(x, u) \cdot u D_i u_h D_j u_h \, dx = \langle w, u \rangle. \tag{2.32}
\]

Let \(\psi\) be as in assumption (2.28) and consider the following test functions

\[
v^m = \varphi(\sigma_1 \exp[\sigma_1 (u_1) - \psi(u^m_1)]), \ldots, \sigma_N \exp[\sigma_N (u_N) - \psi(u^m_N)]),
\]

where \(\varphi \in C^\infty_c(\Omega), \varphi \geq 0\) and \(\sigma_l = \pm 1\) for all \(l\). Therefore, since we have

\[
D_j v^m_k = (\sigma_k D_j \varphi + (\psi'(u_k) D_j u_k - \psi'(u^m_k) D_j u^m_k) \varphi) \exp[\sigma_k (u_k) - \psi(u^m_k)]),
\]

we deduce that for all \(m \in \mathbb{N}\),

\[
\int_{\Omega} \sum_{i=1}^{n} \sum_{j=1}^{N} a^h_{ij}(x, u^m) D_i u^m_h (\sigma_h D_j \varphi + \psi'(u_h) D_j u_h \varphi) \exp[\sigma_h (u_h) - \psi(u^m_h)] \, dx + \\
+ \frac{1}{2} \int_{\Omega} \sum_{i=1}^{n} \sum_{j=1}^{N} \sigma_l D_s a^h_{ij}(x, u^m) \exp[\sigma_l (u_l) - \psi(u^m_l)] \, D_i u^m_h D_j u^m_h \varphi \, dx + \\
- \int_{\Omega} \sum_{i=1}^{n} \sum_{j=1}^{N} a^h_{ij}(x, u^m) D_i u^m_h D_j u^m_h \psi'(u^m_h) \exp[\sigma_h (u_h) - \psi(u^m_h)] \varphi \, dx = \\
= \langle w^m, v^m \rangle.
\]
Let us study the behavior of each term of the previous equality as \( m \to \infty \). First of all, if \( v = (\sigma_1 \varphi, \ldots, \varphi_N) \), we have that \( v^m \to v \) implies

\[
\lim_{m} \langle w^m, v^m \rangle = \langle w, v \rangle.
\] (2.33)

Since \( u^m \to u \), by Lebesgue’s Theorem we obtain

\[
\lim_{m} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x, u^m) D_i u_h^m (D_j (\sigma_h \varphi)) + \\
\varphi \psi'(u_h) D_j u_h \exp[\sigma_h(\psi(u_h) - \psi(u^m_h))] \, dx =
\]

\[
\int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x, u) D_i u_h (D_j (\sigma_h \varphi)) + \varphi \psi'(u_h) D_j u_h \, dx.
\] (2.34)

Finally, note that by assumption (2.28) we have

\[
\sum_{i,j=1}^{n} \sum_{h=1}^{N} \left( \sum_{l=1}^{N} \frac{\sigma_l}{2} D_s a_{ij}^h(x, u^m) \exp[\sigma_l(\psi(u_l) - \psi(u^m_l))] + \\
-a_{ij}^h(x, u^m) \psi'(u^m_h) \exp[\sigma_h(\psi(u_h) - \psi(u^m_h))] \right) D_i u_h^m D_j u_h^m \leq 0.
\]

Hence, we can apply Fatou’s Lemma to obtain

\[
\limsup_{m} \left\{ \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_s a_{ij}^h(x, u^m) \exp[\sigma_l(\psi(u_l) - \psi(u^m_l))] D_i u_h^m D_j u_h^m (\sigma_l \varphi) \, dx + \\
- \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \psi'(u^m_h) \exp[\sigma_h(\psi(u_h) - \psi(u^m_h))] \varphi \, dx \right\} \leq
\]

\[
\leq \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_s a_{ij}^h(x, u) D_i u_h D_j u_h (\sigma_l \varphi) \, dx + \\
- \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x, u) D_i u_h D_j u_h \psi'(u_h) \varphi \, dx,
\]

which, together with (2.33) and (2.34), yields

\[
\int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x, u) D_i u_h D_j v_h \, dx + \\
+ \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_s a_{ij}^h(x, u) \cdot v D_i u_h D_j u_h \, dx \geq \langle w, v \rangle
\]
for all test functions $v = (\sigma_1\varphi, \ldots, \sigma_N\varphi)$ with $\varphi \in C_c^\infty(\Omega, \mathbb{R}^N)$, $\varphi \geq 0$. Since we may exchange $v$ with $-v$ we get

$$
\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a^h_{ij}(x, u) D_i u_h D_j v_h \, dx + \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a^h_{ij}(x, u) \cdot v D_i u_h D_j u_h \, dx = \langle w, v \rangle
$$

for all test functions $v = (\sigma_1\varphi, \ldots, \sigma_N\varphi)$, and since every function $v \in C_c^\infty(\Omega, \mathbb{R}^N)$ can be written as a linear combination of such functions, taking into account Proposition 2.1.2, we infer (2.32). Now, let us prove that

$$
\limsup_m \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a^h_{ij}(x, u^m) D_i u^m_h D_j u^m_h \, dx \leq \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a^h_{ij}(x, u) D_i u_h D_j u_h \, dx. \tag{2.35}
$$

Because of (2.27), Fatou’s Lemma implies that

$$
\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N u \cdot D_s a^h_{ij}(x, u) D_i u_h D_j u_h \, dx \leq \liminf_m \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N u^m \cdot D_s a^h_{ij}(x, u^m) D_i u^m_h D_j u^m_h \, dx.
$$

Combining this fact with (2.32), we deduce that

$$
\limsup_m \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a^h_{ij}(x, u^m) D_i u^m_h D_j u^m_h \, dx = \limsup_m \left[ -\frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N u^m \cdot D_s a^h_{ij}(x, u^m) D_i u^m_h D_j u^m_h \, dx + \langle u^m, u^m \rangle \right] \leq \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N u \cdot D_s a^h_{ij}(x, u) D_i u_h D_j u_h \, dx + \langle w, u \rangle = \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a^h_{ij}(x, u) D_i u_h D_j u_h \, dx,
$$

so that (2.35) is proved. Finally, by (2.26) we have

$$
\nu \| Du^m - Du \|_2^2 \leq \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a^h_{ij}(x, u^m) (D_i u^m_h D_j u^m_h - 2D_i u^m_h D_j u_h + D_i u_h D_j u_h) \, dx.
$$
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Hence, by (2.35) we obtain
\[ \limsup_{m} \|Du^m - Du\|_2 \leq 0 \]
which proves that \( u^m \to u \) in \( H^1_0(\Omega, \mathbb{R}^N) \).

We now come to the \((CPS)_{c}\) condition for system (2.24).

**Theorem 2.2.5.** \( f \) satisfies \((CPS)_{c}\) condition for each \( c \in \mathbb{R} \).

**Proof.** Let \((u^m)\) be a \((CPS)_{c}\) sequence for \( f \). Since \((u^m)\) is bounded in \( H^1_0(\Omega, \mathbb{R}^N) \), from Lemma 2.2.3 we deduce that, up to a subsequence, \((g(x,u^m))\) is strongly convergent in \( H^1(\Omega; \mathbb{R}^N) \). Applying Lemma 2.2.4, we conclude the present proof. \( \square \)

### 2.2.3 Existence of multiple solutions for elliptic systems

We now prove the main result, which is an extension of theorems of [35, 37] and a generalization of [11, Theorem 3.2] to systems in diagonal form.

**Proof of Theorem 2.2.1.** We want to apply [37, Theorem 2.1.6]. First of all, because of Theorem 2.2.5, \( f \) satisfies \((CPS)_{c}\) for all \( c \in \mathbb{R} \). Whence, \((c)\) of [37, Theorem 2.1.6] is satisfied. Moreover we have
\[
\frac{\nu}{2} \int_{\Omega} |Du|^2 dx - \int_{\Omega} G(x,u) dx \leq \frac{1}{2} nNC \int_{\Omega} |Du|^2 dx - \int_{\Omega} G(x,u) dx.
\]
We want to prove that assumptions (a) and (b) of [37, Theorem 2.1.6] are also satisfied. Let us observe that, instead of (b) of [37, Theorem 2.1.6], it is enough to find a sequence \((W_n)\) of finite dimensional subspaces with \( \dim(W_n) \to +\infty \) satisfying the inequality of (b) (see also [83, Theorem 1.2]). Let \( W \) be a finite dimensional subspace of \( H^1_0(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N) \). From (2.30) we deduce that for all \( s \in \mathbb{R}^N \) with \( |s| \geq R \)
\[
G(x,s) \geq \frac{G\left(x,R \frac{s}{|s|}\right)}{R^q} |s|^q \geq b_0(x)|s|^q,
\]
where \( b_0(x) = R^{-q} \inf \{G(x,s) : |s| = R\} > 0 \) a.e. \( x \in \Omega \). Therefore there exists \( a_0 \in L^1(\Omega) \) such that
\[
G(x,s) \geq b_0(x)|s|^q - a_0(x)
\]
a.e. \( x \in \Omega \) and for all \( s \in \mathbb{R}^N \). Since \( b_0 \in L^1(\Omega) \), we may define a norm \( \| \cdot \|_G \) on \( W \) by
\[
\|u\|_G = \left( \int_{\Omega} b_0|u|^q dx \right)^{1/q}.
\]
Since $W$ is finite dimensional and $q > 2$, from (2.36) it follows
\[ \lim_{\|u\|_{Q} \to +\infty} f(u) = -\infty \]
and condition (b) of [37, Theorem 2.1.6] is clearly fulfilled too for a sufficiently large $R > 0$. Let now $(\lambda_h, u_h)$ be the sequence of eigenvalues and eigenvectors for the problem
\[ \Delta u = -\lambda u \quad \text{in } \Omega \\
 u = 0 \quad \text{on } \partial \Omega. \]
Let us prove that there exist $h_0, \alpha > 0$ such that
\[ \forall u \in V^+ : \|Du\|_2 = 1 \implies f(u) \geq \alpha, \]
where $V^+ = \text{span} \{ u_h \in H^1_0(\Omega, \mathbb{R}^N) : h \geq h_0 \}$. In fact, given $u \in V^+$ and $\varepsilon > 0$, we find
\[ a^{(1)}_\varepsilon \in C^\infty_c(\Omega), \quad a^{(2)}_\varepsilon \in L^{2n/(n+2)}(\Omega), \]
such that $\|a^{(2)}_\varepsilon\|_{2n/(n+2)} \leq \varepsilon$ and
\[ |g(x, s)| \leq a^{(1)}_\varepsilon(x) + a^{(2)}_\varepsilon(x) + \varepsilon|s|^{(n+2)/(n-2)}. \]
If $u \in V^+$, it follows that
\[
\begin{align*}
f(u) & \geq \frac{\nu}{2} \|Du\|_2^2 - \int_{\Omega} G(x, u) \, dx \\
& \geq \frac{\nu}{2} \|Du\|_2^2 - \int_{\Omega} \left( (a^{(1)}_\varepsilon + a^{(2)}_\varepsilon) |u| + \frac{n-2}{2m} \varepsilon |u|^{2n/(n-2)} \right) \, dx \\
& \geq \frac{\nu}{2} \|Du\|_2^2 - \|a^{(1)}_\varepsilon\|_2 \|u\|_2 - c_1 \|a^{(2)}_\varepsilon\|_{2n/(n+2)} \|Du\|_2 - \varepsilon c_2 \|Du\|_2^{2n/(n-2)} \\
& \geq \frac{\nu}{2} \|Du\|_2^2 - \|a^{(1)}_\varepsilon\|_2 \|u\|_2 - c_1 \varepsilon \|Du\|_2 - \varepsilon c_2 \|Du\|_2^{2n/(n-2)}. \end{align*}
\]
Then if $h_0$ is sufficiently large, from the fact that $(\lambda_h)$ diverges, for all $u \in V^+$, $\|Du\|_2 = 1$ implies
\[ \|a^{(1)}_\varepsilon\|_2 \|u\|_2 \leq \frac{\nu}{6}. \]
Hence, for $\varepsilon > 0$ small enough, $\|Du\|_2 = 1$ implies that $f(u) \geq \nu/6$.

Finally, set $V^- = \text{span} \{ u_h \in H^1_0(\Omega, \mathbb{R}^N) : h < h_0 \}$, we have the decomposition
\[ H^1_0(\Omega; \mathbb{R}^N) = V^+ \oplus V^- \]
Therefore, since the hypotheses for [37, Theorem 2.1.6] are fulfilled, we can find a sequence $(u^m)$ of weak solution of system (2.24) such that
\[ \lim_m f(u^m) = +\infty, \]
and the theorem is now proven.
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2.2.4 Regularity of weak solutions for elliptic systems

Consider the nonlinear elliptic system

\[ \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x,u) D_i u_h D_j v_k \, dx = \int_{\Omega} b(x,u,Du) \cdot v \, dx \]  \hspace{1cm} (2.37)

for all \( v \in H_0^1(\Omega; \mathbb{R}^N) \). For \( l = 1, \ldots, N \), we choose

\[ b_l(x,u,Du) = \left\{ - \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} D_{si} a_{ij}^{hk}(x,u) D_i u_h D_j u_k + g_l(x,u) \right\}. \]

Assume that there exist \( c > 0 \) and \( q < \frac{n+2}{n-2} \) such that for all \( s \in \mathbb{R}^N \) and a.e. in \( \Omega \)

\[ |g(x,s)| \leq c (1 + |s|^q). \]  \hspace{1cm} (2.38)

Then it follows that for every \( M > 0 \), there exists \( C(M) > 0 \) such that for a.e. \( x \in \Omega \), for all \( \xi \in \mathbb{R}^{nN} \) and \( s \in \mathbb{R}^N \) with \( |s| \leq M \)

\[ |b(x,s,\xi)| \leq c(M) \left( 1 + |\xi|^2 \right). \]  \hspace{1cm} (2.39)

A nontrivial regularity theory for quasilinear systems (see, [62, Chapter VI]) yields the following:

**Theorem 2.2.6.** For every weak solution \( u \in H^1(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N) \) of the system (2.24) there exist an open subset \( \Omega_0 \subseteq \Omega \) and \( s > 0 \) such that

\[ \forall p \in (n, +\infty) : u \in C^{0,1-\frac{n}{p}}(\Omega_0; \mathbb{R}^N), \]

\[ \mathcal{H}^{n-s}(\Omega \setminus \Omega_0) = 0. \]

**Proof.** For the proof, see [62, Chapter VI]. \( \square \)

We now consider the particular case when \( a_{ij}^{hk}(x,s) = \alpha_{ij}(x,s) \delta_{hk} \), and provide an almost everywhere regularity result.

**Lemma 2.2.7.** Assume that (2.39) holds. Then the weak solutions \( u \in H_0^1(\Omega, \mathbb{R}^N) \) of the system

\[ \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}(x,u) D_i u_h D_j v_h \, dx + \]

\[ + \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_s a_{ij}(x,u) \cdot v D_i u_h D_j u_h \, dx = \int_{\Omega} g(x,u) \cdot v \, dx \]  \hspace{1cm} (2.40)

for all \( v \in C_c^\infty(\Omega, \mathbb{R}^N) \), belong to \( L^\infty(\Omega, \mathbb{R}^N) \).
Proof. By [114, Lemma 3.3], for each \((CPS)_c\) sequence \((u^m)\) there exist \(u \in H^1_0 \cap L^\infty\) and a subsequence \((u^{m_k})\) with \(u^{m_k} \rightharpoonup u\). Then, given a weak solution \(u\), consider the sequence \((u^m)\) such that each element is equal to \(u\) and the assertion follows.

We can finally state a partial regularity result for our system.

**Theorem 2.2.8.** Assume condition (2.39) and let \(u^2 \in H^1_0(\Omega, \mathbb{R}^N)\) be a weak solution of the system

\[
\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(x,u)D_i u_h D_j v_h \, dx + \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{N} D_s a_{ij}(x,u) \cdot v D_i u_h D_j u_h \, dx = \int_{\Omega} g(x,u) \cdot v \, dx \tag{2.41}
\]

for all \(v \in C^\infty_c(\Omega, \mathbb{R}^N)\). Then there exist an open subset \(\Omega_0 \subseteq \Omega\) and \(s > 0\) such that

\[
\forall p \in (n, +\infty) : u \in C^{0,1-\frac{n}{p}} (\Omega_0; \mathbb{R}^N),
\]

\[
\mathcal{H}^{n-s}(\Omega \setminus \Omega_0) = 0.
\]

**Proof.** It suffices to combine the previous Lemma with Theorem 2.2.6.

### 2.3 Fully nonlinear scalar problems

#### 2.3.1 Introduction

Recently, some results for the quite general problem

\[
\begin{cases}
- \text{div} (\nabla \mathcal{L}(x,u,\nabla u)) + D_s \mathcal{L}(x,u,\nabla u) = g(x,u) & \text{in} \ \Omega \\
u = 0 & \text{on} \ \partial \Omega,
\end{cases}
\tag{2.42}
\]

have been considered in [7, 8] and [88].

The goal of this section is to extend some of the results of [7, 88]. In order to solve (2.42), we shall look for critical points of functionals \(f : W^{1,p}_0(\Omega) \to \mathbb{R}\) given by

\[
f(u) = \int_{\Omega} \mathcal{L}(x,u,\nabla u) \, dx - \int_{\Omega} G(x,u) \, dx.
\tag{2.43}
\]

In general, \(f\) is continuous but not even locally Lipschitzian unless \(\mathcal{L}\) does not depend on \(u\) or \(\mathcal{L}\) is subjected to some very restrictive growth conditions. Then, again we shall refer to non-smooth critical point theory.

We assume that \(\mathcal{L} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}\) is measurable in \(x\) for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\), of class \(C^1\) in \((s, \xi)\) for a.e. \(x \in \Omega\), the function \(\mathcal{L}(x,s,\cdot)\) is strictly convex and \(\mathcal{L}(x,s,0) = 0\) for a.e. \(x \in \Omega\). Furthermore, we shall assume that:

- there exist \(a \in L^1(\Omega)\) and \(b_0, \nu > 0\) such that

\[
\nu |\xi|^p \leqslant \mathcal{L}(x,s,\xi) \leqslant a(x) + b_0 |s|^p + b_0 |\xi|^p,
\tag{2.44}
\]
for a.e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \);

- for each \( \varepsilon > 0 \) there exists \( a_\varepsilon \in L^1(\Omega) \) such that
  \[
  |D_s \mathcal{L}(x, s, \xi)| \leq a_\varepsilon(x) + \varepsilon |s|^p + b_1 |\xi|^p ,
  \]
  (2.45)

for a.e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \), with \( b_1 \in \mathbb{R} \) independent of \( \varepsilon \).

Furthermore, there exists \( a_1 \in L^{p'}(\Omega) \) such that
\[
|\nabla \xi \mathcal{L}(x, s, \xi)| \leq a_1(x) + b_1 |s|^\frac{p}{p'} + b_1 |\xi|^{p-1},
\]
(2.46)

for a.e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \);

- there exists \( R > 0 \) such that
  \[
  |s| \geq R \implies D_s \mathcal{L}(x, s, \xi)s \geq 0 ,
  \]
  (2.47)

for a.e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \);

- \( G : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function such that
  \[
  G(x, s) \leq d(x)|s|^p + b|s|^{p'}
  \]
  (2.48)

\[
\lim_{s \to 0} \frac{G(x, s)}{|s|^p} = 0
\]
(2.49)

for a.e. \( x \in \Omega \) and all \( s \in \mathbb{R} \), where \( d \in L^\frac{n}{n-2}(\Omega) \) and \( b \in \mathbb{R} \). Moreover,
\[
G(x, s) = \int_0^s g(x, \tau) \, d\tau
\]

and there exist \( c_1, c_2 > 0 \) such that
\[
|g(x, s)| \leq c_1 + c_2 |s|^\sigma
\]
for a.e. \( x \in \Omega \) and each \( s \in \mathbb{R} \), where \( \sigma < p^* - 1 \).

- there exist \( q > p \) and \( R' > 0 \) such that for each \( \varepsilon > 0 \) there is \( a_\varepsilon \in L^1(\Omega) \) with
  \[
  |s| \geq R' \implies 0 < qG(x, s) \leq g(x, s)s ,
  \]
  (2.50)

\[
|s| \geq R' \implies q\mathcal{L}(x, s, \xi) - \nabla \xi \mathcal{L}(x, s, \xi) \cdot \xi - D_s \mathcal{L}(x, s, \xi)s \geq q|\xi|^p - a_\varepsilon(x) - \varepsilon |s|^p
\]
(2.51)

for a.e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \).

Under the previous assumptions, the following is our main result.

**Theorem 2.3.1.** The boundary value problem
\[
\begin{aligned}
- \text{div} \left( \nabla \xi \mathcal{L}(x, u, \nabla u) \right) + D_s \mathcal{L}(x, u, \nabla u) &= g(x, u) \quad \text{in } \Omega \\
& \quad \text{in } \Omega \\
u &= 0 \\
& \quad \text{on } \partial \Omega
\end{aligned}
\]

has at least one nontrivial weak solution \( u \in W^{1,p}_0(\Omega) \).
This result is an extension of [7, Theorem 3.3], since instead of assuming that
\[ q \mathcal{L}(x, s, \xi) - \nabla_\xi \mathcal{L}(x, s, \xi) \cdot D_s \mathcal{L}(x, s, \xi)s \geq \nu |\xi|^p, \]
for a.e. \( x \in \Omega \) and for all \( \xi \in \mathbb{R}^n \), we only request condition (2.51). In this way the proof of Lemma (2.3.4) becomes more difficult. The keypoint, to deal with the more general assumption, is constituted by Lemma (2.3.3).

Similarly, in [88, Theorem 1], a multiplicity result for (2.42) is proved, assuming that
\[ q \mathcal{L}(x, s, \xi) - \nabla_\xi \mathcal{L}(x, s, \xi) \cdot D_s \mathcal{L}(x, s, \xi)s \geq \nu |\xi|^p, \]
for a.e. \( x \in \Omega \) and for all \( \xi \in \mathbb{R}^n \), which are both stronger than (2.47) and (2.51). In particular, the first inequality above and the more general condition (2.47) are involved in Theorem (2.1.3).

Finally, let us point out that the growth conditions (2.44) – (2.46) are a relaxation of those of [7, 88], where it is assumed that
\[
|\nabla_\xi \mathcal{L}(x, s, \xi)| \leq a_1(x) + b_1 |s|^{p-1} + b_1 |\xi|^{p-1},
\]
for a.e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \).

2.3.2 The concrete Palais–Smale condition

Let us point out that as a consequence of assumption (2.44) and convexity of \( \mathcal{L}(x, s, \cdot) \), we can find \( M > 0 \) such that for each \( \varepsilon > 0 \) there is \( a_\varepsilon \in L^1(\Omega) \) with
\[
\nabla_\xi \mathcal{L}(x, s, \xi) \cdot \xi \geq \nu |\xi|^p - a(x) - b_0 |s|^p,
\]
\[
|D_s \mathcal{L}(x, s, \xi)| \leq M |\nabla_\xi \mathcal{L}(x, s, \xi) \cdot \xi + a_\varepsilon(x) + \varepsilon |s|^p|\mu,
\]
for a.e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \).

We now come to a local compactness property, which is crucial for the \((CPS)_e\) condition to hold. This result improves [88, Lemma 2], since (2.51) relaxes condition (8) in [88].

**Theorem 2.3.2.** Let \((u_h)\) be a bounded sequence in \(W^{1,p}_0(\Omega)\) and set
\[
\langle w_h, v \rangle = \int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla v \, dx + \int_\Omega D_s \mathcal{L}(x, u_h, \nabla u_h) v \, dx,
\]
for all \( v \in C_0^\infty(\Omega) \). If \((w_h)\) is strongly convergent to some \( w \) in \(W^{-1,p'}(\Omega)\), then \((u_h)\) admits a strongly convergent subsequence in \(W^{1,p}_0(\Omega)\).
Proof. Since \((u_h)\) is bounded in \(W^{1,p}_0(\Omega)\), we find a \(u\) in \(W^{1,p}_0(\Omega)\) such that, up to a subsequence,

\[
\nabla u_h \rightharpoonup \nabla u \text{ in } L^p(\Omega), \quad u_h \rightharpoonup u \text{ in } L^p(\Omega), \quad u_h(x) \rightharpoonup u(x) \text{ for a.e. } x \in \Omega.
\]

By [23, Theorem 2.1], up to a subsequence, we have

\[
\nabla u_h(x) \rightharpoonup \nabla u(x) \text{ for a.e. } x \in \Omega. \tag{2.57}
\]

Therefore, by (2.46) we deduce that

\[
\nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \rightharpoonup \nabla \xi \mathcal{L}(x, u, \nabla u) \text{ in } L^p(\Omega, \mathbb{R}^n).
\]

We now want to prove that \(u\) solves the equation

\[
\forall v \in C^\infty_c(\Omega) : \langle w, v \rangle = \int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla v \, dx + \int_\Omega D_s \mathcal{L}(x, u, \nabla u) v \, dx. \tag{2.58}
\]

To this aim, let us test equation (2.56) with the functions

\[
v_h = \varphi \exp\{-M(u_h + R)^+\}, \quad \varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega), \quad \varphi \geq 0.
\]

It results for each \(h \in \mathbb{N}\)

\[
\int_\Omega \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \varphi \exp\{-M(u_h + R)^+\} \, dx - \langle w_h, \varphi \exp\{-M(u_h + R)^+\} \rangle +
\]

\[
+ \int_\Omega \left[ D_s \mathcal{L}(x, u_h, \nabla u_h) - M \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla (u_h + R)^+ \right] \varphi \exp\{-M(u_h + R)^+\} \, dx = 0.
\]

Of course, for a.e. \(x \in \Omega\), we obtain

\[
\varphi \exp\{-M(u_h + R)^+\} \rightharpoonup \varphi \exp\{-M(u + R)^+\}.
\]

Since by inequality (2.55) and (2.47) for each \(\varepsilon > 0\) and \(h \in \mathbb{N}\) we have

\[
[D_s \mathcal{L}(x, u_h, \nabla u_h) - M \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla (u_h + R)^+] \varphi \exp\{-M(u_h + R)^+\} +
\]

\[
-\varepsilon |u_h|^p \varphi \leq a_\varepsilon(x) \varphi,
\]

Fatou's Lemma implies that for each \(\varepsilon > 0\)

\[
\limsup_h \int_\Omega [D_s \mathcal{L}(x, u_h, \nabla u_h) - M \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla (u_h + R)^+] \varphi \exp\{-M(u_h + R)^+\} +
\]

\[
-\varepsilon |u_h|^p \varphi \, dx \leq \int_\Omega [D_s \mathcal{L}(x, u, \nabla u) - M \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla (u + R)^+] \varphi \exp\{-M(u + R)^+\} +
\]

\[
-\varepsilon |u|^p \varphi \, dx.
\]
Since \( (u_h) \) is bounded in \( L^p(\Omega) \), we find \( c > 0 \) such that for each \( \varepsilon > 0 \)
\[
\limsup_h \int_\Omega \left[ D_s \mathcal{L}(x, u_h, \nabla u_h) - M \nabla_x \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla (u_h + R)^+ \right] \varphi \exp\{-M(u_h + R)^+\} \, dx \\
\leq \int_\Omega \left[ D_s \mathcal{L}(x, u, \nabla u) - M \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla (u + R)^+ \right] \varphi \exp\{-M(u + R)^+\} \, dx - c\varepsilon .
\]

Letting \( \varepsilon \to 0 \), the previous inequality yields
\[
\limsup_h \int_\Omega \left[ D_s \mathcal{L}(x, u_h, \nabla u_h) - M \nabla_x \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla (u_h + R)^+ \right] \varphi \exp\{-M(u_h + R)^+\} \, dx \\
\leq \int_\Omega \left[ D_s \mathcal{L}(x, u, \nabla u) - M \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla (u + R)^+ \right] \varphi \exp\{-M(u + R)^+\} \, dx .
\]

Note that we also have
\[
\varphi \exp\{-M(u_h + R)^+\} \to \varphi \exp\{-M(u + R)^+\} \quad \text{in } W_0^{1,p}(\Omega) .
\]

Moreover
\[
\nabla \varphi \exp\{-M(u_h + R)^+\} \to \nabla \varphi \exp\{-M(u + R)^+\} \quad \text{in } L^p(\Omega, \mathbb{R}^n) ,
\]
so that
\[
\int_\Omega \nabla_x \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \varphi \exp\{-M(u_h + R)^+\} \, dx \\
\to \int_\Omega \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi \exp\{-M(u + R)^+\} \, dx .
\]

Therefore, we may conclude that
\[
\int_\Omega \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi \exp\{-M(u + R)^+\} \, dx - \langle w, \varphi \exp\{-M(u + R)^+\} \rangle +
\]
\[
+ \int_\Omega \left[ D_s \mathcal{L}(x, u, \nabla u) - M \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla (u + R)^+ \right] \varphi \exp\{-M(u + R)^+\} \, dx \geq 0 .
\]

Consider now the test functions
\[
\varphi_k := \varphi H \left( \frac{u}{k} \right) \exp\{M(u + R)^+\} , \quad \varphi \in C_0^\infty(\Omega) , \quad \varphi \geq 0 ,
\]
where \( H \in C^1(\mathbb{R}) \), \( H = 1 \) in \( [-\frac{1}{2}, \frac{1}{2}] \) and \( H = 0 \) in \( ]-\infty, -1] \cup [1, +\infty[ \). It follows that
\[
\int_\Omega \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi_k \exp\{-M(u + R)^+\} \, dx - \langle w, \varphi H \left( \frac{u}{k} \right) \rangle +
\]
\[
+ \int_\Omega \left[ D_s \mathcal{L}(x, u, \nabla u) - M \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla (u + R)^+ \right] \varphi H \left( \frac{u}{k} \right) \, dx \geq 0 .
\]

Furthermore, standard computations yield
\[
\nabla \varphi_k = \exp\{M(u + R)^+\} \left[ \nabla \varphi H \left( \frac{u}{k} \right) + H' \left( \frac{u}{k} \right) \frac{\varphi}{k} \nabla u + M \nabla (u + R)^+ \varphi H \left( \frac{u}{k} \right) \right] .
\]
Since \( \varphi H \left( \frac{u}{k} \right) \) goes to \( \varphi \) in \( W^{1,p}_0(\Omega) \), as \( k \to +\infty \) we have

\[
\left\langle w, \varphi H \left( \frac{u}{k} \right) \right\rangle \to \langle w, \varphi \rangle.
\]

By the properties of \( H \) and the growth conditions on \( \nabla \xi \mathcal{L} \), letting \( k \to +\infty \) yields

\[
\int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi H \left( \frac{u}{k} \right) \to \int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi \ dx,
\]

\[
\int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi H' \left( \frac{u}{k} \right) \frac{\varphi}{k} \ dx \to 0,
\]

\[
\int_\Omega M \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla (u + R)^+ \varphi H \left( \frac{u}{k} \right) \to \int_\Omega M \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla (u + R)^+ \varphi.
\]

Whence, we conclude that for all \( \varphi \in C^\infty_c(\Omega) \)

\[
\varphi \geq 0 \implies \langle w, \varphi \rangle \leq \int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi \ dx + \int_\Omega D_s \mathcal{L}(x, u, \nabla u) \varphi \ dx.
\]

Choosing now as test functions

\[
v_h := \varphi \exp\{-M(u_h - R)^-\},
\]

where as before \( \varphi \geq 0 \), we obtain the opposite inequality so that (2.16) is proven.

In particular, taking into account Proposition 2.1.2, we immediately obtain

\[
\langle w, u \rangle = \int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \ dx + \int_\Omega D_s \mathcal{L}(x, u, \nabla u) u \ dx. \tag{2.59}
\]

The final step is to show that \( (u_h) \) goes to \( u \) in \( W^{1,p}_0(\Omega) \). Consider the function \( \zeta : \mathbb{R} \to \mathbb{R} \) defined by

\[
\zeta(s) = \begin{cases} 
Ms & \text{if } 0 < s < R \\
MR & \text{if } s \geq R \\
-Ms & \text{if } -R < s < 0 \\
MR & \text{if } s \leq -R,
\end{cases} \tag{2.60}
\]

and let us prove that

\[
\limsup_{h} \int_\Omega \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \exp\{\zeta(u_h)\} \ dx \leq \int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \exp\{\zeta(u)\} \ dx. \tag{2.61}
\]

Since by Proposition 2.1.2, \( u_h \exp\{\zeta(u_h)\} \) are admissible test functions for (2.56), we have

\[
\int_\Omega \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \exp\{\zeta(u_h)\} \ dx - \langle w_h, u_h \exp\{\zeta(u_h)\}\rangle +
\]
+ \int_\Omega \left[ D_s \mathcal{L}(x, u_h, \nabla u_h) + \zeta'(u_h) \nabla \zeta \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \right] u_h \exp\{\zeta(u_h)\} \, dx = 0.

Let us observe that (2.57) implies that
\[
\nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \rightarrow \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \quad \text{for a.e. } x \in \Omega.
\]
Since by inequality (2.55) for each \( \varepsilon > 0 \) and \( h \in \mathbb{N} \) we have
\[
\left[ -D_s \mathcal{L}(x, u_h, \nabla u_h) - \zeta'(u_h) \nabla \zeta \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \right] u_h \exp\{\zeta(u_h)\} + R \exp\{MR\} \varepsilon |u_h|^p \leq R \exp\{MR\} a_\varepsilon(x),
\]
Fatou’s Lemma yields
\[
\limsup_h \int_\Omega \left[ -D_s \mathcal{L}(x, u_h, \nabla u_h) - \zeta'(u_h) \nabla \zeta \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \right] u_h \exp\{\zeta(u_h)\} + \varepsilon R \exp\{MR\} |u|^p \, dx \leq \int_\Omega \left[ -D_s \mathcal{L}(x, u, \nabla u) - \zeta'(u) \nabla \zeta \mathcal{L}(x, u, \nabla u) \cdot \nabla u \right] u \exp\{\zeta(u)\} + \varepsilon R \exp\{MR\} |u|^p \, dx.
\]
Therefore, since \( (u_h) \) is bounded in \( L^p(\Omega) \), we find \( c > 0 \) such that for all \( \varepsilon > 0 \)
\[
\limsup_h \int_\Omega \left[ -D_s \mathcal{L}(x, u_h, \nabla u_h) - \zeta'(u_h) \nabla \zeta \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \right] u_h \exp\{\zeta(u_h)\} \, dx \leq \int_\Omega \left[ -D_s \mathcal{L}(x, u, \nabla u) - \zeta'(u) \nabla \zeta \mathcal{L}(x, u, \nabla u) \cdot \nabla u \right] u \exp\{\zeta(u)\} \, dx - c \varepsilon.
\]
Taking into account that \( \varepsilon \) is arbitrary, we conclude that
\[
\limsup_h \int_\Omega \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \exp\{\zeta(u_h)\} \, dx = \limsup_h \left\{ \int_\Omega \left[ -D_s \mathcal{L}(x, u_h, \nabla u_h) - \zeta'(u_h) \nabla \zeta \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \right] u_h \exp\{\zeta(u_h)\} \, dx + \langle w_h, u_h \exp\{\zeta(u_h)\} \rangle \right\} \leq \int_\Omega \left[ -D_s \mathcal{L}(x, u, \nabla u) - \zeta'(u) \nabla \zeta \mathcal{L}(x, u, \nabla u) \cdot \nabla u \right] u \exp\{\zeta(u)\} \, dx + \langle w, u \exp\{\zeta(u)\} \rangle = \int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \exp\{\zeta(u)\} \, dx.
\]
In particular, we have
\[
\int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \exp\{\zeta(u)\} \, dx \leq \liminf_h \int_\Omega \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \exp\{\zeta(u_h)\} \, dx \leq \limsup_h \int_\Omega \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \exp\{\zeta(u_h)\} \, dx \leq \int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \exp\{\zeta(u)\} \, dx,
\]
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namely

\[
\lim_h \int_{\Omega} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \exp\{\zeta(u_h)\} \, dx = \int_{\Omega} \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \exp\{\zeta(u)\} \, dx.
\]

Therefore, by (2.54), generalized Lebesgue’s theorem yields

\[
\lim \sup_h \int_{\Omega} |\nabla u_h|^p \, dx \leq \int_{\Omega} |\nabla u|^p \, dx,
\]

that implies the strong convergence of \((u_h)\) to \(u\) in \(W_{0}^{1,p}(\Omega)\).

**Lemma 2.3.3.** Let \(c \in \mathbb{R}\) and let \((u_h)\) be a \((CPS)_c\)-sequence in \(W_{0}^{1,p}(\Omega)\). Then for each \(\varepsilon > 0\) and \(\varrho > 0\) there exists \(K_{\varrho,\varepsilon} > 0\) such that for all \(h \in \mathbb{N}\)

\[
\int_{\{\|u_h\| \leq \varrho\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leq \varepsilon \int_{\{\varrho < \|u_h\| < K_{\varrho,\varepsilon}\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + K_{\varrho,\varepsilon}
\]

and

\[
\int_{\{\|u_h\| \leq \varrho\}} |\nabla u_h|^p \, dx \leq \varepsilon \int_{\{\varrho < \|u_h\| < K_{\varrho,\varepsilon}\}} |\nabla u_h|^p \, dx + K_{\varrho,\varepsilon}.
\]

**Proof.** Let \(\sigma, \varepsilon > 0\) and \(\varrho > 0\). For all \(v \in W_{0}^{1,p}(\Omega)\), we set

\[
\langle w_h, v \rangle = \int_{\Omega} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla v \, dx + \int_{\Omega} D_s \mathcal{L}(x, u_h, \nabla u_h) v \, dx - \int_{\Omega} g(x, u_h) v \, dx.
\]

(2.62)

Let us now consider \(\vartheta_1 : \mathbb{R} \to \mathbb{R}\) given by

\[
\vartheta_1(s) = \begin{cases} 
  s & \text{if } |s| < \sigma \\
  s + 2\sigma & \text{if } \sigma \leq s < 2\sigma \\
  -s - 2\sigma & \text{if } -2\sigma < s \leq -\sigma \\
  0 & \text{if } |s| \geq 2\sigma.
\end{cases}
\]

(2.63)

Then, testing (2.62) with \(\vartheta_1(u_h) \in L^\infty([-2\sigma, 2\sigma])\), we obtain

\[
\int_{\Omega} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \vartheta_1(u_h) \, dx + \int_{\Omega} D_s \mathcal{L}(x, u_h, \nabla u_h) \vartheta_1(u_h) \, dx \leq
\]

\[
\leq \int_{\Omega} g(x, u_h) \vartheta_1(u_h) \, dx + \|w_h\|_{-1,p'} \|\vartheta_1(u_h)\|_{1,p'}.
\]

Then, it follows that

\[
\int_{\{\|u_h\| \leq \sigma\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx - \int_{\{\sigma < \|u_h\| \leq 2\sigma\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx +
\]
Let $K_0 > 0$ be such that $\|w_h\|_{-1,p'} \leq K_0$. Then, since by (2.54) we have
\[
\nu \|\vartheta_1(u_h)\|_{1,p}^p \leq \int_{\{u_h \leq \sigma\}} \nu |\nabla u_h|^p \, dx + \int_{\{\sigma < |u_h| \leq 2\sigma\}} \nu |\nabla u_h|^p \, dx \leq \\
\leq \int_{\{u_h \leq \sigma\}} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + \int_{\{\sigma < |u_h| \leq 2\sigma\}} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + \\
+ \int_{\{u_h \leq \sigma\}} a(x) \, dx + b_0 \int_{\{u_h \leq \sigma\}} |u_h|^p \, dx + \int_{\{\sigma < |u_h| \leq 2\sigma\}} a(x) \, dx + \\
+ b_0 \int_{\{\sigma < |u_h| \leq 2\sigma\}} |u_h|^p \, dx,
\]
taking into account (2.55), we get for a sufficiently small value of $\sigma > 0$
\[
\left(1 - \sigma M - \frac{1}{4}\right) \int_{\{u_h \leq \sigma\}} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leq \\
\leq \left(1 + \sigma M + \frac{1}{4}\right) \int_{\{\sigma < |u_h| \leq 2\sigma\}} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + \\
+ \int_{\Omega} \left(c_1 + c_2 |2\sigma|^{\frac{n(p-1)+p}{n-p}}\right) \sigma \, dx + \frac{4^{\nu'}}{p'p^{p'} \nu'^p} K_0^{p'} + \\
+ \int a \varphi(x) \, dx + \left[b_0 (2p+1)\sigma^p + \varepsilon \sigma^{p'+1}\right] \mathcal{L}^n(\Omega).
\]
Whence, we have shown an inequality of the type
\[
\int_{\{u_h \leq \sigma\}} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leq \\
\leq K_1 \int_{\{\sigma < |u_h| \leq 2\sigma\}} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + K_2.
\]
Let us now define for each $k \geq 1$ the functions $\vartheta_{2k}, \vartheta_{2k-1} : \mathbb{R} \to \mathbb{R}$ by setting
\[
\vartheta_{2k}(s) = \begin{cases} 
0 & \text{if } |s| \leq k\sigma \\
 s - k\sigma & \text{if } k\sigma < s < (k + 1)\sigma \\
 s + k\sigma & \text{if } -(k + 1)\sigma < s < -k\sigma \\
 -s + (k + 2)\sigma & \text{if } (k + 1)\sigma < s < (k + 2)\sigma \\
 -s - (k + 2)\sigma & \text{if } -(k + 2)\sigma < s < -(k + 1)\sigma \\
0 & \text{if } |s| \geq (k + 1)\sigma.
\end{cases}
\]
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and

\[ \vartheta_{2k-1}(s) = \begin{cases} 
\frac{s}{k} & \text{if } |s| \leq k \sigma \\
-s + (k+1) \sigma & \text{if } k \sigma < s < (k+1) \sigma \\
-s - (k+1) \sigma & \text{if } -(k+1) \sigma \leq s < -k \sigma \\
s + (k+1) \sigma & \text{if } -(k+1) \sigma < s \leq -(k+1) \sigma \\
0 & \text{if } |s| \geq (k+1) \sigma.
\end{cases} \]

Therefore, by iterating on \( k \), we obtain the \( k \)-th inequality

\[ \int_{\{u_h \leq k \sigma\}} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leq \]

\[ \leq K_1(k) \int_{\{k \sigma < |u_h| \leq (k+1) \sigma\}} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + K_2(k). \]

Let now choose \( k \geq 1 \) such that \( k \sigma \geq \rho \) and \( k \sigma \geq R \). Take \( 0 < \delta < 1 \) and let \( \vartheta_{\delta} : \mathbb{R} \to \mathbb{R} \) be the function defined by setting

\[ \vartheta_{\delta}(s) = \begin{cases} 
0 & \text{if } |s| \leq k \sigma \\
-s - k \sigma & \text{if } k \sigma < s < (k+1) \sigma \\
s + k \sigma & \text{if } -(k+1) \sigma < s < -k \sigma \\
-\delta s + \sigma + \delta(k+1) \sigma & \text{if } (k+1) \sigma \leq s < (k+1) \sigma + \frac{\sigma}{\delta} \\
-\delta s - \sigma - \delta(k+1) \sigma & \text{if } -(k+1) \sigma - \frac{\sigma}{\delta} < s \leq -(k+1) \sigma \\
0 & \text{if } |s| \geq (k+1) \sigma + \frac{\sigma}{\delta}.
\end{cases} \]

As before, we get

\[ \int_{\Omega} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \vartheta_{\delta}(u_h) \, dx + \int_{\Omega} \mathcal{D}_{s} \mathcal{L}(x, u_h, \nabla u_h) \vartheta_{\delta}(u_h) \, dx \leq \]

\[ \leq \int_{\Omega} g(x, u_h) \vartheta_{\delta}(u_h) \, dx + \frac{1}{\mathcal{P} \mathcal{P} \mathcal{P}} \|w_h\|_{-1, \mathcal{P}}^{\mathcal{P}} + \delta \|\vartheta_{\delta}(u_h)\|_{1, \mathcal{P}}^{\mathcal{P}}. \]

Taking into account (2.47), by computations, we deduce that

\[ \int_{\Omega} \mathcal{D}_{s} \mathcal{L}(x, u_h, \nabla u_h) \vartheta_{\delta}(u_h) \, dx \geq 0. \]
Moreover we have as before

\[
\|\vartheta_\delta(u_h)\|_{1,p}^p \leq \int_{\{|u_h| \leq (k+1)\sigma\}} |\nabla u_h|^p \, dx + \int_{\{|u_h| \geq (k+1)\sigma\}} |\nabla u_h|^p \, dx \leq \\
\leq \frac{1}{\nu} \int_{\{|u_h| \leq (k+1)\sigma\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + \\
+ \frac{1}{\nu} \int_{\{|u_h| > (k+1)\sigma\}} \frac{a(x)}{\nu} \int_{\{|u_h| \leq (k+1)\sigma\}} |u_h|^p \, dx + \\
+ \frac{1}{\nu} \int_{\{|u_h| > (k+1)\sigma\}} \frac{b_0}{\nu} \int_{\{|u_h| \geq (k+1)\sigma\}} |u_h|^p \, dx,
\]

so that

\[
\left(1 - \frac{\delta}{\nu}\right) \int_{\{|u_h| \leq (k+1)\sigma\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leq \\
\leq \left(\delta + \frac{\delta}{\nu}\right) \int_{\{|u_h| > (k+1)\sigma\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + \\
+ \int_{\Omega} \left(c_1 + c_2 \frac{\nu}{\delta} + \frac{\sigma}{\delta} \left(\frac{\nu}{\delta} \right)^{n-p} \right) \nabla u_h \, dx + \frac{1}{\nu} \int_{\{|u_h| \geq (k+1)\sigma\}} \sigma^p \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + K_3(k, \delta).
\]

Therefore, we get

\[
\int_{\{|u_h| \leq (k+1)\sigma\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leq \\
\leq \frac{\nu \delta + \delta}{\nu - \delta} \int_{\{|u_h| > (k+1)\sigma\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + K_3(k, \delta).
\]

Combining this inequality with (2.64) we conclude that

\[
\int_{\{|u_h| \leq \delta\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leq \int_{\{|u_h| \leq k\sigma\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leq \\
\leq K_1(k) \int_{\{|u_h| \leq (k+1)\sigma\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + K_2(k) \leq \\
\leq K_1(k) \frac{\nu \delta + \delta}{\nu - \delta} \int_{\{|u_h| > (k+1)\sigma\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + K_1(k)K_3(k, \delta) + K_2(k) \leq \\
\leq \varepsilon \int_{\{|u_h| > \delta\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + K_0, \varepsilon,
\]

where we have fixed \(\delta > 0\) in such a way that \(K_1(k)\frac{\nu \delta + \delta}{\nu - \delta} \leq \varepsilon\). \(\square\)
The next result is an extension of [88, Lemma 1], since (2.51) relaxes (9) of [88].

Lemma 2.3.4. Let $c \in \mathbb{R}$. Then each $(CPS)_c$--sequence for $f$ is bounded in $W^{1,p}_0(\Omega)$.

Proof. First of all, we can find $a_0 \in L^1(\Omega)$ such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$

$$qG(x, s) \leq sg(x, s) + a_0(x).$$

Now, let $(u_h)$ be a $(CPS)_c$--sequence for $f$ and let for all $v \in C^\infty_c(\Omega)$

$$\langle w, v \rangle = \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla v \, dx +$$

$$+ \int_\Omega D_s L(x, u, \nabla u) u_h \, dx - \int_\Omega g(x, u_h) v \, dx.$$ (2.65)

According to Proposition 2.1.2 and Lemma 2.3.3, for each $\varepsilon > 0$ we have

$$-\|w_h\|_{-1,p'} \|u_h\|_{1,p} \leq \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx +$$

$$+ \int_\Omega D_s L(x, u_h, \nabla u_h) u_h \, dx - \int_\Omega g(x, u_h) u_h \, dx \leq$$

$$\leq \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + \int_\Omega D_s L(x, u_h, \nabla u_h) u_h \, dx +$$

$$- q \int_\Omega G(x, u_h) \, dx + \int_\Omega a_0 \, dx \leq$$

$$\leq (1 + \varepsilon) \int_{\{|u_h| > R'\}} \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + \int_\Omega D_s L(x, u_h, \nabla u_h) u_h \, dx +$$

$$- q \int_\Omega L(x, u_h, \nabla u_h) \, dx + q f(u_h) + \int_\Omega a_0 \, dx + K_{R', \varepsilon}.$$ (2.66)

On the other hand, from Lemma 2.3.3 and (2.51), for each $\varepsilon > 0$ we obtain

$$\int_\Omega D_s L(x, u_h, \nabla u_h) u_h \, dx =$$

$$= \int_{\{|u_h| \leq R'\}} D_s L(x, u_h, \nabla u_h) u_h \, dx + \int_{\{|u_h| > R'\}} D_s L(x, u_h, \nabla u_h) u_h \, dx \leq$$

$$\leq \varepsilon MR' \int_{\{|u_h| > R'\}} \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx +$$

$$- \int_{\{|u_h| > R'\}} \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + q \int_\Omega L(x, u_h, \nabla u_h) \, dx + \int_\Omega a_\varepsilon(x) \, dx +$$

$$+ \varepsilon \int_{\{|u_h| > R'\}} |u_h|^p \, dx - \nu \int_{\{|u_h| > R'\}} |\nabla u_h|^p \, dx + K_{R', \varepsilon}.$$
Taking into account Poincare and Young’s inequalities, by (2.46) we find \( c > 0 \) and \( C_{R', \varepsilon} > 0 \) with
\[
\int_{\Omega} D_s \mathcal{L}(x, u_h, \nabla u_h) u_h \, dx \leq \\
\leq \varepsilon c \int_{\{ |u_h| > R' \}} |\nabla u_h|^p \, dx - \int_{\{ |u_h| > R' \}} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + \\\n+ q \int_{\Omega} \mathcal{L}(x, u_h, \nabla u_h) \, dx + \int_{\Omega} a_\varepsilon(x) \, dx - \nu \int_{\{ |u_h| > R' \}} |\nabla u_h|^p \, dx + C_{R', \varepsilon}.
\]
Therefore, for a sufficiently small \( \varepsilon > 0 \), there exists \( \vartheta \) with \( \vartheta |x| \nabla u_h|_p \leq \|w_h\|_{-1,p'} \|u_h\|_{1,p} + qf(u_h) + \\\n+ \int_{\Omega} a_0 \, dx + \int_{\Omega} a_\varepsilon \, dx + K_{R', \varepsilon} + C_{R', \varepsilon}.
\]
Moreover, it is
\[
\int_{\Omega} |\nabla u_h|^p \, dx \leq (1 + \varepsilon) \int_{\{ |u_h| > R' \}} |\nabla u_h|^p \, dx + K_{R', \varepsilon}.
\]
Since \( w_h \to 0 \) in \( W^{-1,p'}(\Omega) \), the assertion follows.

### 2.3.3 Existence of a weak solution

**Lemma 2.3.5.** Under assumptions (2.49) we have
\[
\int_{\Omega} G(x, u_h) \, dx \overline{|u_h|_{1,p}^p} \to 0 \quad \text{as } h \to +\infty,
\]
for each \((u_h)\) that goes to 0 in \( W^{1,p}_0(\Omega) \).

**Proof.** Let \((u_h) \subseteq W^{1,p}_0(\Omega)\) with \( u_h \to 0 \) in \( W^{1,p}_0(\Omega) \). We can find \((\varrho_h) \subseteq \mathbb{R}\) and a sequence \((w_h) \subseteq W^{1,p}(\Omega)\) such that \( u_h = \varrho_h w_h \), \( \varrho_h \to 0 \) and \( \|w_h\|_{1,p} = 1 \). Taking into account (2.49), it follows
\[
\lim_{h} \frac{G(x, u_h(x))}{\|u_h\|_{1,p}^p} = 0 \quad \text{for a.e. } x \in \Omega.
\]
Moreover, for a.e. \( x \in \Omega \) we have
\[
\frac{G(x, u_h(x))}{\|u_h\|_{1,p}^p} \leq d|w_h|^p + b \varrho_h^{\frac{p}{p'-r}} |w_h|^r.
\]
If \( w \) is the weak limit of \((w_h)\), since \( d|w_h|^p \to d|w|^p \) in \( L^1(\Omega) \) and \( b \varrho_h^{\frac{p}{p'-r}} |w_h|^r \to 0 \) in \( L^1(\Omega) \), (a variant of) Lebesgue’s Theorem concludes the proof.
We finally conclude with the proof of the main result of this section.

*Proof of Theorem (2.3.1).* From Lemma (2.3.4) and Theorem (2.3.2) it follows that \( f \) satisfies the \((CPS)_c\) condition for each \( c \in \mathbb{R} \). By (2.44) and (2.50) it easily follows that

\[
\forall u \in W^{1,p}_0(\Omega) \setminus \{0\}: \quad \lim_{t \to +\infty} f(tu) = -\infty.
\]

Finally from Lemma (2.3.5) and (2.44) we deduce that 0 is a strict local minimum for \( f \).

From Theorem (1.2.5) the assertion follows.

**Remark 2.3.6.** As proved by D. Arcoya and L. Boccardo in [7], each weak solution of (2.42) belongs to \( W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) provided that \( \mathcal{L} \) and \( g \) satisfy suitable conditions. Then, some nice regularity results hold for various classes of integrands \( \mathcal{L} \). (see [72]).
Chapter 3

Functionals with broken symmetry

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We refer the reader to [97, 98, 111, 112, 104]. Some parts of these publications have been slightly modified to give this collection a more uniform appearance.
Chapter 3. Functionals with broken symmetry

3.1 Quasilinear elliptic systems

3.1.1 Introduction

In critical point theory, an open problem concerning existence, is the role of symmetry in obtaining multiple critical points for even functionals.

Around 1980, the semilinear scalar problem

\[
\begin{aligned}
- \sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u) &= g(x,u) + \varphi \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

with \( g \) superlinear and odd in \( u \) and \( \varphi \in L^2(\Omega) \), has been object of a very careful analysis by A. Bahri and H. Berestycki in [16], M. Struwe in [114], G.-C. Dong and S. Li in [55] and by P. H. Rabinowitz in [92] via techniques of classical critical point theory. Around 1990, A. Bahri and P. L. Lions in [18, 19] improved the previous results via a Morse–Index type technique.

Later on, since 1994, several efforts have been devoted to study existence for quasilinear scalar problems of the type

\[
\begin{aligned}
- \sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2} \sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_i u D_j u &= g(x,u) \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

We refer the reader to [11, 35, 34, 37, 114] and to [7, 88, 100] for a more general setting.

In this case the associated functional \( f : H^1_0(\Omega, \mathbb{R}^N) \to \mathbb{R} \) given by

\[
f(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x,u)D_i u D_j u \, dx - \int_{\Omega} G(x,u) \, dx,
\]

is not even locally Lipschitz unless the \( a_{ij} \)'s do not depend on \( u \) or \( n = 1 \). Consequently, techniques of non–smooth critical point theory have to be applied.

It seems now natural to ask whether some existence results for perturbed even functionals still hold in a quasilinear setting, both scalar \((N = 1)\) and vectorial \((N \geq 2)\).

In [101] it has recently been proved that diagonal quasilinear elliptic systems of the type \((k = 1, \ldots, N)\)

\[
\begin{aligned}
- \sum_{i,j=1}^{n} D_j(a_{ij}^k(x,u)D_i u_k) + \frac{1}{2} \sum_{i,j=1}^{n} D_s a_{ij}^k(x,u)D_i u_k D_j u_k &= D_s D_j^k G(x,u) \quad \text{in } \Omega,
\end{aligned}
\]

possess a sequence \((u^m)\) of weak solutions in \( H^1_0(\Omega, \mathbb{R}^N) \) under suitable assumptions, including symmetry, on coefficients \( a_{ij}^k \) and \( G \). In order to prove this result, we looked for critical points of the functional \( f_0 : H^1_0(\Omega, \mathbb{R}^N) \to \mathbb{R} \) defined by

\[
f_0(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x,u)D_i u_h D_j u_h \, dx - \int_{\Omega} G(x,u) \, dx.
\]
3.1. Quasilinear elliptic systems

In this section we want to investigate the effects of destroying the symmetry of system (3.2) and show that for each \( \varphi \in L^2(\Omega, \mathbb{R}^N) \) the perturbed problem

\[
- \sum_{i,j=1}^{n} D_j(a_{ij}^k(x, u)D_i u_k) + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_s a_{ij}^h(x, u)D_i u_h D_j u_h = D_s G(x, u) + \varphi_k \quad \text{in } \Omega,
\]

still has infinitely many weak solutions. Of course, to this aim, we shall study the associated functional

\[
f(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x, u)D_i u_h D_j u_h \, dx - \int_{\Omega} G(x, u) \, dx - \int_{\Omega} \varphi \cdot u \, dx. \tag{3.5}
\]

In the next, \( \Omega \) will denote an open and bounded subset of \( \mathbb{R}^n \). In order to adapt the perturbation argument of [92], we shall consider the following assumptions:

- the matrix \( (a_{ij}^h(x, s)) \) is measurable in \( x \) for each \( s \in \mathbb{R}^N \) and of class \( C^1 \) in \( s \) for a.e. \( x \in \Omega \) with
  
  \[ a_{ij}^h(x, s) = a_{ji}^h(x, s). \]

Moreover, there exist \( \nu > 0 \) and \( C > 0 \) such that

\[
\sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x, s)\xi_i \xi_j \geq \nu |\xi|^2, \quad \left| a_{ij}^h(x, s) \right| \leq C, \quad \left| D_s a_{ij}^h(x, s) \right| \leq C, \tag{3.6}
\]

\[
\sum_{i,j=1}^{n} \sum_{h=1}^{N} s \cdot D_s a_{ij}^h(x, s)\xi_i \xi_j \geq 0, \tag{3.7}
\]

for a.e. \( x \in \Omega \) and for all \( s \in \mathbb{R}^N \) and \( \xi \in \mathbb{R}^{nN} \);

- (if \( N \geq 2 \)) there exists a bounded Lipschitz function \( \psi : \mathbb{R} \to \mathbb{R} \) such that

\[
\sum_{i,j=1}^{n} \sum_{h=1}^{N} \left( \frac{1}{2} D_s a_{ij}^h(x, s) \cdot \exp_\sigma(r, s) + a_{ij}^h(x, s) D_s (\exp_\sigma(r, s))_h \right) \xi_i \xi_j \leq 0, \tag{3.8}
\]

for a.e. \( x \in \Omega \), for all \( \xi \in \mathbb{R}^{nN} \), \( \sigma \in \{-1, 1\}^N \) and \( r, s \in \mathbb{R}^N \), where

\[
(\exp_\sigma(r, s))_i := \sigma_i \exp[\sigma_i(\psi(r_i) - \psi(s_i))].
\]

for each \( i = 1, \ldots, N \).

- the function \( G(x, s) \) is measurable in \( x \) for all \( s \in \mathbb{R}^N \), of class \( C^1 \) in \( s \) for a.e. \( x \in \Omega \) with \( G(x, 0) = 0 \) and \( g(x, \cdot) \) denotes the gradient of \( G \) with respect of \( s \).
Chapter 3. Functionals with broken symmetry

- there exist \( q > 2 \) and \( R > 0 \) such that
  \[
  |s| \geq R \implies 0 < qG(x, s) \leq s \cdot g(x, s),
  \]
  for a.e. \( x \in \Omega \) and all \( s \in \mathbb{R}^N \);

- there exists \( \gamma \in [0, q - 2] \) such that
  \[
  \sum_{i,j=1}^{n} \sum_{h=1}^{N} s \cdot D_s a_{ij}^h(x, s) \xi_i^h \xi_j^h \leq \gamma \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x, s) \xi_i^h \xi_j^h,
  \]
  for a.e. \( x \in \Omega \) and all \( s \in \mathbb{R}^N \) and \( \xi \in \mathbb{R}^{nN} \).

Under the previous assumptions, the following is our main result.

**Theorem 3.1.1.** Assume that there exists \( \sigma \in \left[1, \frac{q \sigma + q \sigma + (q - 1)(n + 2)}{q \sigma + q \sigma - (q - 2)} \right] \) such that
\[
|g(x, s)| \leq a + b|s|^\sigma,
\]
with \( a, b \in \mathbb{R} \) and that for a.e. \( x \in \Omega \) and for each \( s \in \mathbb{R}^N \)
\[
a_{ij}^h(x, -s) = a_{ij}^h(x, s), \quad g(x, -s) = -g(x, s).
\]
Then there exists a sequence \( (u^m) \subseteq H_0^1(\Omega, \mathbb{R}^N) \) of solutions to the system

\[
- \sum_{i,j=1}^{n} D_j (a_{ij}^h(x, u) D_i u_k) +
\]
\[
+ \frac{1}{2} \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_{s_i} a_{ij}^h(x, u) D_i u_h D_j u_h = D_{s_k} G(x, u) + \varphi_k \quad \text{in} \; \Omega,
\]
such that
\[
\lim_{m} f(u^m) = +\infty.
\]

This is clearly an extension of the results of [16, 55, 92, 114] to the quasilinear case, both scalar \((N = 1)\) and vectorial \((N \geq 2)\).

Let us point out that in the case \( N = 1 \) a stronger version of the previous result can be proven. Indeed, we may completely drop assumption (b) and replace Lemma 2.2.4 with [37, Lemma 2.2.4].

To our knowledge, in the case \( N > 1 \) only very few multiplicity results have been obtained so far via non-smooth critical point theory (see [11, 101, 114]).

### 3.1.2 Symmetry perturbed functionals

Given \( \varphi \in L^2(\Omega, \mathbb{R}^N) \), we shall now consider the functional \( f : H_0^1(\Omega, \mathbb{R}^N) \to \mathbb{R} \) defined by

\[
f(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x, u) D_i u_h D_j u_h \, dx - \int_{\Omega} G(x, u) \, dx - \int_{\Omega} \varphi \cdot u \, dx.
\]
3.1. Quasilinear elliptic systems

If $\varphi \neq 0$, clearly $f$ is not even. Note that by (3.9) we find $c_1, c_2, c_3 > 0$ such that

$$\frac{1}{q} (s \cdot g(x, s) + c_1) \geq G(x, s) + c_2 \geq c_3 |s|^q.$$  \hfill (3.12)

**Lemma 3.1.2.** Assume that $u \in H^1_0(\Omega, \mathbb{R}^N)$ is a weak solution to (3.4). Then there exists $\sigma > 0$ such that

$$\int_{\Omega} (G(x, u) + c_2) \, dx \leq \sigma \left( f(u)^2 + 1 \right)^{\frac{1}{2}}.$$  

**Proof.** If $u \in H^1_0(\Omega, \mathbb{R}^N)$ is a weak solution to (3.4), taking into account (3.10), we deduce that

$$f(u) = f(u) - \frac{1}{2} f'(u)(u) =$$

$$= \int_{\Omega} \left[ \frac{1}{q} g(x, u) \cdot u - G(x, u) - \frac{1}{2} \varphi \cdot u \right] \, dx +$$

$$- \frac{1}{4} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_i u^h_{ij}(x, u) \cdot u D_i u_h D_j u_h \, dx \geq$$

$$\geq \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\Omega} (g(x, u) \cdot u + c_1) \, dx - \frac{1}{2} \| \varphi \|_2 \| u \|_2 +$$

$$- \frac{1}{4} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} a^h_{ij}(x, u) D_i u_h D_j u_h \, dx - c_4 \geq$$

$$\geq \left( \frac{q}{2} - 1 - \frac{\gamma}{2} \right) \int_{\Omega} (G(x, u) + c_2) \, dx - \frac{\gamma}{2} f(u) - \varepsilon \| u \|_q^q - \beta(\varepsilon) \| \varphi \|_2^q - c_5$$

with $\varepsilon \to 0$ and $\beta(\varepsilon) \to +\infty$. Choosing $\varepsilon > 0$ small enough, by (3.12) we have

$$\sigma f(u) \geq \int_{\Omega} (G(x, u) + c_2) \, dx - c_6,$$

where $\sigma = \frac{2+\gamma}{q-2-\gamma}$, and the assertion follows as in [92, Lemma 1.8].

We now want to introduce the modified functional, which is the main tool in order to obtain our result. Let us define $\chi \in C^\infty(\mathbb{R})$ by setting $\chi = 1$ for $s \leq 1$, $\chi = 0$ for $s \geq 2$ and $-2 < \chi' < 0$ when $1 < s < 2$, and let for each $u \in H^1_0(\Omega, \mathbb{R}^N)$

$$\phi(u) = 2\sigma \left( f(u)^2 + 1 \right)^{\frac{1}{2}}, \quad \psi(u) = \chi \left( \phi(u)^{-1} \int_{\Omega} (G(x, u) + c_2) \, dx \right).$$

Finally, we define the modified functional by

$$\tilde{f}(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} a^h_{ij}(x, u) D_i u_h D_j u_h \, dx +$$

$$- \int_{\Omega} G(x, u) \, dx - \psi(u) \int_{\Omega} \varphi \cdot u \, dx.$$  \hfill (3.13)

$$\tilde{f}(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h=1}^{N} a^h_{ij}(x, u) D_i u_h D_j u_h \, dx +$$

$$- \int_{\Omega} G(x, u) \, dx - \psi(u) \int_{\Omega} \varphi \cdot u \, dx.$$  \hfill (3.14)
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The Euler’s equation associated to the previous functional is given by

\[
- \sum_{i,j=1}^{n} D_j (a_{ij}^k(x,u) D_i u_k) +
\]

\[
\frac{1}{2} \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_{sh} a_{ij}^h (x,u) D_i u_h D_j u_h = \tilde{g}(x,u) \quad \text{in} \quad \Omega,
\]

where we set

\[
\tilde{g}(x,u) = g(x,u) + \psi(u) \varphi + \psi'(u) \int_{\Omega} \varphi \cdot u \, dx.
\]

Note that taking into account the previous Lemma, if \( u \in H_0^1(\Omega, \mathbb{R}^N) \) is a weak solution to (3.4), we have that \( \psi(u) = 1 \) and therefore \( \tilde{f}(u) = f(u) \). In the next result, we measure the defect of symmetry of \( \tilde{f} \), which turns out to be crucial in the final comparison argument.

**Lemma 3.1.3.** There exists \( \beta > 0 \) such that for all \( u \in H_0^1(\Omega, \mathbb{R}^N) \)

\[
|\tilde{f}(u) - \tilde{f}(-u)| \leq \beta \left( |\tilde{f}(u)|^{\frac{1}{q}} + 1 \right).
\]

**Proof.** Note first that if \( u \in \text{supp}(\psi) \) then

\[
\left| \int_{\Omega} \varphi \cdot u \, dx \right| \leq \alpha \left( |f(u)|^{\frac{1}{q}} + 1 \right),
\]

where \( \alpha > 0 \) depends on \( \|\varphi\|_2 \). Indeed, by (3.12) we have

\[
\left| \int_{\Omega} \varphi \cdot u \, dx \right| \leq \|u\|_2 \|\varphi\|_2 \leq c \|u\|_q \leq \tilde{c} \left( \int_{\Omega} (G(x,u) + c_2) \, dx \right)^{\frac{1}{q}},
\]

and since \( u \in \text{supp}(\psi) \),

\[
\int_{\Omega} (G(x,u) + c_2) \, dx \leq 4\sigma (f(u)^2 + 1)^{\frac{1}{2}} \leq \bar{c}(|f(u)| + 1),
\]

inequality (3.16) easily follows. Now, since of course

\[
|f(u)| \leq |\tilde{f}(u)| + 2 \left| \int_{\Omega} \varphi \cdot u \, dx \right|,
\]

by (3.16) we immediately get for some \( b > 0 \)

\[
\psi(u) \left| \int_{\Omega} \varphi \cdot u \, dx \right| \leq b \psi(u) \left( |\tilde{f}(u)|^{\frac{1}{q}} + \left| \int_{\Omega} \varphi \cdot u \, dx \right|^{\frac{1}{q}} + 1 \right).
\]

Using Young’s inequality, for some \( b_1, b_2 > 0 \) we have that

\[
\psi(u) \left| \int_{\Omega} \varphi \cdot u \, dx \right| \leq b_1 \left( |\tilde{f}(u)|^{\frac{1}{q}} + 1 \right),
\]
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and

\[ \psi(-u) \left| \int_{\Omega} \varphi \cdot u \, dx \right| \leq b_2 \left( |f(u)|^{\frac{1}{q}} + 1 \right), \]

and since

\[ |\tilde{f}(u) - \tilde{f}(-u)| = (\psi(u) + \psi(-u)) \left| \int_{\Omega} \varphi \cdot u \, dx \right|, \]

the assertion follows.

\[ \square \]

Theorem 3.1.4. There exists \( M > 0 \) such that if \( u \in H^1_0(\Omega, \mathbb{R}^N) \) is a weak solution to (3.15) with \( \tilde{f}(u) \geq M \) then \( u \) is a weak solution to (3.4) and \( \tilde{f}(u) = f(u) \).

Proof. Let us first prove that there exist \( M > 0 \) and \( \tilde{\alpha} > 0 \) such that

\[ \forall M \in [\tilde{M}, +\infty[ : \ \tilde{f}(u) \geq M, \ u \in \text{supt}(\psi) \implies f(u) \geq \tilde{\alpha} M. \] (3.17)

Since we have

\[ f(u) \geq \tilde{f}(u) - \left| \int_{\Omega} \varphi \cdot u \right|, \]

by (3.16) we deduce that

\[ f(u) + \alpha |f(u)|^{\frac{1}{q}} \geq \tilde{f}(u) - \alpha \geq \frac{M}{2} \]

for \( M \geq \tilde{M} \), with \( \tilde{M} \) large enough. Now, if it was \( f(u) \leq 0 \), we would obtain

\[ \frac{\alpha q'}{q} + \frac{1}{q} |f(u)| \geq \alpha |f(u)|^{\frac{1}{q}} \geq \frac{M}{2} + |f(u)|, \]

which is not possible if we take \( \tilde{M} > 2\alpha q'(q')^{-1} \). Therefore it is \( f(u) > 0 \) and

\[ f(u) > \frac{M}{4} \quad \text{or} \quad f(u) \geq \left( \frac{M}{4\alpha} \right)^{q}, \]

and (3.17) is proven. Of course, taking into account the definition of \( \psi \), to prove the Lemma it suffices to show that if \( M > 0 \) is sufficiently large and \( u \in H^1_0(\Omega, \mathbb{R}^N) \) is a weak solution to (3.15) with \( \tilde{f}(u) \geq M \), then

\[ \phi(u)^{-1} \int_{\Omega} (G(x, u) + c_2) \, dx \leq 1. \]

If we set

\[ \vartheta(u) = \phi(u)^{-1} \int_{\Omega} (G(x, u) + c_2) \, dx, \]

it follows that

\[ \psi'(u)(u) = \]

\[ = \chi'(\vartheta(u))\varphi(u)^{-2} \left[ \phi(u) \int_{\Omega} g(x, u) \cdot u \, dx - (2\sigma)^2 \vartheta(u)f(u)f'(u)(u) \right]. \]
Define now \( T_1, T_2 : H_0^1(\Omega, \mathbb{R}^N) \rightarrow \mathbb{R} \) by setting
\[
T_1(u) = \chi'((\partial(u))(2\sigma)^2(\partial(u))\phi(u)^{-2}f(u) \int_\Omega \varphi \cdot u \, dx,
\]
and
\[
T_2(u) = \chi'((\partial(u))\phi(u)^{-1} \int_\Omega \varphi \cdot u \, dx + T_1(u).
\]
Then we obtain
\[
\tilde{f}'(u)(u) = (1 + T_1(u)) \int_\Omega \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x,u) D_i u_h D_j u_h \, dx +
\frac{1}{2} (1 + T_1(u)) \int_\Omega \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_s a_{ij}^h(x,u) \cdot u D_i u_h D_j u_h \, dx +
-(1 + T_2(u)) \int_\Omega g(x,u) \cdot u \, dx - (\psi(u) + T_1(u)) \int_\Omega \varphi \cdot u \, dx.
\]
Consider now the term
\[
\tilde{f}(u) - \frac{1}{2(1 + T_1(u))} \tilde{f}'(u)(u).
\]
If \( \psi(u) = 1 \) and \( T_1(u) = 0 = T_2(u) \), the assertion follows from Lemma 3.1.2. Otherwise, since \( 0 \leq \psi(u) \leq 1 \), if \( T_1(u) \) and \( T_2(u) \) are both small enough the computations we have made in Lemma 3.1.2 still hold true with \( \sigma \) replaced by \( (2 - \varepsilon)\sigma \), for a small \( \varepsilon > 0 \), and again assertion follows as in Lemma 3.1.2.

It then remains to show that if \( M \rightarrow \infty \), then \( T_1(u), T_2(u) \rightarrow 0 \). We may assume that \( u \in \text{supt}(\psi) \), otherwise \( T_i(u) = 0 \), for \( i = 1, 2 \). Therefore, taking into account (3.16), there exists \( c > 0 \) with
\[
|T_1(u)| \leq c \frac{|f(u)|^{\frac{1}{q}} + 1}{|f(u)|}.
\]
Finally, by (3.17) we deduce \( |T_1(u)| \rightarrow 0 \) as \( M \rightarrow \infty \). Similarly, \( |T_2(u)| \rightarrow 0 \).

### 3.1.3 Boundedness of concrete Palais–Smale sequences

**Definition 3.1.5.** Let \( c \in \mathbb{R} \). A sequence \((u^m) \subseteq H_0^1(\Omega, \mathbb{R}^N)\) is said to be a concrete Palais–Smale sequence at level \( c \) ((CPS)\(_c\)–sequence, in short) for \( \tilde{f} \), if \( \tilde{f}(u^m) \rightarrow c \),
\[
\sum_{i,j=1}^{n} \sum_{h=1}^{N} D_s a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m \in H^{-1}(\Omega, \mathbb{R}^N)
\]
eventually as \( m \rightarrow \infty \) and
\[
-\sum_{i,j=1}^{n} D_j (a_{ij}^h(x,u^m) D_i u_h^m) + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_s a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m - \bar{g}_k(x,u^m),
\]
goes to zero strongly in \( H^{-1}(\Omega, \mathbb{R}^N) \), where \( \bar{g}(x,u) = g(x,u) + \psi(u) \varphi + \psi'(u) \int_\Omega \varphi \cdot u \, dx \). We say that \( \tilde{f} \) satisfies the concrete Palais–Smale condition at level \( c \), if every (CPS)\(_c\) sequence for \( \tilde{f} \) admits a strongly convergent subsequence in \( H_0^1(\Omega, \mathbb{R}^N) \).
Lemma 3.1.6. There exists $M > 0$ such that each $(CPS)_c$--sequence $(u^m)$ for $\bar{f}$ with $c \geq M$ is bounded in $H^1_0(\Omega, \mathbb{R}^N)$.

Proof. Let $M > 0$ and $(u^m)$ be a $(CPS)_c$--sequence for $\bar{f}$ with $c \geq M$ in $H^1_0(\Omega, \mathbb{R}^N)$ such that, eventually as $m \to +\infty$

$$M \leq \bar{f}(u^m) \leq K.$$ 

for some $K > 0$. Taking into account [101, Lemma 3], we have $\bar{f}(u^m)(u^m) \to 0$ as $m \to +\infty$. Therefore, for large $m \in \mathbb{N}$ and any $\varrho > 0$, it follows

$$\varrho\|u^m\|_{1,2} + K \geq \bar{f}(u^m) - \varrho \bar{f}(u^m)(u^m) = \left(\frac{1}{2} - \varrho(1 + T_1(u^m))\right) \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x, u^m)D_i u^m_h D_j u^m_h \, dx +$$

$$- \frac{\varrho}{2}(1 + T_1(u^m)) \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_a a_{ij}^h(x, u^m) \cdot u^m D_i u^m_h D_j u^m_h \, dx +$$

$$+ \varrho(1 + T_2(u^m)) \int_\Omega g(x, u^m) \cdot u^m \, dx +$$

$$- \int_\Omega G(x, u^m) \, dx + [\varrho(\psi(u^m) + T_1(u^m)) - \psi(u^m)] \int_\Omega \varphi \cdot u^m \, dx \geq$$

$$\geq \left(\frac{1}{2} - \varrho(1 + T_1(u^m)) - \frac{\varrho^2}{2}(1 + T_1(u^m))\right) \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x, u^m)D_i u^m_h D_j u^m_h \, dx +$$

$$+ \varrho(1 + T_2(u^m)) \int_\Omega g(x, u^m) \cdot u^m \, dx +$$

$$- \int_\Omega G(x, u^m) \, dx + [\varrho(\psi(u^m) + T_1(u^m)) - \psi(u^m)] \int_\Omega \varphi \cdot u^m \, dx \geq$$

$$\geq \varrho \left(1 - \varrho(2 + \gamma)(1 + T_1(u^m))\right) \|u^m\|_{1,2}^2 + (\varrho(1 + T_2(u^m)) - 1) \int_\Omega G(x, u^m) \, dx$$

$$- [\varrho(1 + T_1(u^m)) + 1]\|\varphi\|_2\|u^m\|_2.$$ 

If we choose $M$ sufficiently large, we find $\varepsilon > 0$, $\eta > 0$ and $\varrho \in \left[\frac{1+\eta}{q}, \frac{1-\varepsilon}{q}\right]$ [such that uniformly in $m \in \mathbb{N}$]

$$(1 - \varrho(2 + \gamma)(1 + T_1(u^m))) > \varepsilon, \quad (\varrho(1 + T_2(u^m)) - 1) > \eta.$$ 

Hence we obtain

$$\varrho\|u^m\|_{1,2} + K \geq \frac{\varrho \varepsilon}{2} \|u^m\|_{1,2}^2 + b\eta\|u^m\|^q \ - c\|u^m\|_{1,2},$$

which implies that the sequence $(u^m)$ is bounded in $H^1_0(\Omega, \mathbb{R}^N)$. \hfill \Box

Lemma 3.1.7. Let $c \in \mathbb{R}$. Then there exists $M > 0$ such that for each bounded $(CPS)_c$ sequence $(u^m)$ for $\bar{f}$ with $c \geq M$, the sequence $(\bar{g}(x, u^m))$ admits a convergent subsequence in $H^{-1}(\Omega, \mathbb{R}^N)$. 

Proof. Let \((u^m)\) be a bounded \((CPS)_c\)–sequence for \(\tilde{f} \) with \(c \geq M\). We may assume that \((u^m) \subseteq \text{supp}(\psi)\), otherwise \(\psi(u^m) = 0\) and \(\psi'(u^m) = 0\). Recall that

\[
\tilde{g}(x, u^m) = g(x, u^m) + \psi(u^m)\varphi + \psi'(u^m) \int_\Omega \varphi \cdot u^m \, dx.
\]

Since by [37, Theorem 2.2.7] the maps

\[
H^1_0(\Omega, \mathbb{R}^N) \quad \longrightarrow \quad H^{-1}(\Omega, \mathbb{R}^N)
\]

and

\[
H^1_0(\Omega, \mathbb{R}^N) \quad \longrightarrow \quad H^{-1}(\Omega, \mathbb{R}^N)
\]

are completely continuous, the sequences \((g(x, u^m))\) and \((\psi(u^m)\varphi)\) admit a convergent subsequence in \(H^{-1}(\Omega, \mathbb{R}^N)\). Now, we have

\[
\psi'(u^m) = [\chi'(|\vartheta(u^m)|)\phi(u^m)^{-1}] g(x, u^m) + [4\sigma^2 \chi'(|\vartheta(u^m)|)\phi(u^m)^{-2} \vartheta(u^m) f(u^m)] f'(u^m).
\]

On the other hand it is

\[
f'(u^m) = \tilde{f}'(u^m) + \left[ \int_\Omega \varphi \cdot u^m \, dx \right] \psi'(u^m) + [\psi(u^m) - 1] \varphi.
\]

Therefore we deduce that

\[
\left[ 1 + \left[ 4\sigma^2 \chi'(|\vartheta(u^m)|)\phi(u^m)^{-2} \vartheta(u^m) f(u^m) \int_\Omega \varphi \cdot u^m \, dx \right] \right] \psi'(u^m) = \]

\[
[\chi'(|\vartheta(u^m)|)\phi(u^m)^{-1}] g(x, u^m) + [4\sigma^2 \chi'(|\vartheta(u^m)|)\phi(u^m)^{-2} \vartheta(u^m) f(u^m) \int_\Omega \varphi \cdot u^m \, dx] f'(u^m) - [4\sigma^2 \chi'(|\vartheta(u^m)|)\phi(u^m)^{-2} \vartheta(u^m) f(u^m)(\psi(u^m) - 1)] \varphi.
\]

By assumption we have \(\tilde{f}'(u^m) \to 0\) in \(H^{-1}(\Omega, \mathbb{R}^N)\). Taking into account the definition of \(\chi\), \(\phi\) and \(\vartheta\), all of the square brackets in equation (3.18) are bounded in \(\mathbb{R}\) for some \(M > 0\) and we conclude that also \((\psi'(u^m))\) admits a convergent subsequence in \(H^{-1}(\Omega, \mathbb{R}^N)\). The assertion is now proven.

\[ \square \]

3.1.4 Compactness of concrete Palais–Smale sequences

The next result is the crucial property for Palais–Smale condition to hold.

Lemma 3.1.8. Let \((u^m)\) be a bounded sequence in \(H^1_0(\Omega, \mathbb{R}^N)\) and set

\[
\langle w^m, v \rangle = \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x, u^m) D_i u^m_h D_j v_h \, dx +
\]
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$$+ \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{N} \sum_{h=1}^{n} D_i a_{ij}(x, u^m) \cdot vD_i u_h^m D_j u_h^m \, dx$$

for all $v \in C^\infty_c(\Omega, \mathbb{R}^N)$. Then, if $(u^m)$ is strongly convergent to some $w$ in $H^{-1}(\Omega, \mathbb{R}^N)$, $(u^m)$ admits a strongly convergent subsequence in $H^1_0(\Omega, \mathbb{R}^N)$.

**Proof.** See, [101, Lemma 6].

**Theorem 3.1.9.** There exists $M > 0$ such that $\tilde{f}$ satisfies $(CPS)_c$-condition for $c \geq M$.

**Proof.** Let $(u^m)$ be a $(CPS)_c$ sequence for $f$ with $c \geq M$, where $M > 0$ is as in Lemma 3.1.6. Therefore, $(u^m)$ is bounded in $H^1(\Omega, \mathbb{R}^N)$ and from Lemma 3.1.7 we deduce that, up to subsequences, $(\bar{g}(x, u^m))$ is strongly convergent in $H^{-1}(\Omega, \mathbb{R}^N)$. Therefore, the assertion follows from Lemma 3.1.8.

3.1.5 Existence of multiple solutions

Let $(\lambda_h, u_h)$ be the sequence of eigenvalues and eigenvectors for the problem

$$\begin{cases}
\Delta u = -\lambda u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

and set

$$V_k = \text{span} \{ u_1, \ldots, u_k \in H^1_0(\Omega, \mathbb{R}^N) \}.$$  

We deduce that for all $s \in \mathbb{R}^N$

$$|s| \geq R \implies G(x, s) \geq \frac{G(x, R^q/|s|)}{R^q} |s|^q \geq b_0(x)|s|^q,$$

where

$$b_0(x) = R^{-q}\inf \{ G(x, s) : |s| = R \} > 0.$$

Then it follows that for each $k \in \mathbb{N}$ there exists $R_k > 0$ such that for all $u \in V_k$

$$\|u\|_{L^q} \geq R_k \implies \tilde{f}(u) \leq 0.$$

We can now give the following

**Definition 3.1.10.** For each $k \in \mathbb{N}$ set

$$D_k = V_k \cap B(0, R_k),$$

$$\Gamma_k = \left\{ \gamma \in C(D_k, H^1_0) : \gamma \text{ odd and } \gamma|_{\partial B(0, R_k)} = \text{Id} \right\},$$

and

$$b_k = \inf_{\gamma \in \Gamma_k} \max_{u \in D_k} \tilde{f}(\gamma(u)).$$
Lemma 3.1.11. For each $k \in \mathbb{N}$, $\varrho \in ]0, R_k[$ and $\gamma \in \Gamma_k$

$$\gamma(D_k) \cap \partial B(0, \varrho) \cap V_{k-1}^\perp \neq \emptyset.$$  

Proof. See, [92, Lemma 1.44]. □

Lemma 3.1.12. There exist $\beta > 0$ and $k_0 \in \mathbb{N}$ such that

$$\forall k \geq k_0 : b_k \geq \beta k^{\frac{(n+2)-(n-2)\sigma}{n(\sigma+1)}}.$$  

Proof. Let $\gamma \in \Gamma_k$ and $\varrho \in ]0, R_k[$. By previous Lemma there exists $w \in \gamma(D_k) \cap \partial B(0, \varrho) \cap V_{k-1}^\perp$, and therefore

$$\max_{u \in D_k} \tilde{f}(\gamma(u)) \geq \tilde{f}(w) \geq \inf_{u \in \partial B(0, \varrho) \cap V_{k-1}^\perp} \tilde{f}(u). \quad (3.19)$$  

Given $u \in \partial B(0, \varrho) \cap V_{k-1}^\perp$, by (3.11) we find $\alpha_1, \alpha_2, \alpha_3 > 0$ with

$$\tilde{f}(u) \geq \frac{1}{2} \varrho^2 - \alpha_1 \|u\|_\sigma^{\sigma+1} - \alpha_2 \|\varphi\|_2 \|u\|_2 - \alpha_3.$$  

Now, By Gagliardo-Niremberg inequality, there is $\alpha_4 > 0$ such that

$$\|u\|_\sigma^{\sigma+1} \leq \alpha_4 \|u\|_{1,2}^\vartheta \|u\|_{1,2}^{1-\vartheta},$$  

where $\vartheta = \frac{n(\sigma-1)}{2(\sigma+1)}$. As is well known, it is

$$\|u\|_2 \leq \frac{1}{\sqrt[2]{\lambda_{k-1}}} \|u\|_{1,2},$$  

so that we obtain

$$\tilde{f}(u) \geq \frac{1}{2} \varrho^2 - \alpha_1 \lambda_k^{-\frac{(1-\vartheta)(\sigma+1)}{2}} \varrho^{\sigma+1} - \alpha_2 \|\varphi\|_2 \lambda_k^{-\frac{1}{2}} \varrho - \alpha_3.$$  

Choosing now

$$\varrho = c \lambda_k^{-\frac{(1-\vartheta)(\sigma+1)}{2(\sigma-1)}},$$  

yields

$$\tilde{f}(u) \geq \frac{1}{4} \varrho^2 - \alpha_2 \|\varphi\|_2 \lambda_k^{-\frac{1}{2}} \varrho - \alpha_3.$$  

Now, as is shown in [49], there exists $\alpha_5 > 0$ such that for large $k$

$$\lambda_k \geq \alpha_5 k^{\frac{2}{\sigma}}.$$  

Therefore we find $\beta > 0$ with

$$\tilde{f}(u) \geq \beta k^{\frac{(n+2)-(n-2)\sigma}{n(\sigma-1)}},$$  

and by (3.19) the Lemma is proved. □
Definition 3.1.13. For each $k \in \mathbb{N}$ set
\[ U_k = \left\{ \xi = tu_{k+1} + w : \ t \in [0, R_{k+1}], \ w \in B(0, R_{k+1}) \cap V_k, \ \|\xi\|_{1,2} \leq R_{k+1} \right\} \]
and
\[ A_k = \left\{ \lambda \in C(U_k, H^1_0) : \lambda|_{\partial U_k} \in \Gamma_{k+1} \text{ and } \lambda|_{\partial B(0,R_{k+1}) \cup (B(0,R_{k+1}) \setminus B(0,R_k)) \cap V_k} = \text{Id} \right\} \]
and
\[ c_k = \inf_{\lambda \in A_k} \max_{u \in U_k} \bar{f}(\lambda(u)). \]

We now come to the our main existence tool. Of course, differently from the proof of [92, Lemma 1.57], in this nonsmooth framework, we shall apply [37, Theorem 1.1.13] instead of the classical Deformation Lemma [92, Lemma 1.60].

Lemma 3.1.14. Assume that $c_k > b_k \geq M$, where $M$ is as in Theorem 3.1.9. If $\delta \in ]0, c_k - b_k[ \text{ and } A_k(\delta) = \left\{ \lambda \in A_k : \bar{f}(\lambda(u)) \leq b_k + \delta \text{ for } u \in D_k \right\},$
set
\[ c_k(\delta) = \inf_{\lambda \in A_k(\delta)} \max_{u \in U_k} \bar{f}(\lambda(u)). \]
Then $c_k(\delta)$ is a critical value for $\bar{f}$.

Proof. Let $\overline{\varepsilon} = \frac{1}{2}(c_k - b_k - \delta) > 0$ and assume by contradiction that $c_k(\delta)$ is not a critical value for $\bar{f}$. Therefore, taking into account Lemma 3.1.9, by [37, Theorem 1.1.13], there exists $\varepsilon > 0$ and a continuous map
\[ \eta : H^1_0(\Omega, \mathbb{R}^N) \times [0, 1] \to H^1_0(\Omega, \mathbb{R}^N) \]
such that for each $u \in H^1_0(\Omega, \mathbb{R}^N)$ and $t \in [0, 1]$
\[ \bar{f}(u) \not\in [c_k(\delta) - \overline{\varepsilon}, c_k(\delta) + \overline{\varepsilon}] \implies \eta(u, t) = u, \quad (3.20) \]
and
\[ \eta(c_k(\delta) + \varepsilon, 1) \subseteq \bar{c}(c_k(\delta) - \varepsilon). \quad (3.21) \]
Choose $\lambda \in A_k(\delta)$ so that
\[ \max_{u \in U_k} \bar{f}(\lambda(u)) \leq c_k(\delta) + \varepsilon \quad (3.22) \]
and consider
\[ \eta(\lambda(\cdot), 1) : U_k \to H^1_0(\Omega, \mathbb{R}^N). \]
Observe that if $u \in \partial B(0, R_{k+1})$ or $u \in (B(0, R_{k+1}) \setminus B(0, R_k)) \cap V_k$, by definition $\bar{f}(\lambda(u)) = \bar{f}(u)$. Hence, by (3.20), it is $\eta(\lambda(u), 1) = u$. We conclude that $\eta(\lambda(\cdot), 1) \in A_k$. Moreover, by our choice of $\varepsilon > 0$ and $\delta > 0$ we obtain
\[ \forall u \in D_k : \bar{f}(\lambda(u)) \leq b_k + \delta \leq c_k - \overline{\varepsilon} \leq c_k(\delta) - \varepsilon. \]
Therefore (3.20) implies that $\eta(\lambda(\cdot), 1) \in A_k(\delta)$. On the other hand, again by (3.21) and (3.22)

$$\max_{u \in U_k} f(\eta(u, 1)) \leq c_k(\delta) - \varepsilon,$$

which is not possible, by definition of $c_k(\delta)$. \hfill $\square$

It only remains to prove that we cannot have $c_k = b_k$ for $k$ sufficiently large.

**Lemma 3.1.15.** Assume that $c_k = b_k$ for all $k \geq k_1$. Then, there exist $\gamma > 0$ and $\tilde{k} \geq k_1$ with

$$b_k \leq \gamma \tilde{k}^{q-1}.$$

**Proof.** Choose $k \geq k_1$, $\varepsilon > 0$ and a $\lambda \in A_k$ such that

$$\max_{u \in U_k} f(\lambda(u)) \leq b_k + \varepsilon.$$

Define now $\tilde{\lambda} : D_{k+1} \to H_0^1$ such that

$$\tilde{\lambda}(u) = \begin{cases} 
\lambda(u) & \text{if } u \in U_k \\
-\lambda(-u) & \text{if } u \in -U_k.
\end{cases}$$

Since $\tilde{\lambda}|_{B(0, r_{k+1}) \cap V_k}$ is continuous and odd, it follows $\tilde{\lambda} \in \Gamma_{k+1}$. Then

$$b_{k+1} \leq \max_{u \in D_{k+1}} \tilde{f}(\tilde{\lambda}(u)).$$

By Lemma 3.1.3 we have

$$\max_{u \in -U_k} \tilde{f}(\tilde{\lambda}(u)) \leq b_k + \varepsilon + \beta \left( |b_k + \varepsilon|^\frac{1}{q} + 1 \right),$$

and since $D_{k+1} = U_k \cup (-U_k)$, we get

$$\forall \varepsilon > 0 : \ b_{k+1} \leq b_k + \varepsilon + \beta \left( |b_k + \varepsilon|^\frac{1}{q} + 1 \right),$$

that yields

$$\forall k \geq k_1 : \ b_{k+1} \leq b_k + \beta \left( |b_k|^\frac{1}{q} + 1 \right).$$

The assertion now follows recursively as in [93, Proposition 10.46]. \hfill $\square$

We finally come to the proof of the main result, which extends the theorems of [16, 55, 92, 114] to the quasilinear case, both scalar and vectorial.

**Proof of Theorem 3.1.1.** Observe that inequality

$$1 < \sigma < \frac{qn + (q - 1)(n + 2)}{qn + (q - 1)(n - 2)},$$

implies

$$\frac{q}{q - 1} < \frac{(n + 2) - \sigma(n - 2)}{n(n - 1)}.$$
Remark 3.1.16. In 1988 and 1992, A. Bahri & P. L. Lions showed via a perturbation technique based on Morse theory that, at least in some particular cases, the growth restriction on \( \sigma \) is not essential. More precisely, they proved that the problem

\[
-\Delta u = |u|^\sigma - 1 u - \varphi \quad \text{in } \Omega
\]

has a sequence \((u_h)\) of solutions in \( H^1_0(\Omega) \) for each \( \sigma \in \left( 1, \frac{n+2}{n-2} \right) \) (see [18, 19]).

One knows from Pohozaev’s identity that even when \( \varphi = 0 \) this result is false in general if \( \sigma > \frac{n+2}{n-2} \) so that this theorem seems to be optimal. The problem of whether or not this existence results hold also in the quasilinear case is open.

### 3.2 Problems at exponential growth

#### 3.2.1 Introduction

In 1994 K. Sugimura proved that, given an open bounded domain \( \Omega \) of \( \mathbb{R}^2 \) with smooth boundary \( \partial \Omega \), for each \( \varphi \in L^2(\Omega) \) the semilinear elliptic problem

\[
\begin{aligned}
-\Delta u &= g(x, u) + \varphi \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]  

(3.24)

admits an unbounded sequence of solutions \((u_h) \subset H^1_0(\Omega)\) provided that \(g(x, u)\) is an odd (in \( u \)) superlinear nonlinearity with exponential growth such that

\[
A_1 e^{s|p_1|} - B_1 \leq \int_0^s g(x, \tau) \, d\tau \leq A_2 e^{s|p_2|} - B_2 \quad 0 < p_1 \leq p_2 < \frac{1}{2}
\]

a.e. in \( \Omega \) and for each \( s \in \mathbb{R} \), where \( A_1, A_2 > 0 \) and \( B_1, B_2 \geq 0 \) (see [115]).

The main goal of this section is improving Sugimura’s result and at the same time extending these type of achievements to the case of quasilinear elliptic equations. For a planar domain \( \Omega \), the analogue of the Sobolev embedding \( H^1_0(\Omega) \hookrightarrow L^2(\Omega) \) in dimensions greater than 3 is the Orlicz space embedding

\[
\forall s \geq 1 : \quad H^1_0(\Omega) \ni u \mapsto e^{u^2} \in L^s(\Omega)
\]

for which the Trudinger–Moser inequality holds: there exists \( C_{TM} > 0 \) with

\[
\forall u \in H^1_0(\Omega) : \quad \|u\|_{1,2} \leq 1 \implies \int_\Omega e^{4\pi u^2} \, dx \leq C_{TM} \mathcal{L}^2(\Omega),
\]

(3.25)

where \( \mathcal{L}^2 \) denotes the usual Lebesgue measure in \( \mathbb{R}^2 \) and \( \| \cdot \|_{1,2} \) is the standard norm in \( H^1_0(\Omega) \). In view of a sharp inequality like (3.25) (see Theorem 3.2.9), we shall obtain a multiplicity result for the exponential nonlinearity

\[
\forall s \in \mathbb{R} : \quad g(s) = |s|^{p-2} s e^{|s|^p},
\]

(3.26)

all over the subcritical range \( 1 < p < 2 \).
Let us now briefly recall the historic background of the problem of broken symmetry for elliptic equations. If \( \Omega \) is a bounded domain of \( \mathbb{R}^n \) with \( n \geq 2 \), the multiplicity of solutions for semilinear elliptic problems of the type

\[
\begin{aligned}
-\Delta u &= g(x,u) + \varphi \quad \text{in } \Omega \\
 u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

with \( g \) superlinear, odd in \( u \) and for \( a, b > 0 \)

\[ |g(x,s)| \leq a + b|s|^p, \quad 1 < p < \sigma \leq 2^* - 1 \quad \text{if } n \geq 3, \]

\[ 1 < p < +\infty \quad \text{if } n = 1, 2, \]

has been investigated by the variational techniques developed by Bahri, Berestycki, Rabinowitz and Struwe in the early eighties [16, 55, 92, 114, 117]. Later on, around 1990, Bahri and Lions improved the previous results via a technique based on Morse theory (see [18, 19]).

Very recently further improvements have been achieved by a completely new method devised by P. Bolle (see [25]). When \( n = 2 \), the result of Bahri and Lions [18] is optimal for the power case \( g(x,s) = |s|^{p-1}s \), namely the multiplicity appears for all \( p > 1 \). However, when \( n \geq 3 \), it remains open the problem of whether (3.27) has an infinite number of solutions for all \( \sigma \) all the way up to the exponent \( 2^* - 1 \).

Since 1994, several works have been devoted to the study of quasilinear elliptic equations of the type:

\[
- \sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_iu) + \frac{1}{2} \sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_iuD_ju = g(x,u) \quad \text{in } \Omega \quad (3.28)
\]

where \( \Omega \) is a bounded domain of \( \mathbb{R}^n \) with \( n \geq 3 \). We refer the reader to [35, 37] for the study of multiplicity of solutions of this problem and furthermore to [7] and [100] for an even more general framework. The functional \( f_0: H_0^1(\Omega) \to \mathbb{R} \) associated with (3.28) is given by

\[
f_0(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x,u)D_iuD_ju \, dx - \int_{\Omega} G(x,u) \, dx,
\]

where \( D_s G(x,u) = g(x,u) \). As pointed out in [37], this functional fails to be smooth (\( C^1 \)) for \( n \geq 3 \). On the other hand, also in the case \( n = 2 \), being

\[ \forall s < +\infty : \quad H^1_0(\Omega) \hookrightarrow L^s(\Omega) \quad \text{but} \quad H^1_0(\Omega) \not\hookrightarrow L^\infty(\Omega), \]

it may happen that

\[ \sum_{i,j=1}^{2} D_s a_{ij}(x,u)D_iuD_ju \notin H^{-1}(\Omega), \]

even if \( D_s a_{ij} \in L^\infty \), so that in general \( f_0 \) is continuous but fails to be locally Lipschitzian.

Consequently, techniques of nonsmooth critical point theory have to be employed and the methods of [25] cannot be used since the functional is requested to be of class \( C^2 \). We refer
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the reader to [37, 48, 54, 68] and [69] for the abstract framework that we shall need in the following.

It seemed natural to ask whether also in the quasilinear setting the multiplicity of solutions persists under perturbations. A partial answer to this question has been given in [111] and [97] where it was proved that if

\[ |g(x, s)| \leq a + b|s|^p, \quad 1 < p < \frac{qn + 2(q-1)}{qn - 2(q-1)}, \]

with \( a, b > 0 \) and for each \( i, j = 1, \ldots, n \)

\[ a_{ij}(x, -s) = a_{ij}(x, s), \quad g(x, -s) = -g(x, s) \]

ea.e. in \( \Omega \) and for each \( s \in \mathbb{R} \), then for each \( \varphi \in L^2(\Omega) \) the problem

\[
- \sum_{i,j=1}^{n} D_j(a_{ij}(x, u)D_i u) + \frac{1}{2} \sum_{i,j=1}^{n} D_s a_{ij}(x, u)D_iuD_j u = g(x, u) + \varphi \quad (3.29)
\]

with \( u = 0 \) on \( \partial \Omega \), has an unbounded sequence \((u_h) \subset H^1_0(\Omega)\) of solutions.

A natural question is now whether the multiplicity of solutions appears for the perturbed equation (3.29) when \( g \) possesses the exponential growth (3.26), all along the subcritical range \( 1 < p < 2 \). We are ready to give an answer to this question by stating the main result of the section. In the next, \( \Omega \) will denote a smooth bounded domain of \( \mathbb{R}^2 \). Moreover, we assume that:

(\( \mathcal{H} \)) each \( a_{ij}(x, s) \) is measurable in \( x \) for each \( s \in \mathbb{R} \) and of class \( C^1 \) in \( s \) for a.e. \( x \in \Omega \) with \( a_{ij} = a_{ji}, \ a_{ij} \in L^\infty(\Omega \times \mathbb{R}) \) and \( D_s a_{ij} \in L^\infty(\Omega \times \mathbb{R}) \). Moreover, there exist \( \nu > 0 \) and \( R > 0 \) such that:

\[
\sum_{i,j=1}^{2} a_{ij}(x, s)\xi_i\xi_j \geq \nu|\xi|^2,
\]

\[
|s| \geq R \implies \sum_{i,j=1}^{2} s D_s a_{ij}(x, s)\xi_i\xi_j \geq 0, \quad (3.30)
\]
a.e. in \( \Omega \) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^2 \).

We point out that assumption (3.30) is well known in the current literature both for existence and regularity theory (see e.g. [7, 35, 37, 111, 100, 97]).

Let \( \varphi : \Omega \times \mathbb{R} \to \mathbb{R} \) be a continuous map and let \( \sigma \geq 0 \) be such that

\[ |\varphi(x, s)| \leq a + b|s|^\sigma \quad (a, b > 0) \]
a.e. in \( \Omega \) and for each \( s \in \mathbb{R} \) and define \( f_\varphi : H^1_0(\Omega) \to \mathbb{R} \) by setting

\[
f_\varphi(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{2} a_{ij}(x, u)D_iuD_j u \, dx - \int_{\Omega} \left(e^{|u|^p} - 1\right) \, dx - \int_{\Omega} \Phi(x, u) \, dx
\]
with $D_s \Phi(x,s) = \varphi(x,s)$ for each $x \in \Omega$ and all $s \in \mathbb{R}$.

Under the preceding assumptions, the following is the main result.

**Theorem 3.2.1.** Let $1 < p < 2$ and assume that

$$a_{ij}(x,-s) = a_{ij}(x,s) \quad (i,j = 1,2)$$

a.e. in $\Omega$ and for each $s \in \mathbb{R}$. Then the problem

$$-\sum_{i,j=1}^{2} D_j(a_{ij}(x,u)D_iu) + \frac{1}{2} \sum_{i,j=1}^{2} D_s a_{ij}(x,u)D_iuD_ju = p|u|^{p-2}ue^{\varphi(x,u)}$$

with $u = 0$ on $\partial \Omega$, has a sequence $(u_h) \subset H^1_0(\Omega)$ of solutions such that

$$\lim_{h \to \infty} f_\varphi(u_h) = +\infty.$$

In particular, our result removes any upper bound in the subcritical growth completely. It has to be remarked that Theorem 3.2.1 is new also in the case $D_s a_{ij}(x,s) = 0$ a.e. in $\Omega$ and for each $s \in \mathbb{R}$ (semilinear case).

In the critical case $p = 2$, Adimurthi has conjectured in [1] that the problem

$$\begin{cases}
-\Delta u = ue^{u^2} + \varphi & \text{in } B(0,1) \\
u = 0 & \text{on } \partial B(0,1)
\end{cases}$$

admits at most one positive solution $u \in H^1_0(B(0,1))$, where $B(0,1)$ is the unit ball in $\mathbb{R}^2$. On the other hand, this uniqueness result seems to be out of reach, so far.

### 3.2.2 Recalls from the theory of Orlicz spaces

Let us briefly recall some basic notions about Orlicz spaces that will be required later. For further details, we refer the interested reader to [94].

**Definition 3.2.2.** Let $(\Omega, \Sigma, \mu)$ be an abstract measure space, where $\Omega$ is some point set, $\Sigma$ is a $\sigma-$algebra of its subsets on which a $\sigma-$additive function $\mu : \Sigma \to \mathbb{R}^*_+$ is given and $\mu$ has the finite subset property. Then, if $\Phi : \mathbb{R} \to \mathbb{R}^*_+$ is a Young function, we define

$$\mathcal{O}^\Phi_\mu = \{ u : \Omega \to \mathbb{R}^* \text{ measurable with } \alpha u \in \mathcal{J}^\Phi_\mu \text{ for some } \alpha > 0 \},$$

where

$$\mathcal{J}^\Phi_\mu = \{ u : \Omega \to \mathbb{R}^* \text{ measurable for } \Sigma : \int_\Omega \Phi(|u|)d\mu < +\infty \}.$$

The space $\mathcal{O}^\Phi_\mu$ is called Orlicz space.

The set $\mathcal{O}^\Phi_\mu$ is a vector space. Moreover, for each $u \in \mathcal{O}^\Phi_\mu$ there exists $\beta > 0$ such that

$$\beta u \in \mathcal{B}_\Phi = \left\{ v \in \mathcal{J}^\Phi_\mu : \int_\Omega \Phi(|v|)d\mu \leq 1 \right\},$$

where $\mathcal{B}_\Phi$ is a circled solid subset of $\mathcal{J}^\Phi_\mu$. This property motivates the following
Definition 3.2.3. We define a functional on the Orlicz space $\mathcal{O}^\Phi_\mu$ by setting
\[
\mathcal{N}_\Phi (u) = \inf \left\{ k > 0 : \frac{1}{k} u \in \mathcal{B}_{\Phi} \right\} = \inf \left\{ k > 0 : \int_\Omega \Phi \left( \frac{|u|}{k} \right) \, d\mu \leq 1 \right\}.
\] (3.33)
We say that $\mathcal{N}_\Phi : \mathcal{O}^\Phi_\mu \to \mathbb{R}_+$ is the gauge norm of the Orlicz space $\mathcal{O}^\Phi_\mu$.

It is readily seen that $(\mathcal{O}^\Phi_\mu, \mathcal{N}_\Phi)$ is a Banach space when $\mu$–a.e. equal functions are identified. Besides the gauge norm, the space $\mathcal{O}^\Phi_\mu$ can be endowed with another norm functional.

Definition 3.2.4. For each $u \in \mathcal{O}^\Phi_\mu$ we set
\[
\|u\|_\Phi = \sup \left\{ \int_\Omega |uv| \, d\mu : v \in \mathcal{O}^\Phi_\mu \text{ such that } \int_\Omega \Psi (|v|) \, d\mu \leq 1 \right\},
\] (3.34)
where $\Psi : \mathbb{R} \to \mathbb{R}_+^*$ is the complementary function to $\Phi$, defined by setting
\[
\forall y \in \mathbb{R} : \Psi (y) = \sup_{x > 0} \left\{ x |y| - \Phi (x) \right\}.
\]
The functional $\|\cdot\|_\Phi$ is called Orlicz norm.

One can prove that $(\mathcal{O}^\Phi_\mu, \|\cdot\|_\Phi)$ is a Banach space when $\mu$–a.e. equal functions are identified, and that the two norms $\|\cdot\|_\Phi$ and $\mathcal{N}_\Phi$ are equivalent. Moreover, there is an useful relationship between the Orlicz and gauge norms, which will be used in the following to obtain a fundamental estimate, namely
\[
\forall u \in \mathcal{O}^\Phi_\mu : \mathcal{N}_\Phi (u) \leq \|u\|_\Phi \leq 2\mathcal{N}_\Phi (u).
\] (3.35)

We end up this section by recalling a result, due to Krasnoselskii and Rutickii, which enables to compute the Orlicz norm $\|\cdot\|_\Phi$.

Theorem 3.2.5. Assume that $(\Phi, \Psi)$ be a complementary pair of Young functions such that $\Phi(x) = 0$ if and only if $x = 0$ where $\Phi$ is strictly increasing. Then
\[
\forall u \in \mathcal{O}^\Phi_\mu : \|u\|_\Phi = \inf_{k > 0} \left\{ \frac{1}{k} \left( 1 + \int_\Omega \Phi (ku) \, d\mu \right) \right\},
\] (3.36)

namely the Orlicz norm $\|\cdot\|_\Phi$ is given in terms of $\Phi$ alone.

Proof. See [94, 112].

This nice alternative formula will be used later on to estimate from below the Orlicz norm.

3.2.3 The perturbation argument

Let us first prove an a priori estimate for weak solutions of (3.31).

Lemma 3.2.6. Assume that $u \in H^1_0 (\Omega)$ is a weak solution of (3.31). Then
\[
\int_\Omega \left( e^{\|u\|^p} - 1 + c \right) \, dx \leq \sigma \left( f^2_\varphi (u) + 1 \right)^{1/2},
\]
for some $\sigma > 0$ and $c > 0$. 

Proof. Let $k \geq 1$ and $\eta_k : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\eta_k(s) = \begin{cases} 
0 & \text{if } s \leq k \\
 s - k & \text{if } k \leq s \leq k + 1 \\
1 & \text{if } s > k + 1.
\end{cases} \tag{3.37}$$

For each $k \geq 1$, we have $f'_{\varphi}(u)(\eta_k(u)) = 0$. Therefore, it results

$$\int_{\{k < u < k+1\}} \sum_{i,j=1}^{2} a_{ij}(x,u) D_i u D_j u \, dx + \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{2} \eta_k(u) D_s a_{ij}(x,u) D_i u D_j u \, dx \geq p(k + 1)^{p-1} \int_{\Omega} \left( e^{|u|^p} - 1 \right) \, dx + \int_{\Omega} \varphi \eta_k(u) \, dx +$$

$$- p(k + 1)^{p-1} \left( e^{(k+1)^p} - 1 \right) \mathcal{L}^2(\Omega).$$

Taking into account that $D_s a_{ij} \in L^\infty(\Omega \times \mathbb{R})$ and $|\eta_k| \leq 1$, inserting the expression of $f_{\varphi}(u)$, we find $C > 0$ and $C_{\delta, \varphi} > 0$ such that

$$\int_{\{k < u < k+1\}} \sum_{i,j=1}^{2} a_{ij}(x,u) D_i u D_j u \, dx + \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{2} \eta_k(u) D_s a_{ij}(x,u) D_i u D_j u \, dx \leq$$

$$\leq \frac{C}{2} \int_{\Omega} \sum_{i,j=1}^{2} a_{ij}(x,u) D_i u D_j u \, dx \leq$$

$$\leq C f_{\varphi}(u) + (1 + \delta) C \int_{\Omega} \left( e^{|u|^p} - 1 \right) \, dx + C_{\delta, \varphi},$$

for each $\delta > 0$. Fixing $\delta > 0$ and choosing $k$ sufficiently large, by combining the two previous estimates we get:

$$C_k f_{\varphi}(u) \geq \int_{\Omega} \left( e^{|u|^p} - 1 \right) \, dx - C'_k,$$

for some $C_k, C'_k > 0$, which easily yields the assertion. \qed

Let us now define $\chi \in C^\infty(\mathbb{R})$ by setting $\chi = 1$ for $s \leq 1$, $\chi = 0$ for $s \geq 2$ and $-2 < \chi' < 0$ when $1 < s < 2$, and let us set

$$\phi(u) = 2\sigma \left( f_{\varphi}^2(u) + 1 \right)^{1/2},$$

$$\psi(u) = \chi \left( \phi(u)^{-1} \int_{\Omega} \left( e^{|u|^p} - 1 + c \right) \, dx \right)$$
for each $u \in H^1_0(\Omega)$. Finally, we define the modified functional by setting

$$
\tilde{f}_\varphi(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^2 a_{ij}(x, u) D_i u D_j u \, dx - \int_\Omega \left( e^{\|u\|_p^p} - 1 \right) \, dx - \psi(u) \int_\Omega \varphi u \, dx.
$$

The Euler’s equation associated with $\tilde{f}_\varphi$ is given by

$$
- \sum_{i,j=1}^2 D_j (a_{ij}(x, u) D_i u) + \frac{1}{2} \sum_{i,j=1}^2 D_s a_{ij}(x, u) D_i u D_j u = \tilde{g}(x, u) \text{ in } \Omega \quad (3.38)
$$

where we have set

$$
\tilde{g}(x, u) = p\|u\|^{p-2} u e^{\|u\|_p^p} + \psi'(u) \varphi + \psi(u) \int_\Omega \varphi u \, dx.
$$

Note that, by Lemma 3.2.6, if $f'_0(u) = 0$, then $\tilde{f}_\varphi(u) = f_\varphi(u)$ and $\tilde{f}'_0(u) = 0$.

**Remark 3.2.7.** If we define $\vartheta : H^1_0(\Omega) \to \mathbb{R}$ by setting:

$$
\vartheta(u) = \varphi(u)^{-1} \int_\Omega \left( e^{\|u\|_p^p} - 1 + c \right) \, dx,
$$

a direct computation yields for each $v \in H^1_0 \cap L^\infty(\Omega)$:

$$
\tilde{f}'_\varphi(u)(v) = (1 + T_1(u)) \int_\Omega \sum_{i,j=1}^2 a_{ij}(x, u) D_i u D_j v \, dx +
$$

$$
+ \frac{1}{2} (1 + T_1(u)) \int_\Omega \sum_{i,j=1}^2 v D_s a_{ij}(x, u) D_i u D_j u \, dx +
$$

$$
- (1 + T_2(u)) \int_\Omega p\|u\|^{p-2} u v e^{\|u\|_p^p} \, dx - (\psi(u) + T_1(u)) \int_\Omega \varphi v \, dx,
$$

where $T_1, T_2 : H^1_0(\Omega) \to \mathbb{R}$ are given by

$$
T_1(u) = \chi'(\vartheta(u))(2\phi)^2 \vartheta(u) \phi(u)^{-2} f'_\varphi(u) \int_\Omega \varphi u \, dx,
$$

$$
T_2(u) = \chi'(\vartheta(u)) \phi(u)^{-1} \int_\Omega \varphi u \, dx + T_1(u).
$$

If $f_\varphi(u) \geq M$ and $M \to +\infty$, then $T_1(u) \to 0$ and $T_2(u) \to 0$ (see [111, 92]).

The following result establishes the links between the modified functional $\tilde{f}_\varphi$ and the original functional $f_\varphi$.

**Theorem 3.2.8.** There exists $\tilde{M} \in \mathbb{R}$ such that the following facts holds:

(a) if $u$ solves (3.38) with $\tilde{f}_\varphi(u) \geq \tilde{M}$, then $u$ solves (3.31) and $\tilde{f}_\varphi(u) = f_\varphi(u)$;

(b) $\tilde{f}_\varphi$ satisfies the concrete Palais–Smale condition at each level $c \geq \tilde{M}$. 
Proof. By Remark 3.2.7 and Lemma 3.2.6, (a) follows arguing as in [111, Theorem 2.3]. Let us now come to (b). Let us first show that each \((CPS)_{\varepsilon}\)-sequence \((u_h) \subset H^1_0(\Omega)\) for \(f_\varepsilon\) with \(\varepsilon \geq \tilde{M}\) is bounded in \(H^1_0(\Omega)\). Let \(k \geq 1\) and \(\eta_k\) be the function defined in (3.37). For each \(k \geq 1\), we have
\[
\frac{\widetilde{f}_\varepsilon'(u_h)(\eta_k(u_h))}{\|u_h\|_{1,2}} \to 0,
\]
as \(h \to +\infty\). In particular, it results
\[
(1 + T_1(u_h)) \int_{\{k < u_h < k+1\}} \sum_{i,j=1}^2 a_{ij}(x,u_h) D_i u_h D_j u_h \, dx + \\
+ \frac{1}{2} (1 + T_1(u_h)) \int_{\Omega} \sum_{i,j=1}^2 \eta_k(u_h) D_s a_{ij}(x,u_h) D_i u_h D_j u_h \, dx = \\
= (1 + T_2(u_h)) \int_{\Omega} p|u_h|^{p-1} |\eta_k(u_h)| e^{\|u_h\|_p} \, dx + \\
+ (T_1(u_h) + \psi(u_h)) \int_{\Omega} \varphi \eta_k(u_h) \, dx + \langle w_h, \eta_k(u_h) \rangle \geq \\
\geq p(k + 1)^{p-1} (1 + T_2(u_h)) \int_{\{u_h \geq k+1\}} e^{\|u_h\|_p} \, dx + \\
+ (T_1(u_h) + \psi(u_h)) \int_{\Omega} \varphi \eta_k(u_h) \, dx + \\
\geq p(k + 1)^{p-1} (1 + T_2(u_h)) \int_{\Omega} (e^{\|u_h\|_p} - 1) \, dx + \\
+ (T_1(u_h) + \psi(u_h)) \int_{\Omega} \varphi \eta_k(u_h) \, dx + \\
- 2p(k + 1)^{p-1} (e^{(k+1)^p} - 1) \mathcal{L}^2(\Omega) + \langle w_h, \eta_k(u_h) \rangle,
\]
where \(w_h \to 0\) in \(H^{-1}(\Omega)\). Inserting now the expression of \(f_\varepsilon(u_h)\), we get
\[
(1 + T_1(u_h)) \int_{\Omega} \sum_{i,j=1}^2 a_{ij}(x,u_h) D_i u_h D_j u_h \, dx + \\
+ \frac{1}{2} (1 + T_1(u_h)) \int_{\Omega} \sum_{i,j=1}^2 \eta_k(u_h) |D_s a_{ij}(x,u_h) D_i u_h D_j u_h \, dx = \\
\geq \frac{p}{2} (k + 1)^{p-1} (1 + T_2(u_h)) \int_{\Omega} \sum_{i,j=1}^2 a_{ij}(x,u) D_i u D_j u \, dx + \\
- p(k + 1)^{p-1} (1 + T_2(u_h)) f_\varepsilon(u_h) - p(k + 1)^{p-1} (1 + T_2(u_h)) \int_{\Omega} \varphi u_h \, dx + \\
+ (T_1(u_h) + \psi(u_h)) \int_{\Omega} \varphi \eta_k(u_h) \, dx + \\
- 2p(k + 1)^{p-1} (e^{(k+1)^p} - 1) \mathcal{L}^2(\Omega) + \langle w_h, \eta_k(u_h) \rangle.
\]
Taking into account that $|\eta_k| \leq 1$, $|\psi| \leq 1$ and $T_1(u_h), T_2(u_h) \to 0$ uniformly in $h$ as $\tilde{M} \to +\infty$, by choosing $k$ large enough we find $C_k > 0$ such that

$$
\nu C_k \int_\Omega |\nabla u_h|^2 \, dx \leq C_k \sum_{i,j=1}^2 a_{i,j}(x,u) D_i u D_j u \, dx \leq
$$

$$
\leq 2p(k+1)^{p-1} f_\varphi(u_h) + 2p(k+1)^{p-1} \|\varphi\|_2 \|u_h\|_2 + 2\|\varphi\|_1
$$

$$
+ 2p(k+1)^{p-1} \left( e^{(k+1)^p} - 1 \right) \mathcal{L}^2(\Omega) + \|w_h\|_{-1,2} \|\eta_k(u_h)\|_{1,2}.
$$

Since $f_\varphi(u_h) \to c$ and $w_h \to 0$ in $H^{-1}(\Omega)$, the above inequality implies that the sequence $(u_h)$ is bounded in $H^1_0(\Omega)$. Now, let $(u_h)$ be a $(CPS)_c$–sequence for $\tilde{f}_\varphi$ with $c \geq \tilde{M}$. Therefore, by the previous step $(u_h)$ is bounded in $H^1_0(\Omega)$. Taking into account that the map $H^1_0(\Omega) \to H^{-1}(\Omega)$

$$
u \mapsto p |u_h|^{p-2} u \eta_k(u_h)
$$

maps bounded sets of $H^1_0(\Omega)$ to relatively compact sets of $H^{-1}(\Omega)$ (see [112]), arguing as in [111, Lemma 3.3] we deduce that, up to subsequences, $(\tilde{g}(x,u_h))$ is strongly convergent in $H^{-1}(\Omega)$. Then, by [37, Theorem 2.2.4], there exists a further subsequence $(u_{h_k})$ strongly convergent in $H^1_0(\Omega)$. \qed

### 3.2.4 The growth estimate from below

Following [92], we shall build a min–max class for $\tilde{f}_\varphi$ and then we shall compare the growths from below and from above of the associated min–max values. Sugimura proved in [115] the following logarithmic estimate from below on the growth of the critical values $b_k$ (see Definition 3.2.11) for problem (3.24)

$$
\forall k \geq k_0 : \ b_k \geq k (\log k)^{\frac{2}{p}-2}, \quad p \in (0,1/2).
$$

Instead, we shall obtain the much stronger estimate:

$$
\forall k \geq k_0 : \ b_k \geq k^2.
$$

Let us now recall the celebrated Trudinger–Moser inequality for a smooth bounded domain $\Omega \subset \mathbb{R}^2$ in its general form: there exists $C_{TM} > 0$ such that

$$
\forall u \in H^1_0(\Omega) : \ |u|_{1,2} \leq 1 \implies \int_\Omega e^{\alpha u^2} \, dx \leq C_{TM} \mathcal{L}^2(\Omega),
$$

for each $\alpha \in [0,4\pi]$. See the works of Trudinger and Moser [86, 121].

The following result is one of the main tools for getting the optimal estimate from below.

**Theorem 3.2.9.** For each $1 < p < 2$ there exists $0 < \vartheta \leq 1$ such that

$$
\forall u \in H^1_0(\Omega) : \ |u|_{1,2} > 1 \implies \int_\Omega \left( e^{|u|^p} - 1 \right) \, dx \leq C_0 |u|_{1,2}^{1/\vartheta} \quad (3.39)
$$

where $\vartheta$ depends only on $R = |u|_{1,2}$ and $C_0 > 0$ is independent of $p,R$. 
Proof. Let us give an outline of the proof. First we introduce a suitable Orlicz space on the bounded domain $\Omega$, rescaling the usual Lebesgue measure in order to give an estimate from above on the gauge norm. Here the Trudinger–Moser inequality plays an important role. Then we introduce the Orlicz norm and we give an estimate from below on this norm, using (3.36). Finally, combining the two estimates with (3.35) will yield (3.39). Let us define a map $\Phi : \mathbb{R} \to \mathbb{R}_+$ by setting
\[
\forall x \in \mathbb{R} : \quad \Phi(x) = e^{|x|^p} - 1.
\]
It is easily seen that $\Phi$ is a Young function, so that we can introduce an associated Orlicz space $\mathcal{O}^\Phi$. Let $(\Omega, \Sigma, \nu)$ be the bounded domain of $\mathbb{R}^2$ endowed with the usual $\sigma-$algebra $\Sigma$ of measurable subsets and with a suitable rescaled Lebesgue measure $\nu$, which will be determined later. Hence, by definition (3.33), the gauge norm $\mathcal{N}_{\Phi} : \mathcal{O}^\Phi \to \mathbb{R}_+$ is given by
\[
\mathcal{N}_{\Phi}(u) = \inf \left\{ k > 0 : \int_{\Omega} \left( e^{\frac{|u|}{k}} - 1 \right) d\nu \leq 1 \right\}. \tag{3.40}
\]
We observe first that the Trudinger–Moser inequality implies
\[
\int_{\Omega} \left( e^{|u|^p} - 1 \right) dx \leq C_{TM} \mathcal{L}^2(\Omega)
\]
for any $u \in H^1_0(\Omega)$ such that $\|u\|_{1,2} \leq (4\pi)^{1/2}$ and for $C_{TM} > C_{TM}$. Hence
\[
\int_{\Omega} \left( e^{\frac{|u|}{k}} - 1 \right) dx \leq C_{TM} \mathcal{L}^2(\Omega) \tag{3.41}
\]
for any $u \in H^1_0(\Omega)$ and $k > 0$ such that $\|u\|_{1,2} \leq k (4\pi)^{1/2}$. Inequality (3.41) suggests us the choice of a new measure $\nu$, defined as
\[
\forall A \in \Sigma : \quad \nu(A) = \frac{\mathcal{L}^2(A)}{C_{TM} \mathcal{L}^2(\Omega)}.
\]
Replacing $dx$ by $dv$, inequality (3.41) allows us to estimate the gauge norm from above, namely, by (3.40) we have
\[
\forall u \in H^1_0(\Omega) : \quad \mathcal{N}_{\Phi}(u) \leq \frac{\|u\|_{1,2}}{(4\pi)^{1/2}}. \tag{3.42}
\]
To get the estimate from below on the gauge norm $\mathcal{N}_{\Phi}$, we consider now the Orlicz norm $\|\cdot\|_{\Phi}$ which by (3.36), may be written as
\[
\|u\|_{\Phi} = \min_{k > 0} \frac{1 + \int_{\Omega} \left( e^{k|u|^p} - 1 \right) d\nu}{k} = \frac{1 + \int_{\Omega} \left( e^{k_0|u|^p} - 1 \right) d\nu}{k_0}, \tag{3.43}
\]
for some $k_0 > 0$ (the minimum point). Indeed, since $e^t - 1 \geq t$ for all $t > 0$,
\[
\frac{1 + \int_{\Omega} \left( e^{k|u|^p} - 1 \right) d\nu}{k} \to +\infty \quad \text{as } k \to 0^+ \text{ and as } k \to +\infty,
\]
so that the infimum in (3.36) is actually a minimum. Therefore, by (3.35) and (3.42), to end up the proof we have to estimate from below the norm $\| \cdot \|_\phi$. We achieve this by comparing the value of $k_0$ with $\left[ \int_\Omega \left( e^{\|u\|^p} - 1 \right) d\nu \right]^{-1}$. If

$$k_0 \leq \frac{1}{\int_\Omega \left( e^{\|u\|^p} - 1 \right) d\nu},$$

we immediately get

$$\|u\|_\phi \geq \int_\Omega \left( e^{\|u\|^p} - 1 \right) d\nu. \quad (3.44)$$

Otherwise, if we assume

$$k_0 > \frac{1}{\int_\Omega \left( e^{\|u\|^p} - 1 \right) d\nu},$$

we can divide the proof into 3 steps, depending on the value of

$$a = \int_\Omega \left( e^{\|u\|^p} - 1 \right) d\nu.$$

- If $a \leq 1$, then there exists a $\tilde{k}$, which does not depend on $u$, such that

$$\|u\|_\phi \geq \frac{1}{\tilde{k}} \int_\Omega \left( e^{\|u\|^p} - 1 \right) d\nu. \quad (3.45)$$

Indeed, the $C^1$ map $\Theta : \mathbb{R} \to \mathbb{R}$ given by

$$\Theta(k) = \frac{1 + \int_\Omega \left( e^{k\|u\|^p} - 1 \right) d\nu}{k},$$

attains its minimum in $k_0$. Then $\Theta'(k_0) = 0$, which yields

$$pk_0^p \int_\Omega |u|^p e^{k_0\|u\|^p} d\nu = 1 + \int_\Omega \left( e^{k_0\|u\|^p} - 1 \right) d\nu \leq 1 + k_0^p \int_\Omega |u|^p e^{k_0\|u\|^p} d\nu.$$

Therefore, it is readily seen that

$$pk_0^{p-1} \int_\Omega \left( e^{\|u\|^p} - 1 \right) d\nu \leq \Theta(k_0) \leq \frac{p}{p-1} \frac{1}{k_0},$$

since $k_0 \geq 1$ by $a \leq 1$. In particular, we obtain:

$$\frac{1}{\int_\Omega \left( e^{\|u\|^p} - 1 \right) d\nu} \geq (p - 1) k_0^p.$$
Inserting this inequality in (3.43) we obtain (3.45).

- If \( a > 1 \) and \( k_0 \geq 1 \) we can repeat the proof as in the case \( a \leq 1 \).
- If \( a > 1 \) and \( k_0 < 1 \), there are only two possibilities: either

\[
\int_{\Omega} (e^{|u|^p} - 1) \, d\nu \leq C
\]  

(3.46)

where \( C > 0 \) is a constant independent of \( R \), or there exists \( \vartheta < 1 \), which depends only on \( R = \|u\|_{1,2} \), in such a way that

\[
k_0 < \frac{1}{\left[ \int_{\Omega} (e^{|u|^p} - 1) \, d\nu \right]^{\vartheta}}.
\]  

(3.47)

We shall prove this alternative later. Relation (3.47) implies

\[
\|u\|_{\phi} > \frac{1}{k_0} > \left[ \int_{\Omega} (e^{|u|^p} - 1) \, d\nu \right]^{\vartheta},
\]  

(3.48)

while (3.46) yields (3.39) directly, for all \( 1 > \vartheta > 0 \). Then, by (3.44), (3.45) and (3.48), for some \( C > 0 \)

\[
\|u\|_{\phi}^{1/\vartheta} \geq \frac{C}{C_{TM} L^2(\Omega)} \int_{\Omega} (e^{|u|^p} - 1) \, dx,
\]  

(3.49)

where \( \vartheta = \vartheta(R) \leq 1 \) depends only on \( R = \|u\|_{1,2} \). On the other hand, combining (3.35) and (3.42) yields

\[
\|u\|_{\phi} \leq \frac{2}{(4\pi)^{1/2}} \|u\|_{1,2} \leq \|u\|_{1,2}.
\]  

(3.50)

The estimate (3.49) on \( \|u\|_{\phi} \), together with (3.50), imply (3.39). To end up the proof of the theorem, it remains to show that either (3.46) or (3.47) is verified. Observe that

\[
a^{-\vartheta} = \left( \int_{\Omega} (e^{|u|^p} - 1) \, d\nu \right)^{-\vartheta} \to 1^- \quad \text{as} \quad \vartheta \to 0^+,
\]  

(3.51)

depending only on \( R = \|u\|_{1,2} \). Indeed, the Trudinger–Moser inequality yields, after some computations,

\[
1 < a \leq \int_{\{ |u|^p \geq 1 \}} \left( e^{\frac{|u|^2}{\|u\|_{1,2}^2}} - 1 \right) \, d\nu + \int_{\{ |u|^p \geq 1 \}} \left( e^{|u|^p} - 1 \right) \, d\nu
\]  

\[\leq 1 + c e^{R^{2p-2}},
\]  

(3.52)

where \( c > 0 \) is a constant independent of \( R \). Inequality (3.52) yields (3.51) directly.

Therefore, it suffices to show that for any \( R > 0 \) either (3.46) holds or there exists a constant \( \varepsilon = \varepsilon(R) \in (0, 1) \) such that

\[
\forall u \in H^1_0(\Omega) : \quad \|u\|_{1,2} = R \implies k_0 \leq 1 - \varepsilon.
\]  

(3.53)
3.2. Problems at exponential growth

By (3.51), if (3.53) is verified then inequality (3.47) holds. Let us first show that

\[ \int_{\Omega} (e^{k\rho|u|^p} - 1) \, d\nu \to \int_{\Omega} (e^{|u|^p} - 1) \, d\nu \quad \text{as } k \to 1^- . \]  

(3.54)

Let \( k = 1 - \eta \), with \( \eta \to 0^+ \). Then we have

\[ \left| \int_{\Omega} (e^{|u|^p} - 1) \, d\nu - \int_{\Omega} (e^{k\rho|u|^p} - 1) \, d\nu \right| \leq \int_{\Omega} e^{|u|^p} \left\{ 1 - e^{-\eta \rho|u|^p} \right\} \, d\nu \]

\[ \leq \left\{ \int_{\Omega} e^{2|u|^p} \, d\nu \right\}^{1/2} \cdot 2\eta \rho \cdot \|u\|_{2p}^p . \]  

(3.55)

The last integral term in inequality (3.55) can be estimated as in (3.52), obtaining (3.54). Analogously one can show that

\[ \int_{\Omega} |u|^p e^{k\rho|u|^p} \, d\nu \to \int_{\Omega} |u|^p e^{|u|^p} \, d\nu \quad \text{as } k \to 1^- . \]  

(3.56)

Let us assume now that (3.53) is not verified. Therefore, recalling that \( k_0 < 1 \), there exists \( R_0 > 0 \) such that for any \( \varepsilon \in (0, 1) \) there exists \( u_\varepsilon \in H_0^1(\Omega) \) with \( \|u_\varepsilon\|_{1,2} = R_0 \), such that \( 1 > k_0 > 1 - \varepsilon \). By definition, \( \Theta'(k_0) = 0 \), so that

\[ ph_0 \int_{\Omega} |u_\varepsilon|^p e^{k_0|u_\varepsilon|^p} \, d\nu \leq 1 + \int_{\Omega} (e^{|u_\varepsilon|^p} - 1) \, d\nu = 1 + a . \]

Therefore

\[ 1 + a \geq p (1 - \varepsilon)^p \int_{\Omega} |u_\varepsilon|^p e^{k_0|u_\varepsilon|^p} \, d\nu \]

\[ \geq p (1 - p\varepsilon) \left\{ \int_{\Omega} |u_\varepsilon|^p e^{|u_\varepsilon|^p} \, d\nu - \varepsilon C(R_0) \right\} = (p - p^2\varepsilon) (a - \varepsilon C(R_0)) \]

by (3.56), which implies that

\[ a \leq \frac{1 + (\varepsilon p - \varepsilon^2 p^2) C(R_0)}{-p^2\varepsilon + p - 1} \]  

(3.57)

for \( 0 < \varepsilon < \frac{p-1}{p^2} \). From (3.57) one can obtain the following upper bound on \( a \):

\[ a < \frac{4}{p - 1} \quad \text{if } 0 < \varepsilon < \min \left\{ \frac{1}{pC(R_0)}, \frac{p - 1}{p^2}, \frac{p - 1}{2p^2}, \frac{1}{p} \right\} ; \]

hence, if (3.53) is not verified, (3.46) holds. Let us assume now that (3.53) holds. By (3.51) there exists \( \vartheta = \vartheta(R) \) with \( 0 < \vartheta < 1 \), such that \( a^{-\vartheta} > k_0 \), that is (3.47). □
Remark 3.2.10. Observe that if (3.39) holds with $\vartheta$, then it holds for any $0 < \vartheta' < \vartheta$, since $R > 1$. Therefore, from now on we can assume that $0 < \vartheta < 1/4$ without loss of generality. The reason of this choice will be explained later.

Let now $(u_k, \lambda_k) \subset H_0^1(\Omega) \times \mathbb{R}$ be (orthonormalized) sequence of solutions to

$$
\begin{align*}
-\Delta u &= \lambda u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

and define recursively

$$
\mathcal{Y}_0 = \langle u_0 \rangle, \quad \forall k \geq 1 : \mathcal{Y}_{k+1} = \mathcal{Y}_k \oplus \mathbb{R}u_{k+1}.
$$

Since each $\mathcal{Y}_k$ is finite dimensional, one can find $\beta_1, \beta_2, \beta_3 > 0$ such that

$$
\forall u \in \mathcal{Y}_k : \bar{f}_\varphi(u) \leq \beta_1 \|u\|_{1,2}^2 - \beta_2 \|u\|_{1,2}^q - \beta_3,
$$

for each $q > 2$. In particular, for each $k \in \mathbb{N}$ there exists $R_k > 0$ such that

$$
\|u\|_{1,2} \geq R_k \implies \bar{f}_\varphi(u) \leq \bar{f}_\varphi(0) \leq 0
$$

for all $u \in \mathcal{Y}_k$ and $R_k \leq R_{k+1}$.

Definition 3.2.11. For each $k \in \mathbb{N}$ set $D_k = \mathcal{Y}_k \cap B(0, R_k)$,

$$
\Gamma_k = \left\{ \gamma \in C(D_k, H_0^1(\Omega)) : \gamma \text{ odd and } \gamma\big|_{\partial B(0,R_k)} = \text{Id} \right\},
$$

and

$$
b_k = \inf_{\gamma \in \Gamma_k} \max_{u \in D_k} \bar{f}_\varphi(\gamma(u)).
$$

Lemma 3.2.12 (Intersection lemma). For any $\gamma \in \Gamma_k$ and each $R < R_k$

$$
\forall k \geq 1 : \gamma(D_k) \cap \partial B(0, R) \cap \mathcal{Y}_{k-1}^+ \neq \emptyset. \tag{3.58}
$$

Proof. See [92, Lemma 1.44].

Observe that for all $q > 2$ and each $a_1 > 0$ there exists $a_2 > 0$ with

$$
e^{\|s\|^p} - 1 \geq a_1 \|s\|^{q} - a_2 \tag{3.59}
$$

for each $s \in \mathbb{R}$.

Lemma 3.2.13. There exist $\beta > 0$ and $k_0 \in \mathbb{N}$ such that

$$
\forall k \geq k_0 : b_k \geq \beta k^2.
$$
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Proof. Let us first note that we have

$$\forall u \in H_0^1(\Omega): \ \tilde{J}_\varphi(u) \geq \mathcal{J}_\varphi(u)$$

where we have set

$$\mathcal{J}_\varphi(u) = \frac{\nu}{2} \int_\Omega |Du|^2 \, dx - \int_\Omega (e^{|u|^p} - 1) \, dx - \psi(u) \int_\Omega \varphi \, u \, dx.$$  

Therefore, it suffices to get the desired estimate for values

$$b_k = \inf_{\gamma \in \Gamma_k} \max_{u \in \mathcal{D}_k} \mathcal{J}_\varphi(\gamma(u))$$

which, for simplicity, we avoid to rename. If $\gamma \in \Gamma_k$ and $R < R_k$, by the Intersection Lemma, we find

$$w \in \gamma(D_k) \cap \partial B_R \cap \mathcal{Y}_{k-1}$$

so that

$$\max_{u \in D_k} \mathcal{J}_\varphi(\gamma(u)) \geq \mathcal{J}_\varphi(\gamma(w)) \geq \inf_{u \in \partial B_R \cap \mathcal{Y}_{k-1}} \mathcal{J}_\varphi(u). \quad (3.60)$$

Therefore, to obtain a lower bound for $b_k$ we have to estimate $\mathcal{J}_\varphi(u)$ from below, with $u \in \partial B_R \cap \mathcal{Y}_{k-1}$ and $R < R_k$. This estimate will be obtained applying the interpolation inequality:

$$||u||_r \leq ||u||_{1,2}^{1-a} ||u||_{1,2}^a, \quad 1 \leq s \leq r < \infty, \quad a = 1 - \frac{s}{r}. \quad (3.61)$$

From now on, suppose $u \in \partial B_R \cap \mathcal{Y}_{k-1}$ and $1 < R < R_k$. First, observe that for any $\beta > 0$ there exists a constant $c = c(\beta, p) > 0$ such that

$$\forall t \in [0, +\infty[ : \ e^{tp} - 1 \leq t^\beta e^{tp} + c.$$

Therefore, by Hölder inequality, it results

$$\int_\Omega (e^{|u|^p} - 1) \, dx \leq \int_\Omega |u|^\beta e^{|u|^p} \, dx + c \mathcal{L}^2(\Omega)$$

$$\leq ||u||_{1,2}^{\beta} \left( \int_\Omega e^{\frac{\alpha}{\alpha-1}|u|^p} \, dx \right)^\frac{\alpha-1}{\alpha} + c_1$$

for some $c_1$, where we put

$$\alpha = \frac{1 - \vartheta^2}{1 - 4\vartheta^2} > 1, \quad \beta = \frac{3(1 - 2\vartheta)}{1 - \vartheta} > 0;$$

combining (3.39) with the previous inequality, and noting that $\frac{\alpha-1}{\alpha} < 1$, we obtain

$$\int_\Omega (e^{|u|^p} - 1) \, dx \leq ||u||_{1,2}^{\beta} \left( \int_\Omega e^{\frac{\alpha}{\alpha-1}|u|^p} - 1 \, dx \right)^\frac{\alpha-1}{\alpha} + c_1$$

$$\leq ||u||_{1,2}^{\beta} C_{\alpha, \vartheta} R^{\frac{\alpha-1}{\alpha}} + c_1 \quad (3.62)$$
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where \( \vartheta = \vartheta (R) \),

\[
C_{\alpha, \vartheta} = c_2 \left( \frac{\alpha}{\alpha - 1} \right)^{(\alpha - 1)/\alpha \vartheta} \tag{3.63}
\]

and \( c_2 \geq \max \{1, C_0\} \). Note that condition \( 1 < R < R_k \) can be always satisfied, by choosing \( R_k \) large enough. Applying now inequality (3.61) with

\[
r = \alpha \beta = \frac{3(1 + \vartheta)}{1 + 2 \vartheta} \geq 2
\]

and \( s = 2 \), we obtain

\[
\|u\|_{\alpha \beta} \leq \|u\|_{1,2}^{1-a} \|u\|_{1,2}^a \leq \lambda_k^{1-a} \|u\|_{1,2}^a \tag{3.64}
\]

where we have used the relation

\[
\forall u \in \mathcal{V}_{k-1} : \|u\|_2 \leq \frac{1}{\lambda_k^{1/2}} \|u\|_{1,2}.
\]

Combining (3.62) with (3.64) yields

\[
\int_{\Omega} \left( e^{\|u\|^p} - 1 \right) dx \leq C_{\alpha, \vartheta} \frac{1}{\lambda_k^\alpha} R^{3^{1-2\vartheta^2} - 1} + c_1.
\]

On the other hand, using (3.59) we have

\[
\int_{\Omega} \psi(u) \varphi dx \leq \|\varphi\|_2 \|u\|_2 \leq c \|\varphi\|_2 \|u\|_q
\]

\[
\leq c \|\varphi\|_2 \frac{a_2^{1/q} \mathcal{L}^2(\Omega)^{1/2}}{a_1^{1/q}} \left\{ \frac{\int_{\Omega} \left( e^{\|u\|^p} - 1 + a_2 \right) dx}{a_2 \mathcal{L}^2(\Omega)} \right\}^{1/q}
\]

\[
\leq C_{\varphi} \int_{\Omega} \left( e^{\|u\|^p} - 1 \right) dx + C_{1, \varphi},
\]

where we can assume \( C_{\varphi} > 1 \) and \( C_{1, \varphi} > 0 \) without loss of generality. Hence

\[
\mathcal{J}_{\varphi}(u) \geq R^2 \left[ \frac{1}{2} - \frac{C_{\alpha, \vartheta, \varphi}}{\lambda_k^{1-\vartheta^2}} R^{1-4\vartheta^2} \right] - C_{2, \varphi} \tag{3.65}
\]

where \( C_{\alpha, \vartheta, \varphi} = C_{\alpha, \vartheta} C_{\varphi} \) and \( C_{2, \varphi} = c_1 C_{\varphi} + C_{1, \varphi} > 0 \). Observe that \( \frac{1-4\vartheta^2}{1-\vartheta^2} > 0 \) for all \( 0 < \vartheta < 1/2 \); hence, we can choose \( R = R(k) \) such that

\[
\lambda_k^{1/\alpha} = \lambda_k^{(1-4\vartheta^2)/(1-\vartheta^2)} = 4C_{\alpha, \vartheta, \varphi} R^{1-4\vartheta^2}. 
\]
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Since \( \lambda_k \geq c_4k \) for large \( k \) (being \( n = 2 \)), \( R \) is subjected to the lower bound

\[
R^2 \geq \left[ \frac{c_4^{1/\alpha}}{4C_{\alpha,\varphi}} \right]^{2\alpha} k^2, \tag{3.66}
\]

where we may assume that \( 0 < c_4 < 1 \) without loss of generality; we remark that \( \vartheta = \vartheta(k) \).

Combining (3.65) with (3.66) yields the following estimate from below:

\[
J_\varphi(u) \geq \left[ \frac{c_4^{1/\alpha}}{4C_{\alpha,\varphi}} \right]^{2\alpha} k^2, \tag{3.67}
\]

which holds for \( k \) large enough. It remains to prove that the constant comparing in the right hand side of inequality (3.67), which depends on \( \vartheta \), may be bounded from below uniformly. By (3.63), recalling that \( 0 < c_4 < 1 \) and \( C_{\alpha,\varphi} \geq 1 \)

\[
\left[ \frac{c_4^{1/\alpha}}{4C_{\alpha,\varphi}} \right]^{2\alpha} \geq \left[ \frac{c_4}{4c_2C_{\varphi}} \right]^{2\alpha} \cdot \left( \frac{\alpha - 1}{\alpha} \right)^{2(\alpha - 1)p\vartheta}. \tag{3.68}
\]

But \( \frac{\alpha - 1}{\alpha} = \frac{3\vartheta^2}{1 - \vartheta^2} \) so that

\[
\left( \frac{\alpha - 1}{\alpha} \right)^{2(\alpha - 1)p\vartheta} \to 1, \quad \left[ \frac{c_4}{4c_2C_{\varphi}} \right]^{2\alpha} \to C_1 > 0
\]

as \( \vartheta \to 0 \). Therefore, we obtain that

\[
\left[ \frac{c_4^{1/\alpha}}{4C_{\alpha,\varphi}} \right]^{2\alpha} \geq C
\]

for all \( \vartheta \) small enough, where \( C > 0 \) is a constant independent on \( \vartheta \). By (3.60),

\[
b_k = \inf_{\gamma \in D_k} \max_{u \in D_k} J_\varphi(\gamma(u)) \geq \inf_{u \in \partial BR \cap Y_{k+1}^{-1}} J_\varphi(u). \tag{3.69}
\]

By combining (3.67) with (3.68), for \( k \) large enough there exists \( R = R(k) \in (0, R_k) \) such that for all \( u \in \partial BR \cap Y_{k+1}^{-1} \)

\[
J_\varphi(u) \geq Ck^2,
\]

and the proof is now complete. \( \Box \)

3.2.5 The growth estimate from above

Definition 3.2.14. We denote by \( U_k \) the set of \( \xi = tu_{k+1} + w \) such that

\[
0 \leq t \leq R_{k+1}, \quad w \in B(0, R_{k+1}) \cap \mathcal{Y}_k, \quad \|\xi\|_{1,2} \leq R_{k+1}.
\]

We denote by \( \Lambda_k \) the set of \( \lambda \in C(U_k, H_0^1(\Omega)) \) such that

\[
\lambda|_{\partial\Omega} \in \Gamma_{k+1}, \quad \lambda|_{\partial B(0,R_{k+1}) \cup \mathcal{Y}_k \cup B(0,R_k) \cap \mathcal{Y}_k} = Id
\]
and we set
\[ c_k = \inf_{\lambda \in A_k} \max_{u \in U_k} \tilde{f}_\varphi(\lambda(u)). \]

The next is our main existence tool.

**Lemma 3.2.15.** Assume that \( c_k > b_k \geq \wbar{M} \) for \( k \) large. If \( \delta \in ]0, c_k - b_k[ \) and
\[ A_k(\delta) = \{ \lambda \in A_k : \tilde{f}_\varphi(\lambda(u)) \leq b_k + \delta \text{ for } u \in D_k \}, \]
set
\[ c_k(\delta) = \inf_{\lambda \in A_k(\delta)} \max_{u \in U_k} \tilde{f}_\varphi(\lambda(u)). \]

Then \( c_k(\delta) \) is a critical value for \( \tilde{f}_\varphi \).

**Proof.** See [111, Lemma 5.5]. Of course, in this nonsmooth framework, we apply [37, Theorem 1.1.13] instead of the deformation Lemma for smooth functionals (see e.g. Lemma 1.60 of [92]).

**Lemma 3.2.16.** Let \( c_k = b_k \) for \( k \) large. Then there exist \( \gamma > 0 \) and \( k_1 \in \mathbb{N} \) such that
\[ \forall k \geq k_1 : b_k \leq \gamma k^{q/(q-1)} \]
for each \( q > 2 \).

**Proof.** Let \( q > 2 \). Following [111, Lemma 2.2], there exists \( \alpha_{\varphi,q} > 0 \) such that
\[ |\tilde{f}_\varphi(u) - \tilde{f}_\varphi(-u)| \leq \alpha_{\varphi,q} \left( |\tilde{f}_\varphi(u)|^{1/q} + 1 \right) \]
for each \( u \in H^1_0(\Omega) \). At this point argue as in [111, Lemma 5.6].

### 3.2.6 Proof the main result

Let us consider values of \( k \) such that \( c_k \geq b_k \geq \wbar{M} \). By assertion (b) of Theorem 3.2.8 the functional \( \tilde{f}_\varphi \) satisfies the concrete Palais–Smale condition at level \( c_k \). Since \( q/(q-1) < 2 \), by combining Lemma 3.2.13 and Lemma 3.2.16 we deduce that \( c_k > b_k \), so that we may apply Lemma 3.2.15 and obtain that \( c_k(\delta) \) is a critical value for \( \tilde{f}_\varphi \). Therefore, by (a) of Theorem 3.2.8, \( f_\varphi \) admits a diverging sequence of critical values (hence of weak solutions of (3.31)). To cover the case of a general nonlinearity \( \varphi \), it suffices to apply slight adaptations to several of the Lemmas (see [92]).
3.3 Systems with non–homogeneous boundary data

3.3.1 Introduction

Since the early seventies, many authors have widely investigated existence and multiplicity of solutions for semilinear elliptic problems with Dirichlet boundary conditions, especially by means of variational methods (see [113] and references therein). In particular, if \( \varphi \) is a real \( L^2 \) function on a bounded domain \( \Omega \subset \mathbb{R}^n \), \( p > 2 \) and \( p < 2^* \) if \( n \geq 3 \) (here, \( 2^* = \frac{2n}{n-2} \)), the following model problem

\[
\begin{align*}
-\Delta u &= |u|^{p-2} u + \varphi \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

has been extensively studied, even when the nonlinear term is more general.

If \( \varphi = 0 \), the problem is symmetric, so multiplicity results have been achieved via the equivariant Lusternik–Schnirelman theory and the notion of genus for \( \mathbb{Z}_2 \)-symmetric sets (see [93] and references therein).

On the contrary, if \( \varphi \neq 0 \), the problem loses its \( \mathbb{Z}_2 \)-symmetry and a natural question is whether the infinite number of solutions persists under perturbation of the odd equation. In this case, a detailed analysis was carried on by Rabinowitz in [92], Struwe in [114], Bahri and Berestycki in [16], Dong and Li in [55] and Tanaka in [117]: the existence of infinitely many solutions was obtained via techniques of classical critical point theory provided that a suitable restriction on the growth of the exponent \( p \) is assumed.

Furthermore, Bahri and Lions have improved some of such results via a technique based on Morse theory (see [18, 19]); while, more recently, Paleari and Squassina have extended some of the above mentioned achievements to the quasilinear case by means of techniques of nonsmooth critical point theory (see [111]).

Other perturbation results were obtained by Bahri and Berestycki in [16] and by Ambrosetti in [4] when \( p > 2 \) is any but subcritical: in particular, they proved that for each \( \nu \in \mathbb{N} \) there exists \( \varepsilon > 0 \) such that \( (\mathcal{P}_{0,\varphi,1}) \) has at least \( \nu \) distinct solutions provided that \( ||\varphi||_2 < \varepsilon \).

The success in looking for solutions of a non–symmetric problem as \( (\mathcal{P}_{0,\varphi,1}) \) made quite interesting to study the problem

\[
\begin{align*}
-\Delta u &= |u|^{p-2} u + \varphi \quad \text{in } \Omega \\
\quad u &= \chi \quad \text{on } \partial \Omega,
\end{align*}
\]

where, in general, the boundary condition \( \chi \) is different from zero. Some multiplicity results for \( \mathcal{P}_{\chi,\varphi,1} \) have been proved in [31, 32] provided that

\[
2 < p < 2 \frac{n+1}{n}, \quad \chi \in C(\partial \Omega, \mathbb{R}) \cap H^{1/2}(\partial \Omega, \mathbb{R}), \quad \varphi \in L^2(\Omega, \mathbb{R}).
\]

The upper bound to \( p \) seems to be a natural extension of the assumption \( 2 < p < 4 \) considered by Ekeland, Ghoussoub and Tehrani in [56] in order to solve such a problem when \( n = 1 \) (in this case, the range \( p < 2 \) was covered by Clarke and Ekeland in a previous paper [45]).

We stress that an improvement of the results in [31, 56] has been reached with a different technique by Bolle in [24] and Bolle, Ghoussoub and Tehrani in [25]. From one hand, they...
prove that if $\Omega \subset \mathbb{R}^n$ is a $C^2$ bounded domain and

$$2 < p < \frac{2n}{n-1}, \quad \chi \in C^2(\partial \Omega, \mathbb{R}), \quad \varphi \in C(\overline{\Omega}, \mathbb{R}),$$

then $(\mathcal{P}_{\chi,\varphi,1})$ has infinitely many classical solutions. On the other hand, they show that in the case $n = 1$ it suffices to assume $p > 2$, namely the result becomes optimal.

It remains open, even for $\chi = 0$, the problem of whether $(\mathcal{P}_{\chi,\varphi,1})$ has an infinite number of solutions for $p$ all the way up to $2^*$. For $\chi = 0$, the most satisfactory result remains the one contained in the celebrated paper [19] of Bahri and Lions where they prove that this fact is true for a subset of $\varphi$ dense in $L^2(\Omega, \mathbb{R})$.

Let us fix $N \geq 1$. The purpose of this section is to show the multiplicity of solutions for the following semilinear elliptic system

$$-\sum_{i,j=1}^{n} \sum_{h=1}^{N} D_j (a_{ijh}(x) D_i u_h) = b(x)|u|^{p-2}u_k(x) \quad \text{in } \Omega$$

$$u = \chi \quad \text{on } \partial \Omega$$

$$(\mathcal{P}_{\chi,\varphi,N})$$

taken any $\chi \in H^{1/2}(\partial \Omega, \mathbb{R}^N)$. Clearly, $(\mathcal{P}_{\chi,\varphi,N})$ reduces to the problem $(\mathcal{P}_{\chi,\varphi,1})$ if $N = 1$, $a_{ijh} = \delta_{ijh}$ and $b(x) = 1$.

To our knowledge no other result can be found in the literature about multiplicity for systems of semilinear elliptic equations with non–homogeneous boundary conditions; on the contrary, some multiplicity results are known in the case of Dirichlet boundary conditions (see [44] for the semilinear case and [111, 101] for some extensions to the quasilinear case).

It is well known that the functional $f : \mathcal{U}_\chi \to \mathbb{R}$ associated with $(\mathcal{P}_{\chi,\varphi,N})$ is given by

$$f(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ijh}(x) D_i u_h D_j u_k \, dx - \frac{1}{p} \int_\Omega b(x)|u|^p \, dx - \int_\Omega \varphi \cdot u \, dx$$

where $\mathcal{U}_\chi = \{ u \in H^1(\Omega, \mathbb{R}^N) : u = \chi \text{ a.e. on } \partial \Omega \}$.

In the next, $\Omega$ will denote a Lipschitz bounded domain of $\mathbb{R}^n$ with $n \geq 3$ while we shall always assume that the coefficients $a_{ijh}$ and $b$ belong to $C(\overline{\Omega}, \mathbb{R})$ with $a_{ijh} = a_{jih}$ and $b > 0$. Moreover, there exists $\nu > 0$ such that

$$\sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ijh}(x) \xi_i \xi_j \eta_h \eta_k \geq \nu |\xi|^2 |\eta|^2$$

(3.69)

for all $x \in \Omega$ and $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^N$ (Legendre–Hadamard condition).

Here, we state our main results.

**Theorem 3.3.1.** Let $p \in [2, 2^{n+1}/n[$. Then for each $\varphi \in L^2(\Omega, \mathbb{R}^N)$ and $\chi \in H^{1/2}(\partial \Omega, \mathbb{R}^N)$ the system $(\mathcal{P}_{\chi,\varphi,N})$ admits a sequence $(u^m)_m$ of solutions in $\mathcal{U}_\chi$ such that $f(u^m) \to +\infty$. 
In order to prove Theorem 3.3.1, we use some perturbation arguments developed in [16, 92, 114]; so the condition $p < 2 \frac{n+1}{n}$ is quite natural.

An improvement of such a “control” can be obtained by means of the Bolle’s techniques, but more assumptions need. In fact, all the weak solutions must be regular and the system has to be diagonal, i.e. $a_{ij}^{hk} = \delta_{ij}^{hk}$.

More precisely, we can prove the following theorem.

**Theorem 3.3.2.** Let $p \in [2, \frac{2n}{n-1}]$, $\partial \Omega$ is of class $C^2$, $\chi \in C^2(\partial \Omega, \mathbb{R}^N)$, $\varphi \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ for some $\alpha \in [0, 1]$ and $a_{ij}^{hk} = \delta_{ij}^{hk}$. Then $(\mathcal{P}_{x, \varphi, N})$ has a sequence $(u^m)_m$ of classical solutions such that $f(u^m) \to +\infty$.

Clearly, Theorems 3.3.1 and 3.3.2 extend the results of [31, 32] and [25] to semilinear elliptic systems. We underline that (3.69) is weaker than the strong ellipticity condition.

Let us point out that, in general, whereas De Giorgi’s famous example of an unbounded weak solution of a linear elliptic system shows (cf. [53]), we can not hope to find everywhere regular solutions for coefficients $a_{ij}^{hk} \in L^\infty(\Omega, \mathbb{R})$. Anyway, if $a_{ij}^{hk} \in C(\overline{\Omega}, \mathbb{R})$ and (3.69) holds we have that if $u$ solves $(\mathcal{P}_{x, \varphi, N})$ then

$$u \in C^{0,\alpha}(\Omega, \mathbb{R}^N)$$

for each $\alpha \in ]0, 1[$ (see [62]); but if we look for classical solutions, namely $u$ of class $C^2$ on $\overline{\Omega}$, the coefficients $a_{ij}^{hk}$ have to be sufficiently smooth while $\varphi \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ for some $\alpha \in [0, 1]$ and $\chi \in C^2(\partial \Omega, \mathbb{R}^N)$ (see [72] and references therein).

### 3.3.2 Reduction to homogeneous boundary conditions

As a first step, let us reduce $(\mathcal{P}_{x, \varphi, N})$ to a Dirichlet type problem. To this aim, let us denote by $\phi \in \mathcal{M}_\chi$ the only solution of the linear system

$$
\begin{cases}
- \sum_{i,j=1}^n \sum_{h=1}^N D_j(a_{ij}^{hk}(x)D_i\phi_h) = 0 & \text{in } \Omega \\
\phi = \chi & \text{on } \partial \Omega \\
k = 1, \ldots, N
\end{cases}
$$

(3.70)

Since $p < 2^*$, it results $\phi \in L^p(\Omega, \mathbb{R}^N)$.

From now on, we shall assume that $b = 1$. Taking into account that there exist two positive constants $m_b$ and $M_b$ such that

$$m_b \leq b(x) \leq M_b \quad \text{for all } x \in \overline{\Omega},$$

the general case can be covered by slight modifies of some lemmas proved in the next sections.

It is easy to show that the following fact holds :

**Proposition 3.3.3.** $u \in \mathcal{M}_\chi$ is a solution of $(\mathcal{P}_{x, \varphi, N})$ if and only if $z \in H^1_0(\Omega, \mathbb{R}^N)$ solves

$$
\begin{cases}
- \sum_{i,j=1}^n \sum_{h=1}^N D_j(a_{ij}^{hk}(x)D_i z_h) = |z + \phi|^{p-2}(z_k + \phi_k) + \varphi_k(x) & \text{in } \Omega \\
z = 0 & \text{on } \partial \Omega \\
k = 1, \ldots, N
\end{cases}
$$
where \( u(x) = z(x) + \phi(x) \) for a.e. \( x \in \Omega \).

Therefore, in order to find solutions of our problem it is enough looking for critical points of the \( C^1 \)-functional \( f_\chi : H^1_0(\Omega, \mathbb{R}^N) \rightarrow \mathbb{R} \) given by

\[
f_\chi(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^N a^{ij}_{kk}(x) D_i u_h D_j u_k \, dx - \frac{1}{p} \int_\Omega |u + \phi|^p \, dx - \int_\Omega \varphi \cdot u \, dx
\]

(we refer the reader to [93, 113] for some recalls of classical critical point theory).

**Lemma 3.3.4.** There exists \( A > 0 \) such that if \( u \in H^1_0(\Omega, \mathbb{R}^N) \) is a critical point of \( f_\chi \), then

\[
\int_\Omega |u + \phi|^p \, dx \leq pA \left( f_\chi^2(u) + 1 \right)^\frac{1}{2}.
\]

**Proof.** By Young’s inequality, for each \( \varepsilon > 0 \) there exist \( \alpha_\varepsilon, \beta_\varepsilon > 0 \) such that

\[
|u + \phi|^{p-1} |\phi| \leq \varepsilon |u + \phi|^p + \alpha_\varepsilon |\phi|^p, \quad |u + \phi||\varphi| \leq \varepsilon |u + \phi|^p + \beta_\varepsilon |\varphi|^p,
\]

with \( \frac{1}{p} + \frac{1}{p'} = 1 \).

Therefore, if \( u \) is a critical point of \( f_\chi \), we get

\[
f_\chi(u) = f_\chi(u) - \frac{1}{2} f'_\chi(u)[u]
\]

\[
= \left( \frac{1}{2} - \frac{1}{p} \right) \int_\Omega |u + \phi|^p \, dx - \frac{1}{2} \int_\Omega |u + \phi|^{p-2} (u + \phi) \cdot \phi \, dx - \frac{1}{2} \int_\Omega \varphi \cdot u \, dx
\]

\[
\geq \frac{p-2}{2p} \int_\Omega |u + \phi|^p \, dx - \frac{1}{2} \int_\Omega |u + \phi|^{p-1} |\phi| \, dx - \frac{1}{2} \int_\Omega (|u + \phi||\varphi| + |\varphi||\phi|) \, dx
\]

\[
\geq \left( \frac{p-2}{2p} - \varepsilon \right) \int_\Omega |u + \phi|^p \, dx - \frac{1}{2} \left( \alpha_\varepsilon \|\phi\|^p_p + \beta_\varepsilon \|\varphi\|^p_{p'} + \|\varphi\|_2 \|\phi\|_2 \right).
\]

Choosing \( \varepsilon \) such that \( p - 2 - 2p\varepsilon > 0 \), i.e., \( \varepsilon \in (0, \frac{1}{2} - \frac{1}{p} [ \), we get

\[
pM_\varepsilon f_\chi(u) \geq \int_\Omega |u + \phi|^p \, dx - pM_\varepsilon \gamma_\varepsilon(p, \phi, \varphi),
\]

where \( M_\varepsilon = \frac{2}{p-2-2p\varepsilon} \) and

\[
\gamma_\varepsilon(p, \phi, \varphi) = \frac{1}{2} \left( \alpha_\varepsilon \|\phi\|^p_p + \beta_\varepsilon \|\varphi\|^p_{p'} + \|\varphi\|_2 \|\phi\|_2 \right).
\]

At this point, the assertion follows by \( A \geq \sqrt{2M_\varepsilon} \max\{1, \gamma_\varepsilon(p, \phi, \varphi)\} \). \( \square \)

Now, let \( \eta \in C^\infty(\mathbb{R}, \mathbb{R}) \) be a cut function such that \( \eta(s) = 1 \) for \( s \leq 1 \), \( \eta(s) = 0 \) for \( s \geq 2 \) while \( -2 < \eta'(s) < 0 \) when \( 1 < s < 2 \). For each \( u \in H^1_0(\Omega, \mathbb{R}^N) \) let us define

\[
\zeta(u) = 2pA \left( f_\chi^2(u) + 1 \right)^\frac{1}{2}, \quad \psi(u) = \eta \left( \zeta(u)^{-1} \int_\Omega |u + \phi|^p \, dx \right),
\]

(3.72)
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where \( A \) is as in Lemma 3.3.4. Finally, we introduce the modified functional \( \tilde{f}_\chi : H^1_0(\Omega, \mathbb{R}^N) \to \mathbb{R} \) in order to apply the techniques used in [31]:

\[
\tilde{f}_\chi(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx - \frac{1}{p} \int_\Omega |u|^p \, dx - \psi(u) \int_\Omega \Theta(x, u) \, dx,
\]

with

\[
\Theta(x, u) = \frac{|u + \phi|^p}{p} - \frac{|u|^p}{p} + \varphi \cdot u.
\]

Let us provide an estimate for the loss of symmetry of \( \tilde{f}_\chi \).

**Lemma 3.3.5.** There exists \( \beta > 0 \) such that

\[
|\tilde{f}_\chi(u) - \tilde{f}_\chi(-u)| \leq \beta \left( |\tilde{f}_\chi(u)|^{\frac{p-1}{p}} + 1 \right)
\]

for all \( u \in \text{supt}(\psi) \).

*Proof.* First of all, let us show that there exist \( c_1, c_2 > 0 \) such that there results

\[
\left| \int_\Omega |u + \phi|^p - |u|^p \, dx \right| \leq c_1 |f_\chi(u)|^{\frac{p-1}{p}} + c_2, \quad (3.73)
\]

\[
\left| \int_\Omega |u - \phi|^p - |u|^p \, dx \right| \leq c_1 |f_\chi(u)|^{\frac{p-1}{p}} + c_2, \quad (3.74)
\]

\[
\left| \int_\Omega \varphi \cdot u \, dx \right| \leq c_1 |f_\chi(u)|^{\frac{p-1}{p}} + c_2 \quad (3.75)
\]

for all \( u \in \text{supt}(\psi) \). In fact, taken any \( u \in H^1_0(\Omega, \mathbb{R}) \) it is easy to see that

\[
||u + \phi|^p - |u|^p| \leq p2^{p-2}|u + \phi|^{p-1}|\phi| + p2^{p-2}|\phi|^p, \quad (3.76)
\]

\[
||u - \phi|^p - |u|^p| \leq p2^{p-2}|u + \phi|^{p-1}|\phi| + p2^{p-3}|\phi|^p. \quad (3.77)
\]

Hence, by (3.76) we get

\[
\left| \int_\Omega |u + \phi|^p - |u|^p \, dx \right| \leq p2^{p-2}||\phi||_p \left( \int_\Omega |u + \phi|^p \, dx \right)^{\frac{p-1}{p}} + p2^{p-2}||\phi||_p^p,
\]

while (3.77) implies

\[
\left| \int_\Omega |u - \phi|^p - |u|^p \, dx \right| \leq p2^{p-2}||\phi||_p \left( \int_\Omega |u + \phi|^p \, dx \right)^{\frac{p-1}{p}} + p2^{p-3}||\phi||_p^p.
\]

Moreover, by Hölder and Young’s inequalities it results

\[
\left| \int_\Omega \varphi \cdot u \, dx \right| \leq \left( \int_\Omega |u + \phi|^p \, dx \right)^{\frac{p-1}{p}} + (p - 2) \left( \frac{||\phi||_{p'}}{p - 1} \right)^{\frac{p-1}{p}} + ||\varphi||_2 ||\phi||_2.
\]
If, furthermore, we assume \( u \in \text{supt}(\psi) \), it follows
\[
\int_{\Omega} |u + \phi|^p \, dx \leq 4pA(|f_X(u)| + 1)
\]
which implies (3.73), (3.74) and (3.75).

Then, again by Young’s inequality, simple calculations and (3.73), (3.75) give
\[
|f_X(u)| \leq a_1|f_X(u)| + a_2,
\]
for suitable \( a_1, a_2 > 0 \). The assertion follows by combining inequalities (3.73), (3.74), (3.75) and (3.78).

Now, we want to link the critical points of \( f_X \) to those ones of \( f_X \). To this aim we need more information about \( f_X \).

Taken \( u \in H_0^1(\Omega, \mathbb{R}^N) \), by direct computations we get
\[
\tilde{f}_X^i(u)[u] = (1 + T_i(u)) \int_{\Omega} \sum_{i,j=1}^n \sum_{b,k=1}^N a_{ij}^{bk}(x) D_i u_b D_j u_k \, dx
\]
\[
- (1 - \psi(u)) \int_{\Omega} |u|^p \, dx - (\psi(u) + T_1(u)) \int_{\Omega} \varphi \cdot u \, dx
\]
\[
- (\psi(u) + T_2(u)) \int_{\Omega} |u + \phi|^{p-2}(u + \phi) \cdot u \, dx,
\]
where \( T_1, T_2 : H_0^1(\Omega, \mathbb{R}^N) \to \mathbb{R} \) are defined by setting
\[
T_1(u) = 4p^2 A^2 \eta'(\delta(u)) \delta(u) \zeta(u)^{-2} f_X(u) \int_{\Omega} \Theta(x, u) \, dx,
\]
\[
T_2(u) = pn'(\delta(u)) \zeta(u)^{-1} \int_{\Omega} \Theta(x, u) \, dx + T_1(u),
\]
with \( \delta(u) = \zeta(u)^{-1} \int_{\Omega} |u + \phi|^p \, dx \).

**Remark 3.3.6.** In order to point out some properties of the maps \( T_1 \) and \( T_2 \) defined above, let us remark that by (3.73) and (3.75) there exist \( b_1, b_2 > 0 \) such that for all \( u \in \text{supt}(\psi) \) it is
\[
|T_i(u)| \leq b_1|f_X(u)|^{-\frac{1}{p}} + b_2|f_X(u)|^{-1} \quad \text{for both } i = 1, 2.
\]

Therefore, arguing as in [92] (see also [31, Lemma 2.9]), there exist \( \alpha_0, M_0 > 0 \) such that if \( M \geq M_0 \) then
\[
\tilde{f}_X(u) \geq M, \; u \in \text{supt}(\psi) \implies f_X(u) \geq \alpha_0 M;
\]
whence, it results \( |T_i(u)| \to 0 \) as \( M \to +\infty \) for \( i = 1, 2 \) (trivially, it is \( T_1(u) = T_2(u) = 0 \) if \( u \notin \text{supt}(\psi) \)).

**Theorem 3.3.7.** There exists \( M_1 > 0 \) such that if \( u \) is a critical point of \( \tilde{f}_X \) and \( \tilde{f}_X(u) \geq M_1 \) then \( u \) is a critical point of \( f_X \) and \( f_X(u) = \tilde{f}_X(u) \).
Proof. Let \( u \in H^1_0(\Omega, \mathbb{R}^N) \) be a critical point of \( \tilde{f}_\lambda \). By the definition of \( \psi \) it suffices to show that, if \( \tilde{f}_\lambda(u) \geq M_1 \) for a large enough \( M_1 \), then \( \delta(u) < 1 \), i.e.,

\[
\zeta(u)^{-1} \int_\Omega |u + \phi|^p \, dx < 1.
\]

By (3.79) we have

\[
f_\lambda(u) = f_\lambda(u) - \frac{1}{2(1 + T_1(u))} \tilde{f}_\lambda(u)[u]
\]

\[
= -\frac{1}{p} \int_\Omega |u + \phi|^p \, dx - \int_\Omega \varphi \cdot u \, dx + \frac{1}{2(1 + T_1(u))} \int_\Omega |u|^p \, dx
\]

\[
+ \frac{\psi(u) + T_1(u)}{2(1 + T_1(u))} \int_\Omega \varphi \cdot u \, dx + \frac{\psi(u) + T_2(u)}{2(1 + T_1(u))} \int_\Omega |u + \phi|^{p-2}(u + \phi) \cdot u \, dx
\]

\[
= \left( \frac{1}{2} - \frac{1}{p} \right) \int_\Omega |u + \phi|^p \, dx - \frac{T_1(u) - T_2(u)}{2(1 + T_1(u))} \int_\Omega |u|^p \, dx
\]

\[
+ \frac{1}{2} \left( \frac{\psi(u) + T_2(u)}{1 + T_1(u)} - 1 \right) \int_\Omega (|u + \phi|^p - |u|^p) \, dx
\]

\[
- \frac{\psi(u) + T_2(u)}{2(1 + T_1(u))} \int_\Omega |u + \phi|^{p-2}(u + \phi) \cdot \phi \, dx
\]

\[
- \left( 1 - \frac{\psi(u) + T_1(u)}{2(1 + T_1(u))} \right) \int_\Omega \varphi \cdot u \, dx.
\]

Then, by Remark 3.3.6 it is possible to choose \( M_1 > 0 \) so large that

\[
\left| \frac{1 - \psi(u)}{1 + T_1(u)} \right| \leq 2, \quad \left| \frac{\psi(u) + T_1(u)}{1 + T_1(u)} \right| \leq 2,
\]

\[
\left| \frac{\psi(u) + T_2(u)}{1 + T_1(u)} - 1 \right| \leq 2, \quad \left| \frac{\psi(u) + T_2(u)}{1 + T_1(u)} \right| \leq 2;
\]

so, working as in the proof of [32, Proposition 2.6], we deduce that for each \( \varepsilon > 0 \) there exist \( h_\varepsilon, \tilde{\gamma}_\varepsilon(p, \phi, \varphi) > 0 \) such that

\[
f_\lambda(u) \geq \left( \frac{p-2}{2p} - 2^{p-2} \left| \frac{T_2(u) - T_1(u)}{1 + T_1(u)} \right| - h_\varepsilon \right) \int_\Omega |u + \phi|^p \, dx - \tilde{\gamma}_\varepsilon(p, \phi, \varphi)
\]

where \( h_\varepsilon \to 0 \) as \( \varepsilon \to 0 \). At this point, choosing a priori \( \varepsilon \) and \( M_1 \) in such a way that

\[
2^{p-2} \left| \frac{T_2(u) - T_1(u)}{1 + T_1(u)} \right| + h_\varepsilon \leq \frac{p - 2}{4p},
\]

we obtain

\[
f_\lambda(u) \geq \frac{p - 2}{4p} \int_\Omega |u + \phi|^p \, dx - \tilde{\gamma}_\varepsilon(p, \phi, \varphi),
\]

which completes the proof if, as in Lemma 3.3.4, the constant \( A \) taken in the definition (3.72) is large enough. \( \square \)
3.3.3 The Palais–Smale condition

Let us point out that, in the check of the Palais–Smale condition for semilinear elliptic systems under the assumption (3.69), an important role is played by the so called Gårding’s inequality.

Lemma 3.3.8. Let \((u^m)_m\) be a bounded sequence in \(H^1_0(\Omega, \mathbb{R}^N)\) and let \((w^m)_m\) be a strongly convergent sequence in \(H^{-1}(\Omega, \mathbb{R}^N)\) such that

\[
\int_{\Omega} \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u^m_h D_j v_k \, dx = \langle w^m, v \rangle \quad \text{for all} \quad v \in H^1_0(\Omega, \mathbb{R}^N).
\]

Then \((u^m)_m\) has a subsequence \((u^{m_k})_k\) strongly convergent in \(H^1_0(\Omega, \mathbb{R}^N)\).

Proof. First of all, in our setting the following Gårding type inequality holds: taken \(\nu\) as in (3.69) for each \(\varepsilon \in ]0, \nu[\) there exists \(c_\varepsilon \geq 0\) such that

\[
\int_{\Omega} \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u^m_h D_j u^m_k \, dx \geq (\nu - \varepsilon) \|Du\|_2^2 - c_\varepsilon \|u\|_2^2
\]

for all \(u \in H^1_0(\Omega, \mathbb{R}^N)\) (see [85, Theorem 6.5.1]). Therefore, fixed \(\varepsilon > 0\), we have

\[
\langle w^l - w^m, u^l - u^m \rangle = \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i (u^l_h - u^m_h) D_j (u^l_k - u^m_k) \, dx \\
\geq (\nu - \varepsilon) \|Du^l - Du^m\|_2^2 - c_\varepsilon \|u^l - u^m\|_2^2
\]

for all \(m, l \in \mathbb{N}\). Since \(u^m \rightharpoonup u\) in \(L^2(\Omega, \mathbb{R}^N)\), up to subsequences, we can conclude that \(Du^m \rightharpoonup Du\) in \(L^2(\Omega, \mathbb{R}^N)\). \(\square\)

Now, let \(d \geq 0\) be such that

\[
\int_{\Omega} \left( \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u^m_h D_j u^m_k + d|u|^2 \right) \, dx \geq \frac{\nu}{2} \|Du\|_2^2
\]

(3.80)

for all \(u \in H^1_0(\Omega, \mathbb{R}^N)\).

Lemma 3.3.9. There exists \(M_2 > 0\) such that if \((u^m)_m\) is a \((PS)_c\)-sequence of \(\bar{f}_\chi\) with \(c \geq M_2\), then \((u^m)_m\) is bounded in \(H^1_0(\Omega, \mathbb{R}^N)\).

Proof. Let \(M_2 > 0\) be fixed and consider \((u^m)_m\), a \((PS)_c\)-sequence of \(\bar{f}_\chi\), with \(c \geq M_2\), such that

\[
M_2 \leq \bar{f}_\chi(u^m) \leq K,
\]

for a certain \(K > M_2\).

First of all, let us remark that if there exists a subsequence \((u^{m_k})_k\) such that \(u^{m_k} \notin \text{supt}(\psi)\) for all \(k \in \mathbb{N}\), then it is a Palais–Smale sequence for the symmetric functional

\[
f_0(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx - \frac{1}{p} \int_{\Omega} |u|^p \, dx
\]
in $H^1_0(\Omega, \mathbb{R}^N)$. Whence, it is easier to prove that such a subsequence is bounded. So, we can assume $u^m \in \text{supt}(\psi)$ for all $m \in \mathbb{N}$.

For $m \in \mathbb{N}$ large enough and any $\rho > 0$, taken $d$ as in (3.80) by (3.79) it results

\begin{align*}
K + \rho \| Du^m \|_2 &\geq \tilde{f}_\lambda(u^m) - \rho \tilde{f}_\lambda'(u^m)[u^m] \\
&= \frac{1}{2} (1 - 2\rho (1 + T_1(u^m))) \int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^h(x) D_i u^m D_j u^m + d|u^m|^2) \, dx \\
&\quad - \frac{d}{2} (1 - 2\rho (1 + T_1(u^m))) \| u^m \|_2^2 + \left( \rho (1 - \psi(u^m)) - \frac{1}{p} \right) \int_\Omega |u^m|^p \, dx \\
&\quad + \rho (\psi(u^m) + T_2(u^m)) \int_\Omega |u^m + \phi|^{p-2}(u^m + \phi) \cdot u^m \, dx \\
&\quad + \rho (\psi(u^m) + T_1(u^m)) \int_\Omega \varphi \cdot u^m \, dx - \psi(u^m) \int_\Omega \Theta(x, u^m) \, dx.
\end{align*}

Since it is $p > 2$, we can fix, a priori, a constant $h \in ]1, \frac{p}{2}[\, \text{ such that, taken } \mu \in ]0, 1 - 2\frac{h}{p}[\,, \rho \in ]h, \frac{1-\mu}{2}[\, \text{ and } \bar{\mu} \in ]0, \rho (1 - \frac{1}{h})]\,, \text{ by Remark 3.3.6 if } M_2 \text{ is large enough for all } m \in \mathbb{N} \text{ we have}

$$ |T_1(u^m)| < \min \left\{ 1, \frac{1-\mu}{2\rho} - 1 \right\}, \quad |T_2(u^m)| < 1 - \frac{1}{h} - \frac{\bar{\mu}}{\rho} $$

and then

$$ \mu < 1 - 2\rho (1 + T_1(u^m)) \leq 1, \quad (3.81) $$

$$ \bar{\mu} \leq \rho (1 + T_2(u^m)) - \frac{1}{p}. \quad (3.82) $$

So, by (3.80) and (3.82) we obtain

\begin{align*}
K + \rho \| Du^m \|_2 &\geq \frac{\mu}{4} \| Du^m \|_2^2 - \frac{d}{2} \| u^m \|_2^2 + \left( \rho (1 - T_2(u^m)) - \frac{1}{p} \right) \int_\Omega |u^m|^p \, dx \\
&\quad - (\rho (1 + |T_1(u^m)|) + 1) \int_\Omega |\varphi| |u^m| \, dx - \rho (1 + |T_2(u^m)|) \int_\Omega |u^m + \phi|^{p-1} |\phi| \, dx \\
&\quad + \left( \rho (\psi(u^m) + T_2(u^m)) - \frac{\psi(u^m)}{p} \right) \int_\Omega (|u^m + \phi|^p - |u^m|^p) \, dx.
\end{align*}

Hence, fixed any $\varepsilon > 0$, by (3.71), (3.82) and a suitable choice of the positive constants $a_1$ and $a_2^2$ there results

\begin{align*}
K + \rho \| Du^m \|_2 + \frac{d}{2} \| u^m \|_2^2 &\geq \frac{\mu}{4} \| Du^m \|_2^2 + \left( \bar{\mu} - \varepsilon a_1 \right) \| u^m \|_p \\
&\quad + \left( \rho (\psi(u^m) + T_2(u^m)) - \frac{\psi(u^m)}{p} \right) \int_\Omega (|u^m + \phi|^p - |u^m|^p) \, dx - a_2^2.
\end{align*}

Let us point out that, as $u^m \in \text{supt}(\psi)$, (3.73) and (3.78) imply

$$ \left( \int_\Omega (|u^m + \phi|^p - |u^m|^p) \, dx \right)_{m \in \mathbb{N}} \text{ is bounded.} $$

Whence, $p > 2$ and a suitable choice of $\varepsilon$ small enough allow to complete the proof. \qed
Lemma 3.3.10. Let $M_2$ be as in Lemma 3.3.9 and $c \geq M_2$. Then, taken any $(PS)_c$-sequence $(u^m)_m$ for $\tilde{f}_\chi$, the sequence

$$\tilde{g}(x, u^m) = |u^m|^{-2} u^m + \psi(u^m) \Theta(x, u^m) + \psi'(u^m) \int_\Omega \Theta(x, u^m) \, dx$$

admits a convergent subsequence in $H^{-1}(\Omega, \mathbb{R}^N)$.

Proof. Follow the steps of [111, Lemma 3.3].

Theorem 3.3.11. The functional $\tilde{f}_\chi$ satisfies the Palais–Smale condition at each level $c \in \mathbb{R}$ with $c \geq M_2$, where $M_2$ is as in Lemma 3.3.9.

Proof. Let $(u^m)_m$ be a Palais–Smale sequence for $\tilde{f}_\chi$ at level $c \geq M_2$. Therefore, $(u^m)_m$ is bounded in $H^1_0(\Omega, \mathbb{R}^N)$ and by Lemma 3.3.10, up to a subsequence, $(\tilde{g}(x, u^m))_m$ is strongly convergent in $H^{-1}(\Omega, \mathbb{R}^N)$. Hence, the assertion follows by Lemma 3.3.8 applied to $w^m = \tilde{g}(x, u^m) + \tilde{f}_\chi'(u^m)$ where, by assumption, $\tilde{f}_\chi'(u^m) \to 0$ in $H^{-1}(\Omega, \mathbb{R}^N)$.

3.3.4 Comparison of growths for min–max values

In this section we shall build two min–max classes for $\tilde{f}_\chi$ and then we compare the growth of the associated min–max values.

Let $(\lambda^l, u^l)_l$ be a sequence in $\mathbb{R} \times H^1_0(\Omega, \mathbb{R}^N)$ such that

\[
\begin{aligned}
-\Delta u^l_k &= \lambda^l u^l_k & \text{in } \Omega \\
u^l &= 0 & \text{on } \partial \Omega, \\
k &= 1, \ldots, N,
\end{aligned}
\]

with $(u^l)_l$ orthonormalized. Let us consider the finite dimensional subspaces

$$V_0 := \langle u^0 \rangle; \quad V_{l+1} := V_l \oplus \mathbb{R} u^{l+1} \text{ for any } l \in \mathbb{N}.$$  

Fixed $l \in \mathbb{N}$ it is easy to check that some constants $\beta_1, \beta_2, \beta_3, \beta_4 > 0$ exist such that

$$\tilde{f}_\chi(u) \leq \beta_1 \|u\|_{1,2}^2 - \beta_2 \|u\|_{1,2}^p - \beta_3 \|u\|_{1,2} - \beta_4, \quad \text{for all } u \in V_l.$$ 

Then, there exists $R_l > 0$ such that

$$u \in V_l, \quad \|u\|_{1,2} \geq R_l \quad \implies \quad \tilde{f}_\chi(u) \leq \tilde{f}_\chi(0) \leq 0.$$ 

Definition 3.3.12. For any $l \geq 1$ we set $D_l = V_l \cap B(0, R_l),$

$$\Gamma_l = \left\{ \gamma \in C(D_l, H^1_0(\Omega, \mathbb{R}^N)) : \gamma \text{ odd and } \gamma|_{\partial B(0, R_l)} = Id \right\},$$

and

$$b_l = \inf_{\gamma \in \Gamma_l} \max_{u \in D_l} \tilde{f}_\chi(\gamma(u)).$$

In order to prove some estimates on the growth of the levels $b_l$, a result due to Tanaka (cf. [117]) implies the following lemma.
Lemma 3.3.13. There exist \( \beta > 0 \) and \( l_0 \in \mathbb{N} \) such that
\[
b_l \geq \beta l^{\frac{2p}{n(p-2)}} \quad \text{for all } l \geq l_0.
\]

Proof. By (3.80) and simple calculations \( a_1, a_2 > 0 \) exist such that
\[
\tilde{f}_X(u) \geq \frac{\nu}{4} \|Du\|_2^2 - a_1\|u\|_p^p - a_2 \quad \text{for all } u \in \partial B(0, R_l) \cap V_{l-1}^+.
\]
Then, it is enough to follow the proof of [117, Theorem 1].

Now, let us introduce a second class of min–max values to be compared with \( b_l \).

Definition 3.3.14. Taken \( l \in \mathbb{N} \), define
\[
U_l = \{ \xi = tu^l + w : 0 \leq t \leq R_{l+1}, \ w \in B(0,R_{l+1}) \cap V_l, \ \|\xi\|_{1,2} \leq R_{l+1} \}
\]
and
\[
\Lambda_l = \left\{ \lambda \in C(U_l, H_0^1(\Omega, \mathbb{R}^N)) : \lambda|_{D_l} \in \Gamma_l \quad \text{and} \quad \lambda|_{\partial B(0,R_{l+1}) \cup (B(0,R_{l+1}) \setminus \overline{D_l})} = Id \right\}.
\]
Assume
\[
c_l = \inf_{\lambda \in \Lambda_l} \max_{u \in U_l} \tilde{f}_X(\lambda(u)).
\]

The following result is the concrete version of Theorem 1.1.11.

Lemma 3.3.15. Assume \( c_l > b_l \geq \max\{M_1, M_2\} \). Taken \( \delta \in ]0, c_l - b_l[ \), let us set
\[
\Lambda_l(\delta) = \left\{ \lambda \in \Lambda_l : \tilde{f}_X(\lambda(u)) \leq b_l + \delta \quad \text{for all} \quad u \in D_l \right\},
\]
\[
c_l(\delta) = \inf_{\lambda \in \Lambda_l(\delta)} \max_{u \in U_l} \tilde{f}_X(\lambda(u)).
\]
Then, \( c_l(\delta) \) is a critical value for \( \tilde{f}_X \).

Proof. The proof can be obtained by arguing as in [92, Lemma 1.57].

Now, we prove that the situation \( c_l = b_l \) can not occur for all large \( l \).

Lemma 3.3.16. Assume that \( c_l = b_l \) for all \( l \geq l_1 \). Then there exists \( \gamma > 0 \) with
\[
b_l \leq \gamma l^p.
\]

Proof. Working as in [92, Lemma 1.64] it is possible to prove that
\[
b_{l+1} \leq b_l + \beta \left( |b_l|^{\frac{p-1}{p}} + 1 \right) \quad \text{for all } l \geq l_1.
\]
The assertion follows by [16, Lemma 5.3] .
Proof of Theorem 3.3.1. Observe that the inequality $2 < p < 2\frac{n+1}{n}$ implies

$$p < \frac{2p}{n(p-2)}.$$ 

Therefore, by Lemmas 3.3.13 and 3.3.16 it follows that there exists a diverging sequence $(l_n)_n \subset \mathbb{N}$ such that $c_{l_n} > b_{l_n}$ for all $n \in \mathbb{N}$, then Lemma 3.3.15 implies that $(c_{l_n}(\delta))_n$ is a sequence of critical values for $f_\delta$. Whence, by Theorem 3.3.7 the functional $f_\delta$ has a diverging sequence of critical values. 

Remark 3.3.17. When $p$ goes all the way up to $2^*$, in a similar fashion, one can prove that for each $\nu \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $(\mathcal{P}_{\varepsilon\cdot \varphi, N})$ has at least $\nu$ distinct solutions in $\mathcal{M}_\varepsilon$. This is possible since there exists $\beta > 0$ such that

$$|\tilde{f}_\varepsilon(u) - \tilde{f}_\varepsilon(-u)| \leq \varepsilon \beta \left( |f_\varepsilon(u)|^{\frac{p-1}{p}} + 1 \right),$$

for each $\varepsilon > 0$ and $u \in \text{supt}(\psi)$, where $\tilde{f}_\varepsilon : H^1_0(\Omega, \mathbb{R}^N) \to \mathbb{R}$ is defined by:

$$\tilde{f}_\varepsilon(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx +$$

$$- \frac{1}{p} \int_\Omega |u|^p \, dx - \psi_\varepsilon(u) \int_\Omega \Theta_\varepsilon(x, u) \, dx$$

with

$$\Theta_\varepsilon(x, u) = \frac{|u + \varepsilon \phi|^p}{p} - \frac{|u|^p}{p} + \varepsilon \varphi \cdot u, \quad \psi_\varepsilon(u) = \eta \left( \zeta(u)^{-1} \int_\Omega |u + \varepsilon \phi|^p \, dx \right)$$

(for more details in the scalar case, see [4, 16, 32]).

3.3.5 Bolle’s method for non-symmetric problems

In this section we briefly recall from [24] the theory devised by Bolle for dealing with problems with broken symmetry. The idea is to consider a continuous path of functionals starting from the symmetric functional $f_0$ and to prove a preservation result for min–max critical levels in order to get critical points also for the end–point functional $f_1$.

Let $\mathcal{X}$ be a Hilbert space equipped with the norm $\| \cdot \|$ and $f : [0, 1] \times \mathcal{X} \to \mathbb{R}$ a $C^2$–functional. Set $f_\theta = f(\theta, \cdot)$ if $\theta \in [0, 1]$.

Assume that $\mathcal{X} = \mathcal{X}_- \oplus \mathcal{X}_+$ and let $(e_l)_{l \geq 1}$ be an orthonormal base of $\mathcal{X}_+$ such that we can define an increasing sequence of subspaces as follows:

$$\mathcal{X}_0 := \mathcal{X}_-, \quad \mathcal{X}_{l+1} := \mathcal{X}_l \oplus \mathbb{R} e_{l+1} \text{ if } l \in \mathbb{N}.$$ 

Provided that $\dim(\mathcal{X}_-) < +\infty$, let us set

$$\mathcal{K} = \{ \zeta \in C(\mathcal{X}, \mathcal{X}) : \zeta \text{ is odd and } \zeta(u) = u \text{ if } \|u\| \geq R \}$$

for a fixed $R > 0$ and

$$c_l = \inf_{\zeta \in \mathcal{K}} \sup_{u \in \mathcal{X}_l} f_0(\zeta(u)).$$

Assume that
3.3. Systems with non–homogeneous boundary data

- $f$ satisfies a kind of Palais–Smale condition in $[0, 1] \times \mathcal{X}$: any $((\theta^m, u^m))_m$ such that

\[
(f(\theta^m, u^m))_m \text{ is bounded and } f'_\theta(u^m) \to 0 \text{ as } m \to +\infty \tag{3.83}
\]

converges up to subsequences;

- for any $b > 0$ there exists $C_b > 0$ such that

\[
|f_\theta(u)| \leq b \implies \left| \frac{\partial}{\partial \theta} f(\theta, u) \right| \leq C_b(\|f'_\theta(u)\| + 1)(\|u\| + 1)
\]

for all $(\theta, u) \in [0, 1] \times \mathcal{X}$;

- there exist two continuous maps $\eta_1, \eta_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ which are Lipschitz continuous with respect to the second variable and such that $\eta_1 \leq \eta_2$. Suppose

\[
\eta_1(\theta, f_\theta(u)) \leq \frac{\partial}{\partial \theta} f(\theta, u) \leq \eta_2(\theta, f_\theta(u)) \tag{3.84}
\]

at each critical point $u$ of $f_\theta$;

- $f_\theta$ is even and for each finite dimensional subspace $\mathcal{W}$ of $\mathcal{X}$ it results

\[
\lim_{\|u\| \to +\infty} \sup_{\theta \in [0, 1]} f(\theta, u) = -\infty .
\]

Taken $i = 1, 2$, let us denote by $\psi_i : [0, 1] \times \mathbb{R} \to \mathbb{R}$ the solutions of the problem

\[
\begin{cases}
\frac{\partial}{\partial \theta} \psi_i(\theta, s) = \eta_i(\theta, \psi_i(\theta, s)) \\
\psi_i(0, s) = s .
\end{cases}
\]

Note that $\psi_i(\theta, \cdot)$ are continuous, non-decreasing on $\mathbb{R}$ and $\psi_1 \leq \psi_2$. Set

\[
\overline{\eta}_1(s) = \sup_{\theta \in [0, 1]} \eta_1(\theta, s), \quad \overline{\eta}_2(s) = \sup_{\theta \in [0, 1]} \eta_2(\theta, s) .
\]

In this framework, the following abstract result can be proved.

**Theorem 3.3.18.** There exists $C \in \mathbb{R}$ such that if $l \in \mathbb{N}$ then

(a) either $f_1$ has a critical point $\tilde{c}_l$ with $\psi_2(1, c_l) < \psi_1(1, c_{l+1}) \leq \tilde{c}_l$,

(b) or we have $c_{l+1} - c_l \leq C(\overline{\eta}_1(c_{l+1}) + \overline{\eta}_2(c_l) + 1)$.

**Proof.** See [24, Theorem 3] and [25, Theorem 2.2].
3.3.6 Application to semilinear elliptic systems

In this section we want to prove Theorem 3.3.1 in a simpler fashion by means of the arguments introduced in Section 6.

For \( \theta \in [0, 1] \), let us consider the functional \( f_\theta : H^1_0(\Omega, \mathbb{R}^N) \to \mathbb{R} \) as

\[
f_\theta(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx - \frac{1}{p} \int_\Omega |u + \theta \phi|^p \, dx - \theta \int_\Omega \varphi \cdot u \, dx.
\]

It can be proved that all the previous assumptions are satisfied.

**Lemma 3.3.19.** Let \( ((\theta^m, u^m))_m \subset [0, 1] \times H^1_0(\Omega, \mathbb{R}^N) \) be such that (3.83) holds. Then \( ((\theta^m, u^m))_m \) converges up to subsequences.

**Proof.** Let \( ((\theta^m, u^m))_m \) be such that (3.83) holds. For a suitable \( K > 0 \) and any \( \varrho > 0 \) it is

\[
K + \varrho \|Du^m\|_2 \geq f_{\theta^m}(u^m) - \varrho f'_{\theta^m}(u^m)[u^m]
\]

\[
= \left( \frac{1}{2} - \varrho \right) \int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx + \left( \frac{1}{p} - \varrho \right) \int_\Omega |u^m + \theta^m \phi|^p \, dx
\]

\[- \theta^m \varrho \int_\Omega |u^m + \theta^m \phi|^{p-2}(u^m + \theta^m \phi) \cdot \phi \, dx
\]

for all \( m \) large enough. Then, fixed any \( \varepsilon > 0 \) and taken \( d \) as in (3.80), (3.71) and simple computations imply

\[
\varrho \|Du^m\|_2 + \left( \frac{1}{2} - \varrho \right) d \|u^m\|_2^2 \geq \left( \frac{1}{2} - \varrho \right) \frac{\nu}{2} \|Du^m\|_2^2
\]

\[
+ \frac{1}{2p-1} \left( \varrho(1-\varepsilon) - \frac{1}{p} \right) \|u^m\|_p^p - a_{\varepsilon}
\]

for a certain \( a_{\varepsilon} > 0 \). Hence, if we fix \( \varrho \in \left[ \frac{1}{2}, \frac{1}{2} \right] \) and \( \varepsilon \in \left[ 0, 1 - \frac{1}{2p-1} \right] \), by this last inequality it follows that \( (u^m)_m \) has to be bounded in \( H^1_0(\Omega, \mathbb{R}^N) \).

So, if we assume \( u^m = f'_{\theta^m}(u^m) + |u^m + \theta^m \phi|^{p-2}(u^m + \theta^m \phi) + \theta^m \varphi \) it is easy to prove that \( (u^m)_m \) strongly converges in \( H^{-1}(\Omega, \mathbb{R}^N) \), up to subsequences. Whence, Lemma 3.3.8 implies that \( (u^m)_m \) has a converging subsequence in \( H^1_0(\Omega, \mathbb{R}^N) \).

**Lemma 3.3.20.** For each \( b > 0 \) there exists \( C_b > 0 \) such that

\[
|f_\theta(u)| \leq b \implies |\frac{\partial}{\partial \theta} f(\theta, u)| \leq C_b(|f'_\theta(u)| + 1)(\|u\|_{1,2} + 1)
\]

for all \((\theta, u) \in [0, 1] \times H^1_0(\Omega, \mathbb{R}^N)\).

**Proof.** Fix \( b > 0 \). The condition \( |f_\theta(u)| \leq b \) is equivalent to

\[
\left| \int_\Omega \frac{1}{2} \sum_{i,j=1}^n \sum_{h,k=1}^N a_{ij}^{hk}(x) D_i u_h D_j u_k - \frac{1}{p} |u + \theta \phi|^p - \theta \varphi \cdot u \right| \leq b
\]

(3.85)
which implies that

\[ \theta \int_{\Omega} \varphi \cdot u \, dx \geq \frac{p}{2} \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx \]

\[ - \int_{\Omega} |u + \theta \phi|^p \, dx - (p - 1) \theta \int_{\Omega} \varphi \cdot u \, dx - pb. \]  

So, taken \( d \) as in (3.80), we have

\[ -f'_\phi(u)[u] = - \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx \]

\[ + \int_{\Omega} |u + \theta \phi|^{p-2}(u + \theta \phi) \cdot u \, dx + \theta \int_{\Omega} \varphi \cdot u \, dx \]

\[ \geq \left( \frac{p}{2} - 1 \right) \int_{\Omega} \left( \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k + d |u|^2 \right) \, dx \]

\[ - \left( \frac{p}{2} - 1 \right) d \|u\|_2^2 - \int_{\Omega} |u + \theta \phi|^{p-2}(u + \theta \phi) \cdot \theta \phi \, dx \]

\[ - (p - 1) \theta \int_{\Omega} \varphi \cdot u \, dx - pb \]

\[ \geq \left( p - 2 \right) \frac{\nu}{4} \|Du\|_2^2 - \left( \frac{p}{2} - 1 \right) d \|u\|_2^2 \]

\[ - \int_{\Omega} |u + \theta \phi|^{p-2}(u + \theta \phi) \cdot \theta \phi \, dx - (p - 1) \theta \int_{\Omega} \varphi \cdot u \, dx - pb. \]

By Hölder inequality there exist \( c_1, c_2, c_3 > 0 \) such that

\[ \left| \int_{\Omega} |u + \theta \phi|^{p-2}(u + \theta \phi) \cdot \theta \phi \, dx \right| \leq c_1 \|u + \theta \phi\|_p^{p-1}, \]  

\[ \left| \int_{\Omega} \varphi \cdot u \, dx \right| \leq c_2 \|u + \theta \phi\|_p + c_3 ; \]  

while (3.85) implies

\[ |u + \theta \phi|^p \leq c_4 \|Du\|_2^2 + c_5(b) \]  

for suitable \( c_4, c_5(b) > 0 \). Then, since Young’s inequality yields

\[ c_1 \|u + \theta \phi\|_p^{p-1} \leq \varepsilon \|u + \theta \phi\|_p + \bar{c}_1(\varepsilon), \]  

\[ c_2 \|u + \theta \phi\|_p \leq \varepsilon \|u + \theta \phi\|_p + \bar{c}_2(\varepsilon), \]  

for all \( \varepsilon > 0 \) and certain \( \bar{c}_1(\varepsilon), \bar{c}_2(\varepsilon) > 0 \), it can be proved that \( c_6, c_7(\varepsilon, b) > 0 \) exist such that

\[ -f'_\phi(u)[u] \geq \left( (p - 2) \frac{\nu}{4} - \varepsilon c_6 \right) \|Du\|_2^2 - c_7(\varepsilon, b). \]

So, if \( \varepsilon \) is small enough, some \( \bar{c}_6, \bar{c}_7(b) > 0 \) can be find such that

\[ \bar{c}_6 \|Du\|_2^2 - \bar{c}_7(b) \leq -f'_\phi(u)[u]. \]  

(3.91)
On the other hand, since
\[
\frac{\partial}{\partial \theta} f(\theta, u) = - \int_{\Omega} |u + \theta \phi|^{p-2}(u + \theta \phi) \cdot \phi \, dx - \int_{\Omega} \varphi \cdot u \, dx
\]
by (3.87) and (3.88) it follows
\[
\left| \frac{\partial}{\partial \theta} f(\theta, u) \right| \leq c_8 \|u + \theta \phi\|^{p-1}_p + c_9
\]
and then by (3.90)
\[
\left| \frac{\partial}{\partial \theta} f(\theta, u) \right| \leq \varepsilon \|u + \theta \phi\|_p^p + c_{10}(\varepsilon)
\]
for any \( \varepsilon > 0 \) and \( c_8, c_9, c_{10}(\varepsilon) > 0 \) suitable constants. So, for all \( \varepsilon > 0 \) and a certain \( c_{11}(\varepsilon, b) > 0 \), (3.89) implies
\[
\left| \frac{\partial}{\partial \theta} f(\theta, u) \right| \leq \varepsilon c_4 \|Du\|_2^2 + c_{11}(\varepsilon, b).
\]
Hence, the proof follows by (3.91), (3.93) and a suitable choice of \( \varepsilon \).

**Lemma 3.3.21.** If \( u \in H^1_0(\Omega, \mathbb{R}^N) \) is a critical point of \( f_\theta \), there exists \( \sigma > 0 \) such that
\[
\int_{\Omega} |u + \theta \phi|^p \, dx \leq \sigma (f_\theta^2(u) + 1)^{1/2}.
\]

*Proof.* It suffices to argue as in Lemma 3.3.4.

**Lemma 3.3.22.** At each critical point \( u \) of \( f_\theta \) the inequality (3.84) holds if \( \eta_1, \eta_2 \) are defined in \( (\theta, s) \in [0, 1] \times \mathbb{R} \) as
\[
-\eta_1(\theta, s) = \eta_2(\theta, s) = C \left( s^2 + 1 \right)^{\frac{p-1}{2p}}
\]
for a suitable constant \( C > 0 \).

*Proof.* It is sufficient to combine (3.92) with Lemma 3.3.21.

**New proof of Theorem 3.3.1.** Clearly, \( f_0 \) is an even functional. Moreover, by Lemmas 3.3.19, 3.3.20 and 3.3.22 all the hypotheses of the existence theorem are fulfilled.

Now, consider \((V_l)_l\), the sequence of subspaces of \( H^1_0(\Omega, \mathbb{R}^N) \) introduced in the previous sections. Defined the set of maps \( \mathcal{X} \) with \( \mathcal{X} = H^1_0(\Omega, \mathbb{R}^N) \), assume
\[
c_l = \inf_{\zeta \in \mathcal{X}} \sup_{u \in V_l} f_0(\zeta(u)).
\]
Simple computations allow to prove that, taken any finite dimensional subspace \( \mathcal{W} \) of \( H^1_0 \), some constants \( \beta_1, \beta_2, \beta_3 > 0 \) exist such that
\[
f_\theta(u) \leq \beta_1 \|u\|^2_{1,2} - \beta_2 \|u\|^p_{1,2} - \beta_3 \quad \text{for all } u \in \mathcal{W}.
\]
Then,
\[ \lim_{u \to W} \sup_{\|u\|_{1,2} \to +\infty} \theta \in [0,1] f_\theta(u) = -\infty. \]

Hence, Theorem 3.3.18 applies and, by the choice made in (3.94), the condition (b) implies that there exists \( \tilde{C} > 0 \) such that
\[
|c_{l+1} - c_l| \leq \tilde{C}\left((c_l)^{\frac{p-1}{p}} + (c_{l+1})^{\frac{p-1}{p}} + 1\right), \tag{3.95}
\]
which implies \( c_l \leq \tilde{\gamma} l^p \) for some \( \tilde{\gamma} > 0 \) in view of [16, Lemma 5.3]. Taking into account Lemma 3.3.13 we conclude that (3.95) can not hold provided that
\[
\frac{2p}{n(p-2)} > p,
\]

namely \( p \in ]2, \frac{2n+1}{n}[ \). Whence, the assertion follows by (a) of Theorem 3.3.18.

\[\square\]

### 3.3.7 The diagonal case

Now, we want to prove Theorem 3.3.2. To this aim let us point out that we deal with the problem
\[
\begin{align*}
-\Delta u_k &= |u|^{p-2}u_k + \varphi_k(x) \quad \text{in } \Omega \\
\varphi_k &= \chi \quad \text{on } \partial\Omega \\
k &= 1, \ldots, N
\end{align*}
\tag{3.96}
\]

and want to prove that (3.96) has an infinite number of solutions if \( p \in ]2, \frac{2n+1}{n}[ \).

In this case, the functional \( f_\theta \) defined in the previous section becomes
\[
f_\theta(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{p} \int_\Omega |u + \theta \phi|^p \, dx - \theta \int_\Omega \varphi \cdot u \, dx
\]

where \( \phi \) solves the system (3.70) with \( a^{hk}_{ij} = \delta^{hk}_{ij} \).

By the regularity assumptions we made on \( \partial\Omega \), \( \chi \) and \( \varphi \) the following lemma can be proved.

**Lemma 3.3.23.** There exists \( c > 0 \) such that if \( u \) is a critical point of \( f_\theta \), then
\[
\left| \int_{\partial\Omega} \left( \frac{1}{2} |\nabla w|^2 - |\frac{\partial w}{\partial n}|^2 \right) \, d\sigma \right| \leq c \int_\Omega (|\nabla w|^2 + |w|^p + 1) \, dx
\]

where \( w = u + \theta \phi \).

**Proof.** If \( u \in H^1_0(\Omega, \mathbb{R}^N) \) is such that \( f_\theta'(u) = 0 \), then some regularity theorems imply that \( u \) is a classical solution of the problem
\[
\begin{align*}
-\Delta u_k &= |u + \theta \phi|^{p-2}(u_k + \theta \phi) + \theta \varphi_k \quad \text{in } \Omega \\
\varphi_k &= 0 \quad \text{on } \partial\Omega \\
k &= 1, \ldots, N,
\end{align*}
\]
then, \( w = u + \theta \phi \in C^2(\Omega, \mathbb{R}^N) \) solves the elliptic system

\[
\begin{cases}
-\Delta w_k = |w|^{p-2} w_k + \theta \varphi_k & \text{in } \Omega \\
\frac{1}{2} |w| w_k = \theta \phi_k & \text{on } \partial \Omega \\
k = 1, \ldots, N.
\end{cases}
\] (3.97)

Taken \( \delta > 0 \), let us consider a cut function \( \eta \in C^\infty(\mathbb{R}, \mathbb{R}) \) such that \( \eta(s) = 1 \) for \( s \leq 0 \) and \( \eta(s) = 0 \) for \( s \geq \delta \). Moreover, taken any \( x \in \mathbb{R}^N \), let \( d(x, \partial \Omega) \) be the distance of \( x \) from the boundary of \( \Omega \). Let us point out that, since \( \eta \) is smooth enough, \( \delta \) can be chosen in such a way that \( d(\cdot, \partial \Omega) \) is of class \( C^2 \) on

\[ \overline{\Omega} \cap \{ x \in \mathbb{R}^n : d(x, \partial \Omega) < \delta \}, \]

and \( \hat{\eta}(x) = \nabla d(x, \partial \Omega) \) coincides on \( \partial \Omega \) with the inner normal.

So, defined \( g : \mathbb{R}^N \to \mathbb{R} \) as \( g(x) = \eta(d(x, \partial \Omega)) \), for each \( k = 1, \ldots, N \) let us multiply the \( k \)-th equation in (3.97) by \( g(x) \nabla w_k \cdot \hat{\eta}(x) \). Hence, working as in [25, Lemma 4.2] and summing up with respect to \( k \), we get

\[
\sum_{k=1}^N \int_{\Omega} -\Delta w_k \ g(x) \nabla w_k \cdot \hat{\eta} \ dx = \int_{\partial \Omega} \left( \frac{1}{2} |\nabla w|^2 - |\frac{\partial w}{\partial n}|^2 \right) d\sigma + O\left( \|w\|_2^2 \right),
\]

\[
\sum_{k=1}^N \int_{\Omega} |w|^{p-2} w_k \ g(x) \nabla w_k \cdot \hat{\eta} \ dx = \frac{\theta^p}{p} \int_{\partial \Omega} |\phi|^p \ d\sigma + O\left( \|w\|_p^p \right),
\]

\[
\sum_{k=1}^N \int_{\Omega} \theta \varphi_k(x) \ g(x) \nabla w_k \cdot \hat{\eta} \ dx = \theta^2 \int_{\partial \Omega} \varphi \cdot \phi \ d\sigma + O\left( \|w\|_p \right).
\]

Whence, the proof follows by putting together these identities. \( \square \)

With the stronger assumptions we made in this section, the estimates in Lemma 3.3.22 can be improved.

**Lemma 3.3.24.** At each critical point \( u \) of \( f_{\theta} \) the inequality (3.84) holds if \( \eta_1, \eta_2 \) are defined in \( (\theta, s) \in [0, 1] \times \mathbb{R} \) as

\[-\eta_1(\theta, s) = \eta_2(\theta, s) = C \left( s^2 + 1 \right)^{\frac{1}{4}} \]

for a suitable constant \( C > 0 \).

**Proof.** Let \( u \) be a critical point of \( f_{\theta} \). Then,

\[
\frac{\partial}{\partial \theta} f(\theta, u) = \int_{\partial \Omega} \frac{\partial u}{\partial n} \cdot \phi \ d\sigma + \int_{\Omega} \varphi \cdot (\theta \phi - u) \ dx ;
\]

so, taking into account Lemma 3.3.23, it is enough to argue as in [25, Lemma 4.3]. \( \square \)

**Proof of Theorem 3.3.2.** Arguing as in the proof of Theorem 3.3.1, we have that the proof of Theorem 3.3.2 follows by Theorem 3.3.18 since also in this case the condition \((b)\) can not occur. Let us point out that, by Lemma 3.3.24, the incompatibility condition is \( \frac{2p}{n(p-2)} > 2 \), i.e. \( p \in \left[ 2, \frac{2n}{n-2} \right[ \).
Chapter 4

Problems of jumping type

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We refer the reader to [107, 108]. Some parts of these publications have been slightly modified to give this collection a more uniform appearance.

4.1 Fully nonlinear elliptic equation

4.1.1 Introduction

Let us consider the semilinear elliptic problem

\[
\begin{aligned}
- \sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu) &= g(x,u) + \omega \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial\Omega ,
\end{aligned}
\]

(4.1)
where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $\omega \in H^{-1}(\Omega)$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies
\[
\lim_{s \to -\infty} \frac{g(x,s)}{s} = \alpha, \quad \lim_{s \to +\infty} \frac{g(x,s)}{s} = \beta. \tag{4.2}
\]

Let us denote by $(\mu_h)$ the eigenvalues of the linear operator on $H^1_0(\Omega)$
\[
\left\{ u \mapsto - \sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u) \right\}.
\]

Since 1972, this jumping problem has been widely investigated in the case when some eigenvalue $\mu_h$ belongs to the interval $[\beta, \alpha]$ (see e.g. [79, 81, 95] and references therein), starting from the pioneering paper [5] of Ambrosetti and Prodi.

On the other hand, since 1994, several efforts have been devoted to study existence of weak solutions of the quasilinear problem
\[
\begin{cases}
-\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2} \sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_iuD_ju = g(x,u) + \omega & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{4.3}
\]
via techniques of nonsmooth critical point theory (see e.g. [10, 34, 37, 47, 114]).

In particular, a jumping problem for the previous equation has been treated in [33]. More recently, existence for the Euler’s equations of multiple integrals of calculus of variations
\[
\begin{cases}
-\text{div} (\nabla \mathcal{L}(x,u,\nabla u)) + D_s \mathcal{L}(x,u,\nabla u) = g(x,u) + \omega & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{4.3}
\]
have also been considered in [7] and in [88, 100] via techniques developed in [37]. In this paper we see how the results of [33] may be extended to the more general elliptic problem (4.3). We shall approach the problem from a variational point of view, that is looking for critical points for continuous functionals $f : W^{1,p}_0(\Omega) \to \mathbb{R}$ of type
\[
f(u) = \int_{\Omega} \mathcal{L}(x,u,\nabla u) \, dx - \int_{\Omega} G(x,u) \, dx - \langle \omega, u \rangle.
\]

We point out that, in general, these functionals are not even locally Lipschitzian, so that classical critical point theory fails. Then we shall refer to non-smooth critical point theory. In our main result (Theorem 4.1.1) we shall prove existence of at least two solutions of the problem by means of a classical minmax theorem in its nonsmooth version.

### 4.1.2 The main result

We assume that $\Omega$ is a bounded domain of $\mathbb{R}^n$, $1 < p < n$, $\omega \in W^{-1,p'}(\Omega)$ and $\mathcal{L} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is measurable in $x$ for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and of class $C^1$ in $(s, \xi)$ a.e. in $\Omega$. Moreover, the function $\mathcal{L}(x,s,\cdot)$ is strictly convex and for each $t \in \mathbb{R}$ $\mathcal{L}(x,s,t\xi) = |t|^p \mathcal{L}(x,s,\xi)$ for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. Furthermore, we assume that:
there exist $\nu > 0$ and $b_1 \in \mathbb{R}$ such that:
\[ \nu |\xi|^p \leq \mathcal{L}(x, s, \xi) \leq b_1 |\xi|^p, \]  
(4.4)
for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$;

- there exist $b_2, b_3 \in \mathbb{R}$ such that:
\[ |D_s \mathcal{L}(x, s, \xi)| \leq b_2 |\xi|^p, \]  
(4.5)
for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and
\[ |\nabla_\xi \mathcal{L}(x, s, \xi)| \leq b_3 |\xi|^{p-1}, \]  
(4.6)
for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$;

- there exist $R > 0$ and a bounded Lipschitzian function $\vartheta : \mathbb{R} \to [0, +\infty]$ such that:
\[ |s| \geq R \implies sD_s \mathcal{L}(x, s, \xi) \geq 0, \]  
(4.7)
for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^n$. Without loss of generality, we may take assume that $\vartheta(s) \to \overline{\vartheta}$ as $s \to \pm\infty$;

- $g(x, s)$ is a Carathéodory function and $G(x, s) = \int_0^s g(x, \tau) \, d\tau$. We assume that there exist $a \in L^{\frac{n}{(p-1)p}}(\Omega)$ and $b \in L^\frac{n}{p}(\Omega)$ such that:
\[ |g(x, s)| \leq a(x) + b(x)|s|^{p-1}, \]  
(4.8)
for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$. Moreover, there exist $\alpha, \beta \in \mathbb{R}$ such that
\[ \lim_{s \to -\infty} \frac{g(x, s)}{|s|^{p-2}s} = \alpha, \quad \lim_{s \to +\infty} \frac{g(x, s)}{|s|^{p-2}s} = \beta, \]  
(4.9)
for a.e. $x \in \Omega$.

Let us now suppose that:
\[ \lim_{s \to +\infty} \mathcal{L}(x, s, \xi) = \lim_{s \to -\infty} \mathcal{L}(x, s, \xi) \]  
(both limits exist by (4.6)) and denote by $\mathcal{L}_\infty(x, \xi)$ the common value, that we shall assume to be of the form $a(x)|\xi|^p$ with $a \in L^\infty(\Omega)$. Moreover, assume that
\[ s_h \to +\infty, \xi_h \to \xi \implies \nabla_\xi \mathcal{L}(x, s_h, \xi_h) \to \nabla_\xi \mathcal{L}_\infty(x, \xi). \]  
(4.10)

Let
\[ \lambda_1 = \min \left\{ p \int_\Omega \mathcal{L}_\infty(x, \nabla u) \, dx : u \in W^{1,p}_0(\Omega), \int_\Omega |u|^p \, dx = 1 \right\}, \]  
(4.11)
be the first eigenvalue of \{ $u \mapsto -\div (\nabla_\xi \mathcal{L}_\infty(x, \nabla u))$ \}.

Observe that by [7, Lemma 1.4] the first eigenfunction $\phi_1$ belongs to $L^\infty(\Omega)$ and by [120, Theorem 1.1] is strictly positive.

Under the previous assumptions, we consider problem (4.3) in the case $\omega = t\phi_1^{p-1} + \omega_0$, with $\omega_0 \in W^{-1,p'}(\Omega)$ and $t \in \mathbb{R}$. The following is our main result:
Theorem 4.1.1. If $\beta < \lambda_1 < \alpha$ then there exist $\bar{t} \in \mathbb{R}$ and $\underline{t} \in \mathbb{R}$ such that the problem

$$
\begin{align*}
-\text{div} \, (\nabla_x \mathcal{L}(x, u, \nabla u)) + D_s \mathcal{L}(x, u, \nabla u) &= g(x, u) + t\varphi_1^{p-1} + \omega_0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}
$$

has at least two weak solutions in $W_0^{1,p}(\Omega)$ for $t > \bar{t}$ and no solutions for $t < \underline{t}$.

This result extends [33, Corollary 2.3] dealing with the case $p = 2$ and

$$
\mathcal{L}(x, s, \xi) = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x, s)\xi_i \xi_j - G(x, s)
$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

In this particular case, existence of at least three solutions has been recently proved in [36] assuming $\beta < \mu_1$ and $\alpha > \mu_2$ where $\mu_1$ and $\mu_2$ are the first and second eigenvalue of the operator

$$
\begin{align*}
u \mapsto -\sum_{i,j=1}^n D_j(A_{ij}D_i u)
\end{align*}
$$

In our general setting we only have existence of the first eigenvalue $\lambda_1$ and it is not clear how to define higher order eigenvalues $\lambda_2, \lambda_3, \ldots$. Therefore in our case the comparison of $\alpha$ and $\beta$ with such eigenvalues is still not possible.

4.1.3 The concrete Palais–Smale condition

The following result is one of the main tools of the paper.

Lemma 4.1.2. Let $(u_h)$ be a sequence in $W_0^{1,p}(\Omega)$ and $(\varphi_h) \subseteq ]0, +\infty[$ with $\varphi_h \to +\infty$ be such that

$$
\begin{align*}
\varphi_h = \frac{u_h}{\varphi_h} \rightharpoonup v \quad \text{in } W_0^{1,p}(\Omega).
\end{align*}
$$

Let $\gamma_h \to \gamma$ in $L^{\frac{np}{n+p'}}(\Omega)$ with $|\gamma_h(x)| \leq c(x)$ for some $c \in L^{\frac{p}{p'}}(\Omega)$. Moreover, let

$$
\begin{align*}
\mu_h \to \mu \quad \text{in } L^{\frac{np'}{n+p'}}(\Omega), \quad \delta_h \to \delta \quad \text{in } W^{-1,p'}(\Omega)
\end{align*}
$$

be such that for each $\varphi \in C_c^\infty(\Omega)$:

$$
\begin{align*}
\int_{\Omega} \nabla_x \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \varphi \, dx + \int_{\Omega} D_s \mathcal{L}(x, u_h, \nabla u_h) \varphi \, dx
\end{align*}
$$

$$
= \int_{\Omega} \gamma_h|u_h|^{p-2}u_h \varphi \, dx + \varphi_h^{p-1} \int_{\Omega} \mu_h \varphi \, dx + \langle \delta_h, \varphi \rangle.
$$

Then, the following facts hold:

(a) $(v_h)$ is strongly convergent to $v$ in $W_0^{1,p}(\Omega)$;

(b) $(\gamma_h|v_h|^{p-2}v_h)$ is strongly convergent to $\gamma|v|^{p-2}v$ in $W^{-1,p'}(\Omega)$;
(c) there exist $\eta^+, \eta^- \in L^\infty(\Omega)$ such that:

$$
\eta^+(x) = \begin{cases} 
\exp\{-\bar{\eta}\} & \text{if } v(x) > 0 \\
\exp\{MR\} & \text{if } v(x) < 0,
\end{cases}
$$

$$\exp\{-\bar{\eta}\} \leq \eta^+(x) \leq \exp\{MR\} \quad \text{if } v(x) = 0,$$

and

$$
\eta^-(x) = \begin{cases} 
\exp\{-\bar{\eta}\} & \text{if } v(x) < 0 \\
\exp\{MR\} & \text{if } v(x) > 0,
\end{cases}
$$

$$\exp\{-\bar{\eta}\} \leq \eta^-(x) \leq \exp\{MR\} \quad \text{if } v(x) = 0,$$

and such that for every $\varphi \in W^{1,p}_0(\Omega)$ with $\varphi \geq 0$:

$$
\int_\Omega \eta^+ \nabla \xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla \varphi \, dx \geq \int_\Omega \gamma \eta^+ |v|^{p-2}v \varphi \, dx + \int_\Omega \mu \eta^+ \varphi \, dx,
$$

$$
\int_\Omega \eta^- \nabla \xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla \varphi \, dx \leq \int_\Omega \gamma \eta^- |v|^{p-2}v \varphi \, dx + \int_\Omega \mu \eta^- \varphi \, dx.
$$

**Proof.** Arguing as in [33, Lemma 3.1], (b) immediately follows. Let us now prove (a). Up to a subsequence, $v_h(x) \rightharpoonup v(x)$ for a.e. $x \in \Omega$. Consider now the function $\zeta : \mathbb{R} \to \mathbb{R}$ defined by

$$
\zeta(s) = \begin{cases} 
Ms & \text{if } 0 < s < R \\
MR & \text{if } s \geq R \\
-Ms & \text{if } -R < s < 0 \\
MR & \text{if } s \leq -R,
\end{cases}
$$

(4.14)

where $M \in \mathbb{R}$ is such that for a.e. $x \in \Omega$, each $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$

$$
|D_s \mathcal{L}(x, s, \xi)| \leq M \nabla_x \mathcal{L}(x, s, \xi) \cdot \xi.
$$

(4.15)

By [100, Proposition 3.1], we may choose in (4.13) the functions $\varphi = v_h \exp\{\zeta(u_h)\}$ yielding

$$
\int_\Omega \nabla \xi \mathcal{L}_h(x, u_h, \nabla v_h) \cdot \nabla v_h \exp\{\zeta(u_h)\} \, dx +
$$

$$
\int_\Omega \left[ D_s \mathcal{L}(x, u_h, \nabla u_h) + \zeta'(u_h)\nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \right] v_h \exp\{\zeta(u_h)\} \, dx =
$$

$$
= \int_\Omega \gamma_h |u_h|^{p-2}u_h v_h \exp\{\zeta(u_h)\} \, dx + \varepsilon_h^{p-1} \int_\Omega \mu_h v_h \exp\{\zeta(u_h)\} \, dx +
$$

$$
+ \langle \delta_h, v_h \exp\{\zeta(u_h)\} \rangle.
$$

Therefore, taking into account conditions (4.6) and (4.15), we have

$$
\varepsilon_h^{p-1} \int_\Omega \nabla \xi \mathcal{L}_h(x, u_h, \nabla v_h) \cdot \nabla v_h \exp\{\zeta(u_h)\} \, dx \leq
$$
By (4.6), (4.7) and (4.15) it results that for each \( h \) we obtain
\[
\leq g_h^{p-1} \int_{\Omega} \gamma_h |v_h|^p \exp\{\zeta(u_h)\} \, dx + g_h^{p-1} \int_{\Omega} \mu_h v_h \exp\{\zeta(u_h)\} \, dx + \langle \delta_h, v_h \exp\{\zeta(u_h)\} \rangle.
\]
After division by \( g_h^{p-1} \), using the hypotheses on \( \gamma_h, \mu_h \) and \( \delta_h \), we obtain
\[
\limsup_h \int_{\Omega} \nabla \xi \mathcal{L}(x, u_h, \nabla v_h) \cdot \nabla v_h \exp\{\zeta(u_h)\} \, dx \leq \exp\{MR\} \left( \int_{\Omega} |v|^p \, dx + \int_{\Omega} \mu v \, dx \right).
\] (4.16)

Now, let us consider the function \( \vartheta_1 : \mathbb{R} \rightarrow \mathbb{R} \) given by
\[
\vartheta_1(s) = \begin{cases} 
\vartheta(s) & \text{if } s \geq 0 \\
Ms & \text{if } -R \leq s \leq 0 \\
-MR & \text{if } s \leq -R,
\end{cases}
\] (4.17)

with \( \vartheta \) satisfying (4.7). Considering in (4.13) the functions \( (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} \) with \( k \in \mathbb{N} \), we obtain
\[
\int_{\Omega} \nabla \xi \mathcal{L}(x, u_h, \nabla v_h) \cdot \nabla (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} \, dx +
\frac{1}{g_h^{p-1}} \int_{\Omega} \left[ D_s \mathcal{L}(x, u_h, \nabla u_h) - \vartheta'_1(u_h) \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \right] (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} \, dx
\]
\[
= \int_{\Omega} \gamma_h |v_h|^{p-2} v_h (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} \, dx + \int_{\Omega} \mu_h (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} \, dx +
\frac{1}{g_h^{p-1}} \langle \delta_h, (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} \rangle.
\] (4.18)

By (4.6), (4.7) and (4.15) it results that for each \( h \in \mathbb{N} \)
\[
[D_s \mathcal{L}(x, u_h, \nabla u_h) - \vartheta'_1(u_h) \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h] (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} \leq 0.
\]
Taking into account assumptions (4.10) and (4.5), we may apply [51, Theorem 5] and deduce that
\[\text{a.e. in } \Omega \setminus \{v = 0\} : \nabla v_h(x) \rightarrow \nabla v(x).\]

Being \( u_h(x) \rightarrow +\infty \) a.e. in \( \Omega \setminus \{v = 0\} \), again recalling (4.10), we have
\[\text{a.e. in } \Omega \setminus \{v = 0\} : \nabla \xi \mathcal{L}(x, u_h(x), \nabla v_h(x)) \rightarrow \nabla \xi \mathcal{L}_{\infty}(x, \nabla v(x)).\]

By combining this pointwise convergence with (4.5), we obtain
\[
\nabla \xi \mathcal{L}(x, u_h, \nabla v_h) \rightarrow \nabla \xi \mathcal{L}_{\infty}(x, \nabla v) \quad \text{in } L^p(\Omega).
\] (4.19)

Therefore, for each \( k \in \mathbb{N} \) we have
\[
\lim_h \int_{\Omega} \nabla \xi \mathcal{L}(x, u_h, \nabla v_h) \cdot \nabla (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} =
\]
4.1. Fully nonlinear elliptic equation

\[ = \nabla \xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla (v^+ \wedge k) \exp\{-\overline{\eta}\}, \]

strongly in \(L^1(\Omega)\),

\[ \lim_h (v^+ \wedge k) \exp\{-\partial_1(u_h)\} = (v^+ \wedge k) \exp\{-\overline{\eta}\}, \]

weakly in \(W^{1,p}_0(\Omega)\), using (b)

\[ \lim_h \gamma_h |v_h|^{p-2} v_h (v^+ \wedge k) \exp\{-\partial_1(u_h)\} = \gamma |v|^{p-2} v (v^+ \wedge k) \exp\{-\overline{\eta}\}, \]

strongly in \(L^1(\Omega)\) and

\[ \lim_h \frac{1}{\theta_h^{p-1}} (v^+ \wedge k) \exp\{-\partial_1(u_h)\} = 0, \]

weakly in \(W^{1,p}_0(\Omega)\). Therefore, letting \(h \to +\infty\) in (4.18), for each \(k \in \mathbb{N}\) we get

\[
\int_\Omega \nabla \xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla (v^+ \wedge k) \exp\{-\overline{\eta}\} \, dx \geq \int_\Omega \gamma |v|^{p-2} v (v^+ \wedge k) \exp\{-\overline{\eta}\} \, dx + \\
+ \int_\Omega \mu v^+ \exp\{-\overline{\eta}\} \, dx.
\]

Finally, if we let \(k \to +\infty\), after division by \(\exp\{-\overline{\eta}\}\), we have

\[
\int_\Omega \nabla \xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla v^+ \, dx \geq \int_\Omega \gamma |v|^{p-2}(v^+)^2 \, dx + \int_\Omega \mu v^+ \, dx.
\]

(4.20)

Analogously, if we define a function \(\partial_2 : \mathbb{R} \to \mathbb{R}\) by

\[ \partial_2(s) = \begin{cases} \\
\partial(s) & \text{if } s \leq 0 \\
-Ms & \text{if } 0 \leq s \leq R \\
-MR & \text{if } s \geq R, 
\end{cases} \]

and consider in (4.13) the test functions \((v^- \wedge k) \exp\{-\partial_2(u_h)\}\) with \(k \in \mathbb{N}\), we obtain

\[
\int_\Omega \nabla \xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla v^- \, dx \leq -\int_\Omega \gamma |v|^{p-2}(v^-)^2 \, dx + \int_\Omega \mu v^- \, dx.
\]

(4.21)

Thus, combining (4.20) and (4.21) yields

\[
\int_\Omega \nabla \xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla v \, dx \geq \int_\Omega \gamma |v|^p \, dx + \int_\Omega \mu v \, dx.
\]

(4.22)

Finally, putting together (4.16) and (4.22), we conclude

\[
\limsup_h \int_\Omega \nabla \xi \mathcal{L}(x, u_h, \nabla v_h) \cdot \nabla v_h \exp\{\zeta(u_h)\} \, dx \leq \\
\leq \exp\{MR\} \int_\Omega \nabla \xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla v \, dx.
\]
In particular, by Fatou's Lemma, it results

\[
\exp\{\text{MR}\} \int_\Omega \nabla_{x} \mathcal{L}_\infty(x, \nabla v) \cdot \nabla v \, dx \leq \\
\leq \liminf_{h} \int_\Omega \nabla_{x} \mathcal{L}(x, u_h, \nabla v_h) \cdot \nabla v_h \exp \{ \xi(u_h) \} \, dx \leq \\
\leq \exp\{\text{MR}\} \int_\Omega \nabla_{x} \mathcal{L}_\infty(x, \nabla v) \cdot \nabla v \, dx,
\]

namely, we get

\[
\nabla_{x} \mathcal{L}(x, u_h, \nabla v_h) \cdot \nabla v_h \exp \{ \xi(u_h) \} \to \exp\{\text{MR}\} \nabla_{x} \mathcal{L}_\infty(x, \nabla v) \cdot \nabla v.
\]

Therefore, since \(\nu|\nabla v_h|^p \leq \nabla_{x} \mathcal{L}(x, u_h, \nabla v_h) \cdot \nabla v_h \exp \{ \xi(u_h) \} \), again thanks to Fatou's Lemma, we conclude that

\[
\limsup_{h} \int_\Omega |\nabla v_h|^p \, dx \leq \int_\Omega |\nabla v|^p \, dx,
\]

and the proof of (a) is concluded.

Let us now prove assertion (c). Up to a subsequence, \(\exp \{-\vartheta_1(u_h)\} \) weakly* converges in \(L^\infty(\Omega)\) to some \(\eta^+\). Of course, we have

\[
\eta^+(x) = \begin{cases} 
\exp \{-\overline{\vartheta}\} & \text{if } \nu(x) > 0 \\
\exp\{\text{MR}\} & \text{if } \nu(x) < 0,
\end{cases}
\]

\[
\exp \{-\overline{\vartheta}\} \leq \eta^+(x) \leq \exp\{\text{MR}\} \text{ if } \nu(x) = 0.
\]

Then, let us consider in (4.13) as test functions:

\[
\varphi \exp \{-\vartheta_1(u_h)\}, \quad \varphi \in C^\infty_c(\Omega), \quad \varphi \geq 0.
\]

Whence, like in the previous argument, we obtain

\[
\int_\Omega \eta^+ \nabla_{x} \mathcal{L}_\infty(x, \nabla v) \cdot \nabla \varphi \, dx \geq \int_\Omega \gamma |\nabla v|^{p-2} v \varphi \, dx + \int_\Omega \mu \nu^+ \varphi \, dx,
\]

for any positive \(\varphi \in W^{1,p}_0(\Omega)\). Similarly, by means of the test functions

\[
\varphi \exp \{-\vartheta_2(u_h)\}, \quad \varphi \in C^\infty_c(\Omega), \quad \varphi \geq 0,
\]

we get for any positive \(\varphi \in W^{1,p}_0(\Omega)\)

\[
\int_\Omega \eta^- \nabla_{x} \mathcal{L}_\infty(x, \nabla v) \cdot \nabla \varphi \, dx \leq \int_\Omega \gamma |\nabla v|^{p-2} v \varphi \, dx + \int_\Omega \mu \nu^- \varphi \, dx,
\]

where \(\eta^-\) is the weak* limit of some subsequence of \(\exp \{-\vartheta_2(u_h)\}\).
4.1. Fully nonlinear elliptic equation

Consider now

\[
g_0(x, s) = g(x, s) - \beta |s|^{p-2}s^+ + \alpha |s|^{p-2}s^-,
G_0(x, s) = \int_0^s g_0(x, \tau) \, d\tau.
\]

Of course, \( g_0 \) is a Carathéodory function satisfying for a.e. \( x \in \Omega \) and for all \( s \in \mathbb{R} \)

\[
\lim_{|s| \to \infty} \frac{g_0(x, s)}{|s|^{p-2}} = 0,
|g_0(x, s)| \leq a(x) + \tilde{b}(x)|s|^{p-1},
\]

with \( \tilde{b} \in L^\infty(\Omega) \). Since we are interested in weak solutions \( u \in W^{1,p}_0(\Omega) \) of the equations

\[
-\text{div} (\nabla \mathcal{L}(x, u, \nabla u)) + D_s \mathcal{L}(x, u, \nabla u) = g(x, u) + t \phi_1^{p-1} + \omega_0,
\]

let us define the associated functional \( f_t : W^{1,p}_0(\Omega) \to \mathbb{R} \), by setting

\[
f_t(u) = \int_\Omega \mathcal{L}(x, u, \nabla u) \, dx - \frac{\beta}{p} \int_\Omega (u^+)^p \, dx - \frac{\alpha}{p} \int_\Omega (u^-)^p \, dx + \int_\Omega G_0(x, u) \, dx - |t|^{p-2}t \int_\Omega \phi_1^{p-1}u \, dx - \langle \omega_0, u \rangle.
\]

**Lemma 4.1.3.** Let \((u_h)\) a sequence in \( W^{1,p}_0(\Omega) \) and \( \varrho_h \subseteq ]0, +\infty[ \) with \( \varrho_h \to +\infty \). Assume that the sequence \( \left( \frac{u_h}{\varrho_h} \right) \) is bounded in \( W^{1,p}_0(\Omega) \). Then

\[
\frac{g_0(x, u_h)}{\varrho_h^{p-1}} \to 0 \text{ in } L^{\frac{np}{n+p-1}}(\Omega), \quad \frac{G_0(x, u_h)}{\varrho_h^p} \to 0 \text{ in } L^1(\Omega).
\]

**Proof.** Argue as in [33, Lemma 3.3].

We now recall from [100] a compactness property of \((CPS)_c\) sequences.

**Theorem 4.1.4.** Let \((u_h)\) be a bounded sequence in \( W^{1,p}_0(\Omega) \) and set

\[
\langle w_h, v \rangle = \int_\Omega \nabla \xi \cdot \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla v \, dx + \int_\Omega D_s \mathcal{L}(x, u_h, \nabla u_h) v \, dx,
\]

for all \( v \in C_c^\infty(\Omega) \). If \((w_h)\) is strongly convergent to some \( w \) in \( W^{-1,p'}(\Omega) \), then \((u_h)\) admits a strongly convergent subsequence in \( W^{1,p}_0(\Omega) \).

**Proof.** See [100, Theorem 3.4].

**Lemma 4.1.5.** For each \( c, t \in \mathbb{R} \) the following assertions are equivalent:

(a) \( f_t \) satisfies the \((CPS)_c\) condition;

(b) every \((CPS)_c\) sequence for \( f_t \) is bounded in \( W^{1,p}_0(\Omega) \).
Proof. \((a) \Rightarrow (b)\). It is trivial. \((b) \Rightarrow (a)\). Let \((u_h)\) be a \((CPS)_c\)-sequence for \(f_t\). Since \((u_h)\) is bounded in \(W_0^{1,p}(\Omega)\), and the map

\[
\left\{ u \mapsto g(x,u) + t\phi_1^{p-1} + \omega_0 \right\},
\]

is completely continuous by (4.8), up to a subsequence \((g(x,u_h) + t\phi_1^{p-1} + \omega_0)\) is strongly convergent in \(L^{np/(n+p)}(\Omega)\), hence in \(W^{-1,p'}(\Omega)\).

We now come to one of the main tools of this paper.

**Theorem 4.1.6.** Let \(c,t \in \mathbb{R}\). Then \(f_t\) satisfies the \((CPS)_c\) condition.

Proof. If \((u_h)\) is a \((CPS)_c\)-sequence for \(f_t\), we have \(f_t(u_h) \to c\) and

\[
\forall v \in C_0^\infty(\Omega) : \int_\Omega \nabla \xi \mathcal{L}(x,u_h,\nabla u_h) \cdot \nabla v \, dx + \int_\Omega D_u \mathcal{L}(x,u_h,\nabla u_h)v \, dx +
\]

\[
-\beta \int_\Omega (u_h^+)^{p-1}v \, dx + \alpha \int_\Omega (u_h^-)^{p-1}v \, dx - \int_\Omega g_0(x,u_h)v \, dx - |t|^{p-2}t \int_\Omega \phi_1 v \, dx =
\]

\[
= (\omega_0 + \sigma_h, v),
\]

where \(\sigma_h \to 0\) in \(W^{-1,p'}(\Omega)\). Taking into account Theorem 4.1.4, by Lemma 4.1.5 it suffices to show that \((u_h)\) is bounded in \(W_0^{1,p}(\Omega)\). Assume by contradiction that, up to a subsequence, \(\|u_h\|_{1,p} \to +\infty\) as \(h \to +\infty\) and set \(v_h = u_h\|u_h\|_{1,p}^{-1}\). By Lemma 4.1.3, we can apply Lemma 4.1.2 choosing

\[
\gamma_h(x) = \begin{cases} 
\beta & \text{if } u_h(x) \geq 0 \\
\alpha & \text{if } u_h(x) < 0
\end{cases}
\]

\[
\mu_h = \frac{g_0(x,u_h)}{\|u_h\|_{1,p}^{p-1}}, \quad \delta_h = |t|^{p-2}t \phi_1 + \omega_0 + \sigma_h.
\]

Then, up to a subsequence, \((v_h)\) strongly converges to some \(v\) in \(W_0^{1,p}(\Omega)\). Moreover, putting \(\varphi = v^+\) in (c) of Lemma 4.1.2, we get

\[
\int_\Omega \eta^- \nabla \xi \mathcal{L}_\infty(x,\nabla v^+) \cdot \nabla v^+ \, dx \leq \int_\Omega \beta \eta^- (v^+)^p \, dx,
\]

hence, taking into account (4.11), we have

\[
\lambda_1 \int_\Omega (v^+)^p \, dx \leq \int_\Omega \nabla \xi \mathcal{L}_\infty(x,\nabla v^+) \cdot \nabla v^+ \, dx \leq \beta \int_\Omega (v^+)^p \, dx.
\]

Since \(\beta < \lambda_1\), then \(v^+ = 0\). By using again the first inequality in (c) of Lemma 4.1.2, for each \(\varphi \geq 0\) we get

\[
\int_\Omega \eta^+ \nabla \xi \mathcal{L}_\infty(x,\nabla v) \cdot \nabla \varphi \, dx \geq \alpha \int_\Omega \eta^+ |v|^{p-2}v \varphi \, dx.
\]
namely, since $v \leq 0$, we have

$$
\int_{\Omega} \nabla \xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla \varphi \, dx \geq \alpha \int_{\Omega} |v|^{p-2} v \varphi \, dx.
$$

In a similar way, by the second inequality in (c) of Lemma 4.1.2 we get

$$
\int_{\Omega} \nabla \xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla \varphi \, dx \leq \alpha \int_{\Omega} |v|^{p-2} v \varphi \, dx.
$$

Therefore we get

$$
\int_{\Omega} \nabla \xi \mathcal{L}_\infty(x, \nabla v) \cdot \nabla \varphi \, dx = \alpha \int_{\Omega} |v|^{p-2} v \varphi \, dx,
$$

which, in view of [74, Remark 1, pp. 161] in impossible if $\alpha$ differs from $\lambda_1$.

4.1.4 Min–Max estimates

Let us first introduce the “asymptotic functional” $f_\infty : W^{1,p}_0(\Omega) \to \mathbb{R}$ by setting

$$
f_\infty(u) = \int_{\Omega} \mathcal{L}_\infty(x, \nabla u) \, dx - \frac{\beta}{p} \int_{\Omega} (u^+)^p \, dx - \frac{\alpha}{p} \int_{\Omega} (u^-)^p \, dx - \int_{\Omega} \phi_{1}^{p-1} u \, dx.
$$

Then consider the functional $\tilde{f}_t : W^{1,p}_0(\Omega) \to \mathbb{R}$ given by

$$
\tilde{f}_t(u) = \int_{\Omega} \mathcal{L}(x, tu, \nabla u) \, dx - \frac{\beta}{p} \int_{\Omega} (u^+)^p \, dx - \frac{\alpha}{p} \int_{\Omega} (u^-)^p \, dx +
$$

$$
- \int_{\Omega} \frac{G_0(x, tu)}{t^p} \, dx - \int_{\Omega} \phi_{1}^{p-1} u \, dx - \frac{\langle \omega_0, u \rangle}{t^{p-1}}.
$$

Theorem 4.1.7. The following fact hold:

(a) assume that $(t_h) \subseteq ]0, +\infty[ \text{ with } t_h \to +\infty$ and $u_h \to u$ in $W^{1,p}_0(\Omega)$. Then

$$
\lim_{h} \tilde{f}_{t_h}(u_h) = f_\infty(u).
$$

(b) assume that $(t_h) \subseteq ]0, +\infty[ \text{ with } t_h \to +\infty$ and $u_h \to u$ in $W^{1,p}_0(\Omega)$. Then

$$
f_\infty(u) \leq \liminf_{h} \tilde{f}_{t_h}(u_h).
$$

(c) assume that $(t_h) \subseteq ]0, +\infty[ \text{ with } t_h \to +\infty$, $u_h \to u$ in $W^{1,p}_0(\Omega)$ and

$$
\limsup_{h} \tilde{f}_{t_h}(u_h) \leq f_\infty(u).
$$

Then $(u_h)$ strongly converges to $u$ in $W^{1,p}_0(\Omega)$. 

Proof. (a) It is easy to prove. (b) Since $u_h \to u$ in $L^p(\Omega)$, it is sufficient to prove that
\[ \int_{\Omega} \mathcal{L}_{\infty}(x, \nabla u) \, dx \leq \liminf_{h} \int_{\Omega} \mathcal{L}(x, t_h u_h, \nabla u_h) \, dx. \]

Let us define the Carathéodory function $\mathcal{\widetilde{L}} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by setting
\[ \mathcal{\widetilde{L}}(x, s, \xi) := \begin{cases} \mathcal{L}(x, \tan(s), \xi) & \text{if } |s| < \frac{\pi}{2} \\ \mathcal{L}_{\infty}(x, \xi) & \text{if } |s| \geq \frac{\pi}{2}. \end{cases} \]

Note that $\mathcal{\widetilde{L}} \geq 0$ and $\mathcal{\widetilde{L}}(x, s, \cdot)$ is convex. Up to a subsequence we have
\[ t_h u_h \to z \text{ a.e. in } \Omega \setminus \{u = 0\}, \quad \nabla u_h \to \nabla u \text{ in } L^p(\Omega \setminus \{u = 0\}), \]
and
\[ \arctan(t_h u_h) \to \arctan(z) \text{ in } L^p(\Omega \setminus \{u = 0\}). \]

Therefore, by [67, Theorem 1] we deduce that
\[ \int_{\Omega \setminus \{u=0\}} \mathcal{\widetilde{L}}(x, \arctan(z), \nabla u) \, dx \leq \liminf_{h} \int_{\Omega \setminus \{u=0\}} \mathcal{\widetilde{L}}(x, \arctan(t_h u_h), \nabla u_h) \, dx, \]
that implies
\[ \int_{\Omega} \mathcal{L}_{\infty}(x, \nabla u) \, dx = \int_{\Omega \setminus \{u=0\}} \mathcal{L}_{\infty}(x, \nabla u) \, dx \leq \liminf_{h} \int_{\Omega \setminus \{u=0\}} \mathcal{L}(x, t_h u_h, \nabla u_h) \, dx = \liminf_{h} \int_{\Omega} \mathcal{L}(x, t_h u_h, \nabla u_h) \, dx. \]

Let us now prove (c). As above, we obtain
\[ \liminf_{h} \int_{\Omega} \mathcal{L} \left( x, t_h u_h, \frac{1}{2} \nabla u_h + \frac{1}{2} \nabla u \right) \, dx \geq \int_{\Omega} \mathcal{L}_{\infty}(x, \nabla u) \, dx. \]

Since we have
\[ \lim_{h} \int_{\Omega} \mathcal{L}(x, t_h u_h, \nabla u) \, dx = \int_{\Omega} \mathcal{L}_{\infty}(x, \nabla u) \, dx \]
and
\[ \limsup_{h} \int_{\Omega} \mathcal{L}(x, t_h u_h, \nabla u_h) \, dx \leq \int_{\Omega} \mathcal{L}_{\infty}(x, \nabla u) \, dx, \quad (4.24) \]
we get
\[ \limsup_{h} \int_{\Omega} (\mathcal{L}(x, t_h u_h, \nabla u_h) - \mathcal{L}(x, t_h u_h, \nabla u)) \, dx \leq 0. \]

On the other hand, the strict convexity implies that for each $h \in \mathbb{N}$
\[ \frac{1}{2} \mathcal{L}(x, t_h u_h, \nabla u_h) + \frac{1}{2} \mathcal{L}(x, t_h u_h, \nabla u) - \frac{1}{2} \mathcal{L} \left( x, t_h u_h, \frac{1}{2} \nabla u_h + \frac{1}{2} \nabla u \right) > 0. \]
Therefore, the previous limits yield
\[
\int_{\Omega} \left\{ \frac{1}{2} \mathcal{L}(x, t_h u_h, \nabla u_h) + \frac{1}{2} \mathcal{L}(x, t_h u_h, \nabla u) - \frac{1}{2} \mathcal{L} \left( x, t_h u_h, \frac{1}{2} \nabla u_h + \frac{1}{2} \nabla u \right) \right\} \, dx \to 0.
\]
In particular, up to a subsequence, we have
\[
\frac{1}{2} \mathcal{L}(x, t_h u_h, \nabla u_h) + \frac{1}{2} \mathcal{L}(x, t_h u_h, \nabla u) - \frac{1}{2} \mathcal{L} \left( x, t_h u_h, \frac{1}{2} \nabla u_h + \frac{1}{2} \nabla u \right) \to 0,
\]
a.e. in \( \Omega \). It easily verified that this can be true only if
\[
\nabla u_h(x) \to \nabla u(x) \text{ for a.e. } x \in \Omega.
\]
Then we have
\[
\frac{1}{\nu} \mathcal{L}(x, t_h u_h, \nabla u_h(x)) \to \frac{1}{\nu} \mathcal{L}_\infty(x, \nabla u(x)) \text{ for a.e. } x \in \Omega.
\]
Taking into account (4.24), we deduce
\[
\frac{1}{\nu} \int_{\Omega} \mathcal{L}(x, t_h u_h, \nabla u_h) \, dx \to \frac{1}{\nu} \int_{\Omega} \mathcal{L}_\infty(x, \nabla u) \, dx,
\]
that by \( \nu|\nabla u_h|^p \leq \mathcal{L}(x, t_h u_h, \nabla u_h) \) yields
\[
\lim_{h} \int_{\Omega} |\nabla u_h|^p \, dx = \int_{\Omega} |\nabla u|^p \, dx,
\]
namely the convergence of \( u_h \) to \( u \) in \( W^{1,p}_0(\Omega) \).

**Remark 4.1.8.** Assume that \( \beta < \lambda_1 < \alpha \). Then the following facts hold:

(a) \( f'_\infty(\overline{\phi_1})(\phi_1) = 0 \);

(b) \( \lim_{s \to -\infty} f_\infty(s \phi_1) = -\infty \), where we have set \( \overline{\phi_1} = \frac{\phi_1}{(\lambda_1 - \beta)^{\frac{1}{p+1}}} \).

**Proof.** (a) It is easy to prove. (b) A direct computation yields that for \( s < 0 \)
\[
f_\infty(s \phi_1) = \frac{\lambda_1 - \alpha}{p} |s|^p - s.
\]
Since \( \alpha > \lambda_1 \), assertion (b) follows.

**Lemma 4.1.9.** For every \( M > 0 \) there exists \( \varrho > 0 \) such that for each \( w \in W^{1,p}_0(\Omega) \) with \( \|w - \phi_1\|_{1,p} \leq \varrho \) we have
\[
\int_{\Omega} \mathcal{L}_\infty(x, -\nabla w^-) \, dx \geq M \int_{\Omega} (w^-)^p \, dx.
\]

**Proof.** Argue as in [33, Lemma 4.1].
Lemma 4.1.10. There exists $r > 0$ such that

(a) for each $w \in W_0^{1,p}(\Omega)$, 
\[ \|w - \phi_1\|_{1,p} \leq r \implies f_\infty(w) \geq f_\infty(\phi_1); \]

(b) for each $w \in W_0^{1,p}(\Omega)$, 
\[ \|w - \phi_1\|_{1,p} = r \implies f_\infty(w) > f_\infty(\phi_1). \]

Proof. Let us fix $u \in W_0^{1,p}(\Omega)$ and define $\eta_t : [0, +\infty[ \to \mathbb{R}$ by setting $\eta_t(t) = f_\infty(tu)$. It is easy to verify that $\eta_t$ assumes the minimum value:

\[ M(u) = -\left(1 - \frac{1}{p}\right) \left(\frac{1}{p}\right)^{\frac{1}{p-1}} \left[ \int_\Omega \phi_1^{p-1} u \, dx \right]^{\frac{p}{p-1}} \frac{1}{p} \int_\Omega \mathcal{L}_\infty(x, \nabla u) \, dx - \frac{\beta}{p} \int_\Omega (u^+)^p \, dx - \frac{\alpha}{p} \int_\Omega (u^-)^p \, dx \right]^{\frac{1}{p-1}}. \]

Moreover, a direct computation yields for each $u \neq \phi_1$

\[ f_\infty(\phi_1) < M(u) \quad (4.25) \]

if and only if

\[ p \int_\Omega \mathcal{L}_\infty(x, \nabla u) \, dx > \beta \int_\Omega (u^+)^p \, dx + \alpha \int_\Omega (u^-)^p \, dx + (\lambda_1 - \beta) \left[ \int_\Omega \phi_1^{p-1} u \, dx \right]^p. \quad (4.26) \]

If we now set $W = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega \phi_1^{p-1} u \, dx = 0 \right\}$, we obtain

\[ W_0^{1,p}(\Omega) = \text{span}(\phi_1) \oplus W. \quad (4.27) \]

Let us now prove that (4.26) is fulfilled in a neighborhood of $\phi_1$. Since (4.26) is homogeneous of degree $p$, we may substitute $\phi_1$ with $\phi_1$. Let us first consider the case $p \geq 2$ and $\beta > 0$. In view of (4.27), by strict convexity, there exists $\varepsilon_p > 0$ such that for any $w \in W$

\[ \beta \int_\Omega ((\phi_1 + w)^+)^p \, dx + (\lambda_1 - \beta) \int_\Omega \phi_1^p \, dx \leq \]

\[ \leq \beta \int_\Omega ((\phi_1 + w)^+)^p \, dx + (\lambda_1 - \beta) \int_\Omega |\phi_1 + w|^p \, dx - (\lambda_1 - \beta) \varepsilon_p \int_\Omega |w|^p \, dx \leq \]

\[ \leq \frac{\beta}{\lambda_1} p \int_\Omega \mathcal{L}_\infty(x, \nabla (\phi_1 + w)^+) \, dx + \frac{\lambda_1 - \beta}{\lambda_1} p \int_\Omega \mathcal{L}_\infty(x, \nabla (\phi_1 + w)) \, dx + \]

\[ -(\lambda_1 - \beta) \varepsilon_p \int_\Omega |w|^p \, dx. \]

On the other hand, by Lemma (4.1.9), for a sufficiently large $M$ we get

\[ \alpha \int_\Omega ((\phi_1 + w)^-)^p \, dx \leq \]

\[ \leq \frac{1}{M} \int_\Omega \mathcal{L}_\infty(x, -\nabla (\phi_1 + w^-) \, dx \leq \frac{\beta}{\lambda_1} p \int_\Omega \mathcal{L}_\infty(x, -\nabla (\phi_1 + w^-) \, dx, \]

\[ \frac{1}{M} \int_\Omega \mathcal{L}_\infty(x, -\nabla (\phi_1 + w^-) \, dx \leq \]
for \( \|w\|_{1,p} \) small enough. By combining (4.28) and (4.29) we obtain

\[
\beta \int_{\Omega} ((\phi_1 + w)^+)^p \, dx + \alpha \int_{\Omega} ((\phi_1 + w)^-)^p \, dx + (\lambda_1 - \beta) \int_{\Omega} \phi_1^p \, dx \leq \frac{\lambda_1}{2} \int_{\Omega} |\phi_1 + w|^p \, dx \]

(4.30)

Therefore (4.26) holds in a neighborhood of \( \overline{\phi}_1 \). In view of (4.4) of \([74, \text{Lemma 4.2}]\), the case \( 1 < p < 2 \) may be treated in a similar fashion. Let us now note that

\[
\forall w \in W : \int_{\Omega} |\phi_1 + w|^p \, dx \geq \int_{\Omega} \phi_1^p \, dx.
\]

In the case \( \beta \leq 0 \) we have

\[
\beta \int_{\Omega} ((\phi_1 + w)^+)^p \, dx + \alpha \int_{\Omega} ((\phi_1 + w)^-)^p \, dx + (\lambda_1 - \beta) \int_{\Omega} \phi_1^p \, dx \leq \frac{\lambda_1}{2} \int_{\Omega} |\phi_1 + w|^p \, dx + (\alpha - \beta) \int_{\Omega} ((\phi_1 + w)^-)^p \, dx + (\lambda_1 - \frac{\lambda_1}{2}) \int_{\Omega} \phi_1^p \, dx
\]

so that we reduce to (4.30). \( \square \)

**Proposition 4.1.11.** Let \( \tau > 0 \) as in Lemma 4.1.10. Then there exist \( \tilde{t} \in \mathbb{R}^+ \) and \( \sigma > 0 \) such that for each \( t \geq \tilde{t} \) and \( w \in W_0^{1,p}(\Omega) \)

\[
\|w - \overline{\phi}_1\|_{1,p} = r \implies \tilde{f}_t(w) \geq f_\infty(\overline{\phi}_1) + \sigma.
\]

**Proof.** By contradiction, let \((t_h) \subseteq \mathbb{R}\) and \((w_h) \subseteq W_0^{1,p}(\Omega)\) such that \( t_h \geq h \) and

\[
\|w_h - \overline{\phi}_1\|_{1,p} = r, \quad \tilde{f}_{t_h}(w_h) < f_\infty(\overline{\phi}_1) + \frac{1}{h}.
\]

(4.31)

Up to a subsequence we have \( w_h \rightharpoonup w \) with \( \|w - \overline{\phi}_1\|_{1,p} \leq r \). Then, by (4.31) and (a) of the previous Lemma we get

\[
\limsup_{h} \tilde{f}_{t_h}(w_h) \leq f_\infty(\overline{\phi}_1) \leq f_\infty(w).
\]

(4.32)

In view of (c) of Theorem 4.1.7, \( w_h \) strongly converges to \( w \) and then \( \|w - \overline{\phi}_1\|_{1,p} = r \). By combining (4.32) with (b) of Lemma 4.1.10, we get a contradiction. \( \square \)

**Proposition 4.1.12.** Let \( \sigma \) and \( \overline{\tau} \) be as in the previous proposition. Then there exists \( \tilde{t} \geq \overline{t} \) such that for each \( t \geq \tilde{t} \) there exist \( v_t, w_t \in W_0^{1,p}(\Omega) \) with

\[
\|v_t - \overline{\phi}_1\|_{1,p} < r, \quad f_t(v_t) \leq \frac{\sigma}{2} + f_\infty(\overline{\phi}_1),
\]

(4.33)

\[
\|w_t - \overline{\phi}_1\|_{1,p} > r, \quad f_t(w_t) \leq \frac{\sigma}{2} + f_\infty(\overline{\phi}_1).
\]

(4.34)

Moreover we have \( \sup_{s \in [0,1]} f_t(sv_t + (1 - s)w_t) < +\infty \).
Proof. We argue by contradiction. Set $t = \bar{t} + h$ and suppose that there exists $(t_h) \subseteq \mathbb{R}$ with $t_h \geq \bar{t}$ such that for every $v_h$ and $w_h$ in $W_0^{1,p}(\Omega)$

$$
\|v_h - \bar{\phi}_1\|_{1,p} < r, \quad f_{t_h}(v_h) > \frac{\sigma}{2} + f_{\infty}(\bar{\phi}_1),
$$

$$
\|w_h - \bar{\phi}_1\|_{1,p} > r, \quad f_{t_h}(w_h) > \frac{\sigma}{2} + f_{\infty}(\bar{\phi}_1).
$$

Take now $(z_h)$ going strongly to $\bar{\phi}_1$ in $W_0^{1,p}(\Omega)$. By (a) of Theorem 4.1.7 we have $\bar{f}_{t_h}(z_h) \to f_{\infty}(\bar{\phi}_1)$. On the other hand eventually $\|z_h - \bar{\phi}_1\|_{1,p} < r$ and $f_{t_h}(z_h) \leq \frac{\sigma}{2} + f_{\infty}(\bar{\phi}_1)$, that contradicts our assumptions. Recalling (b) of Remark 4.1.8, by arguing as in the previous step, it is easy to prove (4.34). The last statement is straightforward. \hfill \Box

4.1.5 Proof of the main result

We now come to the proof of the main result of the paper.

Proof of Theorem 4.1.1. From Theorem 4.1.6 we know that $f_t$ satisfies the (CPS)$_c$ condition for any $c \in \mathbb{R}$. By Proposition 4.1.11 and Proposition 4.1.12 we may apply Theorem 1.1.9 with $u_0 = \bar{\phi}_1$ and obtain existence of at least two weak solutions $u \in W_0^{1,p}(\Omega)$ of problem (4.12) for $t > \bar{t}$ for a suitable $\bar{t}$.

Let us now prove that there exists $\bar{t}$ such that (4.12) has no solutions for $t < \bar{t}$. If the assertion was false, then we could find a sequence $(t_h) \subseteq \mathbb{R}$ with $t_h \to -\infty$ and a sequence $(u_h)$ in $W_0^{1,p}(\Omega)$ such that for every $v \in C_c^\infty(\Omega)$

$$
\int_\Omega \nabla \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla v \, dx + \int_\Omega \mathcal{D}_s \mathcal{L}(x, u_h, \nabla u_h) v \, dx = \beta \int_\Omega (u_h^+)^{p-1} v \, dx - \alpha \int_\Omega (u_h^-)^{p-1} v \, dx + \int_\Omega g_0(x, u_h) v \, dx + |t_h|^{p-2} t_h \int_\Omega \phi_1^{-1} v \, dx + \langle \omega_0, v \rangle
$$

Let us first consider the case when, up to a subsequence, $\frac{t_h}{\|u_h\|_{1,p}} \to 0$ and set $v_h = \frac{u_h}{\|u_h\|_{1,p}}$. By applying Lemma 4.1.2 with $\varrho_h = \|u_h\|_{1,p}$, $\delta_h = \omega_0$ and

$$
\gamma_h(x) = \begin{cases} 
\beta & \text{if } u_h(x) \geq 0, \\
\alpha & \text{if } u_h(x) < 0,
\end{cases}
$$

$$
\mu_h = \frac{g_0(x, u_h)}{\|u_h\|_{1,p}^{p-1}} + \frac{|t_h|^{p-2} t_h}{\|u_h\|_{1,p}^{p-2}} \varphi_1^{-1},
$$

up to a subsequence, $(v_h)$ converges strongly to some $v$ in $W_0^{1,p}(\Omega)$. Then using the same argument as in the proof of Theorem 4.1.6 we get a contradiction.

Assume now that there exists $M > 0$ such that $\|u_h\|_{1,p} \leq -Mt_h$. Then setting $w_h = -u_h t_h^{-1}$, $w_h$ weakly converges to some $w \in W_0^{1,p}(\Omega)$. By applying Lemma 4.1.2 with $\varrho_h = -t_h$, $\delta_h = \omega_0$ and

$$
\gamma_h(x) = \begin{cases} 
\beta & \text{if } u_h(x) \geq 0, \\
\alpha & \text{if } u_h(x) < 0,
\end{cases}
$$

$$
\mu_h = -\frac{g_0(x, u_h)}{|t_h|^{p-2} t_h} - \varphi_1^{-1},
$$

we get a contradiction.
we have that \( w_h \) strongly converges to \( w \) in \( W_0^{1,p}(\Omega) \). The choice of the test function \( \varphi = w^+ \) gives, as in the first case, \( w^+ = 0 \). Arguing as in the end of the proof of Theorem 4.1.6 we obtain a contradiction. \( \square \)

**Remark 4.1.13.** Even though we have only considered existence of weak solutions of (4.12), by [7, Lemma 1.4] the weak solutions \( u \in W_0^{1,p}(\Omega) \) of (4.3) belong to \( L^\infty(\Omega) \). Then, some nice regularity results can be found in [72].

### 4.2 Fully nonlinear variational inequalities

#### 4.2.1 Introduction

Starting from the pioneering paper of Ambrosetti and Prodi [5], jumping problems for semilinear elliptic equations of the type

\[
\begin{aligned}
- \sum_{i,j=1}^n D_j(a_{ij}(x)D_iu) &= g(x,u) & \text{in } \Omega \\
u &= 0 & \text{on } \partial\Omega ,
\end{aligned}
\]

have been extensively treated (see e.g. [66, 79, 81, 95]). Moreover, also the case of semilinear variational inequalities with a situation of jumping type has been discussed in [64, 80]. Very recently, quasilinear inequalities of the form:

\[
\begin{aligned}
\int_\Omega \left\{ \sum_{i,j=1}^n a_{ij}(x,u)D_iuD_j(v-u) + \frac{1}{2} \sum_{i,j=1}^n D_s a_{ij}(x,u)D_iuD_j(u-v) \right\} dx + \\
- \int_\Omega g(x,u)(v-u) \, dx &\geq \langle \omega, v-u \rangle & \forall v \in \tilde{K}_\vartheta , \\
u &\in K_\vartheta ,
\end{aligned}
\]

where \( K_\vartheta = \{ u \in H_0^1(\Omega) : u \geq \vartheta \text{ a.e. in } \Omega \} \), \( \tilde{K}_\vartheta = \{ v \in K_\vartheta : (v-u) \in L^\infty(\Omega) \} \) and \( \vartheta \in H_0^1(\Omega) \), have been considered in [63].

When \( \vartheta = -\infty \), namely we have no obstacle and the variational inequality becomes an equation, the problem has been also studied in [33, 36] by A. Canino and has been extended in [107] to the case of fully nonlinear operators.

The purpose of this paper is to study the more general class of nonlinear variational inequalities of the type:

\[
\begin{aligned}
\int_\Omega \left\{ \nabla \xi \mathcal{L}(x,u,\nabla u) \cdot \nabla (v-u) + D_s \mathcal{L}(x,u,\nabla u)(v-u) \right\} dx + \\
- \int_\Omega g(x,u)(v-u) \, dx &\geq \langle \omega, v-u \rangle & \forall v \in \tilde{K}_\vartheta , \\
u &\in K_\vartheta ,
\end{aligned}
\] (4.35)

In the main result we shall prove the existence of at least two solutions of (4.35). The framework is the same of [107], but technical difficulties arise, mainly in the verification of the Palais–Smale condition. This is due to the fact that such condition is proved in [107] using in a crucial way test functions of exponential type. Such test functions are not admissible for the variational inequality, so that a certain number of modifications is required in particular in the proofs of Theorem 4.2.5 and Theorem 4.2.8.
4.2.2 The main result

In the following, $\Omega$ will denote a bounded domain of $\mathbb{R}^n$, $1 < p < n$, $\vartheta \in W_0^{1,p}(\Omega)$ with $\vartheta^- \in L^\infty(\Omega)$, $\omega \in W^{-1,p'}(\Omega)$ and

$$ \mathcal{L} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} $$

is measurable in $x$ for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and of class $C^1$ in $(s, \xi)$ a.e. in $\Omega$. We shall assume that $\mathcal{L}(x, s, \cdot)$ is strictly convex and for each $t \in \mathbb{R}$

$$ \mathcal{L}(x, s, t) = |t|^p \mathcal{L}(x, s, \xi) $$

(4.36)

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. Furthermore, we assume that:

- there exist $\nu > 0$ and $b_1 \in \mathbb{R}$ such that

$$ \nu |\xi|^p \leq \mathcal{L}(x, s, \xi) \leq b_1 |\xi|^p, $$

(4.37)

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$;

- there exist $b_2, b_3 \in \mathbb{R}$ such that

$$ |D_s \mathcal{L}(x, s, \xi)| \leq b_2 |\xi|^p, $$

(4.38)

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and

$$ |\nabla_\xi \mathcal{L}(x, s, \xi)| \leq b_3 |\xi|^{p-1}, $$

(4.39)

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$;

- there exist $R > 0$ and a bounded Lipschitzian function $\psi : [R, +\infty[ \to [0, +\infty[$ such that

$$ s \geq R \implies D_s \mathcal{L}(x, s, \xi) \geq 0, $$

(4.40)

$$ s \geq R \implies D_s \mathcal{L}(x, s, \xi) \leq \psi'(s) \nabla_\xi \mathcal{L}(x, s, \xi), $$

(4.41)

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$. We denote by $\tilde{\psi}$ the limit of $\psi(s)$ as $s \to +\infty$.

- $g(x, s)$ is a Carathéodory function and $G(x, s) = \int_0^s g(x, \tau) \, d\tau$. We assume that there exist $a \in L^{\frac{np}{np-(p-1)p}}(\Omega)$ and $b \in L^{\frac{n}{p}}(\Omega)$ such that

$$ |g(x, s)| \leq a(x) + b(x)|s|^{p-1}, $$

(4.42)

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$. Moreover, there exists $\alpha \in \mathbb{R}$ such that

$$ \lim_{s \to +\infty} \frac{g(x, s)}{s^{p-1}} = \alpha, $$

(4.43)

for a.e. $x \in \Omega$.

Set now:

$$ \lim_{s \to +\infty} \mathcal{L}(x, s, \xi) = \mathcal{L}_\infty(x, \xi) $$
4.2. Fully nonlinear variational inequalities

Let us remark that we are not assuming the strict convexity uniformly in \( x \) so that such \( L_1 \) is pretty general. Moreover, assume that

\[
s_h \to +\infty, \; \xi_h \to \xi \implies \nabla \xi \mathcal{L}(x, s_h, \xi_h) \to \nabla \xi \mathcal{L}_\infty(x, \xi),
\]

for a.e. \( x \in \Omega \). Let now

\[
\lambda_1 = \min \left\{ p \int_\Omega \mathcal{L}_\infty(x, \nabla u) \, dx : \; u \in W^{1,p}_0(\Omega), \; \int_\Omega |u|^p \, dx = 1 \right\},
\]

be the first (nonlinear) eigenvalue of

\[
\{ u \mapsto -\text{div} \left( \nabla \xi \mathcal{L}_\infty(x, \nabla u) \right) \}.
\]

Observe that by [7, Lemma 1.4] the first eigenfunction \( \phi_1 \) belongs to \( L^\infty(\Omega) \) and by [120, Theorem 1.1] is strictly positive.

Our purpose is to study (4.35) when \( \omega = -\bar{t}^{p-1} \phi_1^{p-1} \), namely the family of problems

\[
\begin{aligned}
\int_\Omega \left\{ \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla (v - u) + D_s \mathcal{L}(x, u, \nabla u) (v - u) \right\} \, dx + \\
- \int_\Omega g(x, u) (v - u) \, dx + t^{p-1} \int_\Omega \phi_1^{p-1} (v - u) \, dx \geq 0 \quad \forall v \in \bar{K}_\theta,
\end{aligned}
\]

where

\[
K_\theta = \left\{ u \in W^{1,p}_0(\Omega) : \; u \geq \theta \text{ a.e. in } \Omega \right\}
\]

and \( \bar{K}_\theta = \{ v \in K_\theta : \; (v - u) \in L^\infty(\Omega) \} \).

Under the previous assumptions, the following is our main result:

**Theorem 4.2.1.** Assume that \( \alpha > \lambda_1 \). Then there exists \( \bar{t} \in \mathbb{R} \) such that for all \( t \geq \bar{t} \) the problem \( (P_t) \) has at least two solutions.

This result extends [63, Theorem 2.1] dealing with Lagrangians of the type

\[
\mathcal{L}(x, s, \xi) = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x, s) \xi_i \xi_j - G(x, s)
\]

for a.e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \).

In this particular case, existence of at least three solutions has been proved in [63] assuming \( \alpha > \mu_2 \) where \( \mu_2 \) is the second eigenvalue of the operator

\[
\left\{ u \mapsto - \sum_{i,j=1}^n D_j (A_{ij} D_i u) \right\}.
\]

In our general setting, since \( \mathcal{L}_\infty \) is not quadratic with respect to \( \xi \), we only have the existence of the first eigenvalue \( \lambda_1 \) and it is not clear how to define higher order eigenvalues \( \lambda_2, \lambda_3, \ldots \). Therefore in our case the comparison of \( \alpha \) with higher eigenvalues has no obvious formulation.
4.2.3 The bounded Palais–Smale condition

In this section we shall consider the more general variational inequalities (4.35). To this aim let us now introduce the functional $f : W^{1,p}_{0}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$

$$f(u) = \begin{cases} \int_{\Omega} L(x,u,\nabla u) \, dx - \int_{\Omega} G(x,u) \, dx - \langle \omega,u \rangle & u \in K_{\varnothing} \\ +\infty & u \notin K_{\varnothing}. \end{cases}$$

Definition 4.2.2. Let $c \in \mathbb{R}$. A sequence $(u_{h})$ in $K_{\varnothing}$ is said to be a concrete Palais–Smale sequence at level $c$, ($(CPS)_{c}$–sequence, for short) for $f$, if $f(u_{h}) \rightarrow c$ and there exists a sequence $(\varphi_{h})$ in $W^{-1,p'}(\Omega)$ such that $\varphi_{h} \rightarrow 0$ and

$$\int_{\Omega} \nabla_{\xi} L(x,u_{h},\nabla u_{h}) \cdot \nabla(v-u_{h}) \, dx + \int_{\Omega} D_{v} L(x,u_{h},\nabla u_{h})(v-u_{h}) \, dx + \int_{\Omega} g(x,u_{h})(v-u_{h}) \, dx - \langle \omega,v-u_{h} \rangle \geq \langle \varphi_{h},v-u_{h} \rangle \quad \forall v \in K_{\varnothing}.$$

We say that $f$ satisfies the concrete Palais–Smale condition at level $c$, ($(CPS)_{c}$, for short), if every $(CPS)_{c}$–sequence for $f$ admits a strongly convergent subsequence in $W^{1,p}_{0}(\Omega)$.

Theorem 4.2.3. Let $u \in K_{\varnothing}$ be such that $|df|(u) < +\infty$. Then there exists $\varphi$ in $W^{-1,p'}(\Omega)$ such that $\|\varphi\|^{-1,p'} \leq |df|(u)$ and

$$\int_{\Omega} \nabla_{\xi} L(x,u,\nabla u) \cdot \nabla(v-u) \, dx + \int_{\Omega} D_{v} L(x,u,\nabla u)(v-u) \, dx + \int_{\Omega} g(x,u)(v-u) \, dx - \langle \omega,v-u \rangle \geq \langle \varphi,v-u \rangle \quad \forall v \in K_{\varnothing}.$$

Proof. Argue as in [63, Theorem 4.6].

Proposition 4.2.4. Let $c \in \mathbb{R}$ and assume that $f$ satisfies the $(CPS)_{c}$ condition. Then $f$ satisfies the $(PS)_{c}$ condition.

Proof. It is an easy consequence of Theorem 4.2.3. 

Let us note that by combining (4.37) with the convexity of $L(x,s,\cdot)$, we get

$$\nabla_{\xi} L(x,s,\xi) : \xi \geq \nu|\xi|^{p} \quad (4.46)$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^{n}$. Moreover, there exists $M > 0$ such that

$$|D_{v} L(x,s,\xi)| \leq M \nabla_{\xi} L(x,s,\xi) : \xi \quad (4.47)$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^{n}$.

Moreover, we point out that assumption (4.40) may be strengthened without loss of generality. Suppose that $\theta(x) > -R$ for a.e. $x \in \Omega$ and define

$$\widetilde{L}(x,s,\xi) = \begin{cases} L(x,s,\xi) & s > -R \\ L(x,-R,\xi) & s \leq -R. \end{cases}$$
Such \( \widetilde{\mathcal{L}} \) satisfy our assumptions. On the other hand, if \( u \) satisfies

\[
\begin{aligned}
\int_{\Omega} \left\{ \nabla_{\xi} \widetilde{\mathcal{L}}(x, u, \nabla u) \cdot (v - u) + D_s \widetilde{\mathcal{L}}(x, u, \nabla u)(v - u) \right\} \, dx + \\
- \int_{\Omega} g(x, u)(v - u) \, dx + t^{p-1} \int_{\Omega} \phi_1^{p-1}(v - u) \, dx \geq 0 \quad \forall v \in \widetilde{K}_\theta,
\end{aligned}
\]

(\( \bar{P}_1 \))

then \( u \) satisfies (\( P_1 \)). Therefore, up to substituting \( \mathcal{L} \) with \( \widetilde{\mathcal{L}} \), we can assume that \( \mathcal{L} \) satisfies (4.40) for any \( s \in \mathbb{R} \) with \( |s| > R \). (Actually \( \widetilde{\mathcal{L}} \) is only locally Lipschitz in \( s \) but one might always define \( \widetilde{\mathcal{L}}(x, s, \xi) = \mathcal{L}(x, \sigma(s), \xi) \) for a suitable smooth function \( \sigma \)).

Now, we want to provide in Theorem 4.2.6 a very useful criterion for the verification of (CPS)\( _c \) condition. Let us first prove a local compactness property for (CPS)\( _c \)-sequences.

**Theorem 4.2.5.** Let \( (u_h) \) be a sequence in \( K_\theta \) and \( (\varphi_h) \) a sequence in \( W^{-1,p'}(\Omega) \) such that \( (u_h) \) is bounded in \( W^{1,p}_0(\Omega) \), \( \varphi_h \to \varphi \) and

\[
\int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot (v - u_h) \, dx + \\
+ \int_{\Omega} D_s \mathcal{L}(x, u_h, \nabla u_h)(v - u_h) \, dx \geq \langle \varphi_h, v - u_h \rangle \quad \forall v \in \overline{K}_\theta.
\]

(4.48)

Then it is possible to extract a subsequence \( (u_{h_k}) \) strongly convergent in \( W^{1,p}_0(\Omega) \).

**Proof.** Up to a subsequence, \( (u_h) \) converges to some \( u \) weakly in \( W^{1,p}_0(\Omega) \), strongly in \( L^p(\Omega) \) and a.e. in \( \Omega \). Moreover, arguing as in step I of [63, Theorem 4.18] it follows that

\[
\nabla u_h(x) \to \nabla u(x) \quad \text{for a.e. } x \in \Omega.
\]

We divide the proof into several steps.

**I** Let us prove that

\[
\limsup_h \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot (u_h - \nabla u_h) \cdot \exp \{-M(u_h - R)^-\} \, dx \leq \\
\leq \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla(-u^-) \cdot \exp \{-M(u - R)^-\} \, dx
\]

(4.49)

where \( M > 0 \) is defined in (4.47) and \( R > 0 \) has been introduced in hypothesis (4.40).

Consider the test functions

\[
v = u_h + \zeta \exp \{-M(u_h + R)^+\}
\]

in (4.48) where \( \zeta \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) and \( \zeta \geq 0 \). Then

\[
\int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla(\zeta \exp \{-M(u_h + R)^+\}) \, dx + \\
+ \int_{\Omega} [D_s \mathcal{L}(x, u_h, \nabla u_h) - M \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla(u_h + R)^+] \zeta \exp \{-M(u_h + R)^+\} \, dx \\
\geq \langle \varphi_h, \zeta \exp \{-M(u_h + R)^+\} \rangle.
\]
From (4.40) and (4.47) we deduce that
\[
[D_s \mathcal{L}(x, u_h, \nabla u_h) - M \nabla \zeta \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla(u_h + R)^+] \zeta \exp \{-M(u_h + R)^+\} \leq 0,
\]
so that by the Fatou’s Lemma we get
\[
\int_{\Omega} \nabla \zeta \mathcal{L}(x, u, \nabla u) \cdot \nabla \zeta \exp \{-M(u + R)^+\} \, dx +
\int_{\Omega} [D_s \mathcal{L}(x, u, \nabla u) - M \nabla \zeta \mathcal{L}(x, u, \nabla u) \cdot \nabla(u + R)^+] \zeta \exp \{-M(u + R)^+\} \, dx \geq
\geq \langle \varphi, \zeta \exp \{-M(u + R)^+\} \rangle \quad \forall \zeta \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega), \zeta \geq 0. \tag{4.50}
\]
Now, let us consider the functions
\[
\eta_k = \eta \exp \{M(u + R)^+\} \partial_k(u),
\]
where \(\eta \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)\) with \(\eta \geq 0\) and \(\partial_k \in C^\infty(\mathbb{R})\) is such that \(0 \leq \partial_k(s) \leq 1\), \(\partial_k = 1\) on \([-k, k]\), \(\partial_k = 0\) outside \([-2k, 2k]\) and \(|\partial_k'| \leq c/k\) for some \(c > 0\).

Putting them in (4.50), for each \(k \geq 1\) we obtain
\[
\int_{\Omega} \nabla \zeta \mathcal{L}(x, u, \nabla u) \cdot \nabla(\eta \partial_k(u)) \, dx + \int_{\Omega} D_s \mathcal{L}(x, u, \nabla u) \eta \partial_k(u) \, dx \geq
\geq \langle \varphi, \eta \partial_k(u) \rangle \quad \forall \eta \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega), \eta \geq 0.
\]
Passing to the limit as \(k \to +\infty\) we obtain
\[
\int_{\Omega} \nabla \zeta \mathcal{L}(x, u, \nabla u) \cdot \nabla \eta \, dx + \int_{\Omega} D_s \mathcal{L}(x, u, \nabla u) \eta \, dx \geq
\geq \langle \varphi, \eta \rangle \quad \forall \eta \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega), \eta \geq 0. \tag{4.51}
\]
Taking \(\eta = (\vartheta^- - u^-) \exp \{-M(u - R)^-\} \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)\) in (4.51) we get
\[
\int_{\Omega} \nabla \zeta \mathcal{L}(x, u, \nabla u) \cdot \nabla(\vartheta^- - u^-) \exp \{-M(u - R)^-\} \, dx \geq
\geq - \int_{\Omega} [D_s \mathcal{L}(x, u, \nabla u) - M \nabla \zeta \mathcal{L}(x, u, \nabla u) \cdot \nabla(u - R)^-]
(\vartheta^- - u^-) \exp \{-M(u - R)^-\} \, dx + \langle \varphi, (\vartheta^- - u^-) \exp \{-M(u - R)^-\} \rangle. \tag{4.52}
\]
On the other hand, taking
\[
v = u_h - (\vartheta^- - u_h^-) \exp \{-M(u_h - R)^-\} \geq u_h - (\vartheta^- - u_h^-) = u_h^+ - \vartheta^-\]
in (4.48) we obtain
\[
\int_{\Omega} \nabla \zeta \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla(\vartheta^- - u_h^-) \exp \{-M(u_h - R)^-\} \, dx +
\int_{\Omega} [D_s \mathcal{L}(x, u_h, \nabla u_h) - M \nabla \zeta \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla(u_h - R)^-] (\vartheta^- - u_h^-) \exp \{-M(u_h - R)^-\} \, dx \leq \langle \varphi_h, (\vartheta^- - u_h^-) \exp \{-M(u_h - R)^-\} \rangle. \tag{4.53}
\]
From (4.40) and (4.47) we deduce that
\[ D_s \mathcal{L}(x, u_h, \nabla u_h) - M \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla (u_h - R)^- \geq 0. \]

From (4.53), using Fatou’s Lemma and (4.52) we obtain
\[
\limsup_h \int_{\Omega} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla (\vartheta^- - u_h^-) \exp \{-M(u_h - R)^-\} \, dx \leq \int_{\Omega} \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla (\vartheta^- - u^-) \exp \{-M(u - R)^-\} \, dx. \tag{4.54}
\]

Since
\[
\lim_h \int_{\Omega} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \vartheta^- \exp \{-M(u_h - R)^-\} \, dx = \int_{\Omega} \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \vartheta^- \exp \{-M(u - R)^-\} \, dx,
\]
then from (4.54) we deduce (4.49).

II) Let us now prove that
\[
\limsup_h \int_{\Omega} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h^+ \exp \{-M(u_h - R)^-\} \, dx \leq \int_{\Omega} \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u^+ \exp \{-M(u - R)^-\} \, dx. \tag{4.55}
\]

We consider the test functions
\[ v = u_h - \left[(u_h^+ - \vartheta^+) \wedge \mathcal{C}\right] \exp \{-M(u_h - R)^-\} \geq \vartheta + (\vartheta^- - u_h^-) \]
in (4.48). By Fatou’s Lemma, we get
\[
\int_{\Omega} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla (u_h^+ - \vartheta^+) \exp \{-M(u_h - R)^-\} \, dx + \int_{\Omega} [D_s \mathcal{L}(x, u_h, \nabla u_h) - M \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla (u_h - R)^-] (u_h^+ - \vartheta^+) \exp \{-M(u_h - R)^-\} \, dx \leq \langle \varphi_h, (u_h^+ - \vartheta^+) \exp \{-M(u_h - R)^-\} \rangle \tag{4.56}
\]
from which we deduce that
\[
[D_s \mathcal{L}(x, u_h, \nabla u_h) - M \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla (u_h - R)^-] (u_h^+ - \vartheta^+) \exp \{-M(u_h - R)^-\}
\]
belongs to \( L^1(\Omega) \). Using Fatou’s Lemma in (4.56) we obtain
\[
\limsup_h \int_{\Omega} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla (u_h^+ - \vartheta^+) \exp \{-M(u_h - R)^-\} \, dx \leq \]
\[
\leq - \int_{\Omega} [D_s \mathcal{L}(x, u, \nabla u) - M \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla (u - R)^-] (u^+ - \vartheta^+) \exp \{-M(u - R)^-\} \, dx + \langle \varphi, (u^+ - \vartheta^+) \exp \{-M(u - R)^-\} \rangle, \tag{4.57}
\]
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from which we also deduce that

$$ [D_s \mathcal{L}(x, u, \nabla u) - M \nabla x \mathcal{L}(x, u, \nabla u) \cdot \nabla (u - R)^- ] (u^+ - \vartheta^+) \exp \{ -M (u - R)^- \} \] (4.58)$$

belongs to \( L^1(\Omega) \). Now, taking \( \eta_k = \left[ (u^+ - \vartheta^+) \wedge k \right] \exp \{ -M (u - R)^- \} \) in (4.51), we have

$$ \int_\Omega \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla \left[ (u^+ - \vartheta^+) \wedge k \right] \exp \{ -M (u - R)^- \} \, dx + $$

$$ + \int_\Omega \left[ D_s \mathcal{L}(x, u, \nabla u) - M \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla (u - R)^- \right] \left[ (u^+ - \vartheta^+) \wedge k \right] \exp \{ -M (u - R)^- \} \, dx \geq \langle \varphi, (u^+ - \vartheta^+) \wedge k \rangle \exp \{ -M (u - R)^- \} \right \rangle \). (4.59)$$

Using (4.58) and passing to the limit as \( k \to +\infty \) in (4.59), it results

$$ \int_\Omega \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla (u^+ - \vartheta^+) \exp \{ -M (u - R)^- \} \, dx + $$

$$ + \int_\Omega \left[ D_s \mathcal{L}(x, u, \nabla u) - M \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla (u - R)^- \right] (u^+ - \vartheta^+) \exp \{ -M (u - R)^- \} \, dx \geq \langle \varphi, (u^+ - \vartheta^+) \wedge k \rangle \exp \{ -M (u - R)^- \} \right \rangle \). (4.60)$$

Combining (4.60) with (4.57) we obtain

$$ \limsup_h \int_\Omega \nabla_x \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla (u_h^+ - \vartheta^+) \exp \{ -M (u_h - R)^- \} \, dx \leq $$

$$ \leq \int_\Omega \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla (u^+ - \vartheta^+) \exp \{ -M (u - R)^- \} \, dx \] (4.61)$$

Since

$$ \lim_h \int_\Omega \nabla_x \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \vartheta^+ \exp \{ -M (u_h - R)^- \} \, dx = $$

$$ = \int_\Omega \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla \vartheta^+ \exp \{ -M (u - R)^- \} \, dx$$

from (4.61) we deduce (4.55).

III) Let us finally prove that \( u_h \to u \) strongly in \( W_0^{1,p}(\Omega) \). We claim that

$$ \limsup_h \int_\Omega \nabla_x \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \exp \{ -M (u_h - R)^- \} \, dx \leq $$

$$ \leq \int_\Omega \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla u \exp \{ -M (u - R)^- \} \, dx$$
In fact using (4.49) and (4.55) we get

$$
\limsup_h \int_\Omega \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \exp \{-M(u_h - R)^-\} \, dx \\
\leq \limsup_h \int_{\Omega \cap \{ u_h \geq 0 \}} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \exp \{-M(u_h - R)^-\} \, dx + \\
+ \limsup_h \int_{\Omega \cap \{ u_h < 0 \}} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla (-u_h^+) \exp \{-M(u_h - R)^-\} \, dx \\
\leq \int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \exp \{-M(u - R)^-\} \, dx
$$

(4.62)

From (4.62) using the Fatou Lemma we get

$$
\lim_h \int_\Omega \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \exp \{-M(u_h - R)^-\} \, dx = \\
= \int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \exp \{-M(u - R)^-\} \, dx.
$$

Therefore, since by (4.46) we have

$$
\nu \exp\{-M(R + \|\partial^-\|_\infty)\} \|\nabla u_h\|^p \leq \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \exp \{-M(u_h - R)^-\},
$$

it follows that

$$
\lim_h \int_\Omega \|\nabla u_h\|^p \, dx = \int_\Omega \|\nabla u\|^p \, dx,
$$

namely the strong convergence of \((u_h)\) to \(u\) in \(W^{1,p}_0(\Omega)\).

**Theorem 4.2.6.** For every \(c \in \mathbb{R}\) the following assertions are equivalent:

(a) \(f\) satisfies the \((CPS)_c\) condition;

(b) every \((CPS)_c\)–sequence for \(f\) is bounded in \(W^{1,p}_0(\Omega)\).

**Proof.** Since the map \(u \mapsto g(x, u)\) is completely continuous from \(W^{1,p}_0(\Omega)\) to \(L^{\frac{np}{n+p'}}(\Omega)\), the proof goes like [63, Theorem 4.37].

**4.2.4 The Palais–Smale condition**

Let us now set

$$
g_0(x, s) = g(x, s) - \alpha(s^+)^{p-1}, \quad G_0(x, s) = \int_0^s g_0(x, t) \, dt.
$$

Of course, \(g_0\) is a Carathéodory function satisfying

$$
\lim_{s \to +\infty} \frac{g_0(x, s)}{s^{p-1}} = 0, \quad |g_0(x, s)| \leq a(x) + b(x)|s|^{p-1},
$$
for a.e. \( x \in \Omega \) and all \( s \in \mathbb{R} \) where \( a \in L_{\text{loc}}^{\frac{n}{n-1}+p}(\Omega) \) and \( b \in L^p(\Omega) \). Then \((P_1)\) is equivalent to finding \( u \in K_\partial \) such that

\[
\int_{\Omega} \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla (v-u) \, dx + \int_{\Omega} D_s \mathcal{L}(x, u, \nabla u)(v-u) \, dx + \\
- \alpha \int_{\Omega} (u^+)^{p-1}(v-u) \, dx - \int_{\Omega} g_0(x, u)(v-u) \, dx + t^{p-1} \int_{\Omega} \phi_1^{p-1}(v-u) \, dx \geq 0 \quad \forall v \in K_\partial.
\]

Let us define the functional \( f : W^{1,p}_0(\Omega) \to \mathbb{R} \cup \{+\infty\} \) by setting

\[
f(u) = \begin{cases} 
\int_{\Omega} \mathcal{L}(x, u, \nabla u) \, dx - \frac{\alpha}{p} \int_{\Omega} u^{+p} \, dx - \int_{\Omega} G_0(x, u) \, dx + t^{p-1} \int_{\Omega} \phi_1^{p-1} u \, dx & \text{if } u \in K_\partial \\
+\infty & \text{if } u \notin K_\partial.
\end{cases}
\]

In view of Theorem 4.2.3, any critical point of \( f \) is a weak solutions of \((P_1)\). Let us introduce a new functional \( f_t : W^{1,p}_0(\Omega) \to \mathbb{R} \cup \{+\infty\} \) by setting for each \( t > 0 \)

\[
f_t(u) = \begin{cases} 
\int_{\Omega} \mathcal{L}(x, tu, \nabla u) \, dx - \frac{\alpha}{p} \int_{\Omega} u^{+p} \, dx - \frac{1}{p} \int_{\Omega} G_0(x, tu) \, dx + \int_{\Omega} \phi_1^{1-p} u \, dx & \text{if } u \in K_t \\
+\infty & \text{if } u \notin K_t,
\end{cases}
\]

where we have set

\[
K_t = \left\{ u \in W^{1,p}_0(\Omega) : tu \geq \vartheta \text{ a.e. in } \Omega \right\}.
\]

From Theorem 4.2.3 it follows that if \( u \) is a critical point of \( f_t \) then \( tu \) satisfies \((P_1)\).

**Lemma 4.2.7.** Let \((u_h)\) a sequence in \( W^{1,p}_0(\Omega) \) and \( \varrho_h \subseteq [0, +\infty] \) with \( \varrho_h \to +\infty \). Assume that the sequence \((\frac{u_h}{\varrho_h})\) is bounded in \( W^{1,p}_0(\Omega) \). Then

\[
\frac{g_0(x, u_h)}{\varrho_h^{p-1}} \to 0 \quad \text{in} \quad L^{\frac{n}{n-1}+p}(\Omega), \quad \frac{G_0(x, u_h)}{\varrho_h^p} \to 0 \quad \text{in} \quad L^1(\Omega).
\]

**Proof.** Argue as in [33, Lemma 3.3]. \( \square \)

In view of (4.46) and (3.38), we can extend \( \psi \) to \([-N, +\infty]\) where \( N \) is such that \( \|\vartheta\|_{\infty} \leq N \), so that assumption (4.41) becomes

\[
s \geq -N \implies D_s \mathcal{L}(x, s, \xi) \leq \psi'(s) \nabla \xi \mathcal{L}(x, s, \xi) \cdot \xi.
\]

**Theorem 4.2.8.** Let \( \alpha > \lambda_1, c \in \mathbb{R} \) and let \((u_h)\) in \( K_\partial \) be a \((CPS)_c\)-sequence for \( f \). Then \((u_h)\) is bounded in \( W^{1,p}_0(\Omega) \).

**Proof.** By Definition 4.2.2, there exists a sequence \((\varphi_h)\) in \( W^{-1,p'}(\Omega) \) with \( \varphi_h \to 0 \) and

\[
\int_{\Omega} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla (v-u_h) \, dx + \int_{\Omega} D_s \mathcal{L}(x, u_h, \nabla u_h)(v-u_h) \, dx + \\
- \alpha \int_{\Omega} (u_h^+)^{p-1}(v-u_h) \, dx - \int_{\Omega} g_0(x, u_h)(v-u_h) \, dx + t^{p-1} \int_{\Omega} \phi_1^{p-1}(v-u_h) \, dx \geq \\
\geq \langle \varphi_h, v-u_h \rangle \quad \forall v \in K_\partial : (v-u_h) \in L^\infty(\Omega). \quad (4.64)
\]
We set now \( g_h = \|u_h\|_{1,p} \), and suppose by contradiction that \( g_h \to +\infty \). If we set \( z_h = g_h^{-1} u_h \), up to a subsequence, \( z_h \) converges to some \( z \) weakly in \( W_0^{1,p}(\Omega) \), strongly in \( L^p(\Omega) \) and a.e. in \( \Omega \). Note that \( z \geq 0 \) a.e. in \( \Omega \).

We shall divide the proof into several steps.

1) We firstly prove that

\[
\int_{\Omega} \nabla_{\xi} \mathcal{L}_{\infty}(x, \nabla z) \cdot \nabla z \, dx \geq \alpha \int_{\Omega} z^p \, dx .
\]

(4.65)

Consider the test functions \( v = u_h + (z \wedge k) \exp \{-\psi(u_h)\} \), where \( \psi \) is the function defined in (4.41). Putting such \( v \) in (4.64) and dividing by \( g_h^{p-1} \), we obtain

\[
\int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u_h, \nabla z_h) \cdot \nabla (z \wedge k) \exp \{-\psi(u_h)\} \, dx + \\
\frac{1}{g_h^{p-1}} \int_{\Omega} \left[ D_{\xi} \mathcal{L}(x, u_h, \nabla u_h) - \psi'(u_h) \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \right] (z \wedge k) \exp \{-\psi(u_h)\} \, dx \geq \\
\alpha \int_{\Omega} (z_h^+)^{p-1} (z \wedge k) \exp \{-\psi(u_h)\} \, dx + \int_{\Omega} \frac{g_0(x, u_h)}{g_h^{p-1}} (z \wedge k) \exp \{-\psi(u_h)\} \, dx + \\
- t^{p-1} \int_{\Omega} \frac{g_1^{p-1}}{g_h^{p-1}} (z \wedge k) \exp \{-\psi(u_h)\} \, dx + \frac{1}{g_h^{p-1}} \langle \varphi_h, (z \wedge k) \exp \{-\psi(u_h)\} \rangle .
\]

Observe now that the first term

\[
\int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u_h, \nabla z_h) \cdot \nabla (z \wedge k) \exp \{-\psi(u_h)\} \, dx
\]

passes to the limit, yielding

\[
\int_{\Omega} \nabla_{\xi} \mathcal{L}_{\infty}(x, \nabla z) \cdot \nabla (z \wedge k) \exp \{-\psi(u_h)\} \, dx .
\]

Indeed, by taking into account assumptions (4.44) and (4.39), we may apply [51, Theorem 5] and deduce that, up to a subsequence,

\[
\text{a.e. in } \Omega \setminus \{z = 0\}: \nabla z_h(x) \to \nabla z(x) .
\]

Since of course \( u_h(x) \to +\infty \) a.e. in \( \Omega \setminus \{z = 0\} \), again recalling (4.44), we have

\[
\text{a.e. in } \Omega \setminus \{z = 0\}: \nabla_{\xi} \mathcal{L}(x, u_h(x), \nabla z_h(x)) \to \nabla_{\xi} \mathcal{L}_{\infty}(x, \nabla z(x)) .
\]

Since by (4.39) the sequence \( (\nabla_{\xi} \mathcal{L}(x, u_h(x), \nabla z_h(x))) \) is bounded in \( L^p(\Omega) \), the assertion follows. Note also that the term

\[
\frac{1}{g_h^{p-1}} \langle \varphi_h, (z \wedge k) \exp \{-\psi(u_h)\} \rangle ,
\]

goes to 0 even if \( 1 < p < 2 \). Indeed, in this case, one could use the Cerami–Palais–Smale condition, which yields \( g_h \varphi_h \to 0 \) in \( W_0^{1,p}(\Omega) \).
Then, passing to the limit as \( h \to +\infty \), we get

\[
\int_{\Omega} \nabla \xi \mathcal{L}_\infty(x, \nabla z) \cdot \nabla (z \land k) \exp \{-\overline{\psi}\} \, dx \geq \alpha \int_{\Omega} z^{p-1}(z \land k) \exp \{-\overline{\psi}\} \, dx.
\]

Passing to the limit as \( k \to +\infty \), we obtain (4.65).

II) Let us prove that \( z_h \to z \) strongly in \( W^{1,p}_0(\Omega) \), so that of course \( \|z\|_{1,p} = 1 \). Consider the function \( \zeta : [-R, +\infty[ \to \mathbb{R} \) defined by

\[
\zeta(s) = \begin{cases} 
MR & \text{if } s \geq R \\
Ms & \text{if } |s| < R
\end{cases}
\]

(4.66)

where \( M \in \mathbb{R} \) is such that for a.e. \( x \in \Omega \), each \( s \in \mathbb{R} \) and \( \xi \in \mathbb{R}^n \)

\[
|D_s \mathcal{L}(x, s, \xi)| \leq M \nabla \xi \mathcal{L}(x, s, \xi) \cdot \xi.
\]

If we choose the test functions

\[
v = u_h - \frac{u_h - \vartheta}{\exp(MR)} \exp(\zeta(u_h))
\]

in (4.64), we have

\[
\int_{\Omega} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla (u_h - \vartheta) \exp\{\zeta(u_h)\} \, dx + \\
+ \int_{\Omega} [D_s \mathcal{L}(x, u_h, \nabla u_h) + \zeta'(u_h) \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h] (u_h - \vartheta) \exp\{\zeta(u_h)\} \, dx \leq \\
\leq \alpha \int_{\Omega} (u_h^+)^{p-1} (u_h - \vartheta) \exp\{\zeta(u_h)\} \, dx + \int_{\Omega} g_0(x, u_h) (u_h - \vartheta) \exp\{\zeta(u_h)\} \, dx + \\
- \ell^{p-1} \int_{\Omega} \varphi_h^{p-1} (u_h - \vartheta) \exp\{\zeta(u_h)\} \, dx + \langle \varphi_h, (u_h - \vartheta) \exp\{\zeta(u_h)\} \rangle.
\]

Note that it results

\[
[D_s \mathcal{L}(x, u_h, \nabla u_h) + \zeta'(u_h) \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h] (u_h - \vartheta) \geq 0.
\]

Therefore, after division by \( \ell^p_h \) we get

\[
\int_{\Omega} \nabla \xi \mathcal{L}(x, u_h, \nabla z_h) \cdot \nabla \left( z_h - \frac{\vartheta}{\ell_h} \right) \exp\{\zeta(u_h)\} \, dx \leq \\
\leq \alpha \int_{\Omega} (z_h^+)^{p-1} \left( z_h - \frac{\vartheta}{\ell_h} \right) \exp\{\zeta(u_h)\} \, dx + \frac{1}{\ell^{p-1}_h} \int_{\Omega} g_0(x, u_h) \left( z_h - \frac{\vartheta}{\ell_h} \right) \exp\{\zeta(u_h)\} \, dx + \\
- \frac{\ell^{p-1}}{\ell^{p-1}_h} \int_{\Omega} \varphi_h^{p-1} \left( z_h - \frac{\vartheta}{\ell_h} \right) \exp\{\zeta(u_h)\} \, dx + \frac{1}{\ell^{p-1}_h} \left\langle \varphi_h, \left( z_h - \frac{\vartheta}{\ell_h} \right) \exp\{\zeta(u_h)\} \right\rangle,
\]
which yields

\[ \limsup_h \int_{\Omega} \nabla_\xi L(x, u_h, \nabla z_h) \cdot \nabla z_h \exp\{\zeta(u_h)\} \, dx \leq \alpha \exp\{MR\} \int_{\Omega} z^p \, dx. \]  

(4.67)

By combining (4.67) with (4.65) we get

\[ \limsup_h \int_{\Omega} \nabla_\xi L(x, u_h, \nabla z_h) \cdot \nabla z_h \exp\{\zeta(u_h)\} \, dx \leq \exp\{MR\} \int_{\Omega} \nabla_\xi L_\infty(x, \nabla z) \cdot \nabla z \, dx \]

In particular, by Fatou’s Lemma, it results

\[ \exp\{MR\} \int_{\Omega} \nabla_\xi L_\infty(x, \nabla z) \cdot \nabla z \, dx \leq \liminf_h \int_{\Omega} \nabla_\xi L(x, u_h, \nabla z_h) \cdot \nabla z_h \exp\{\zeta(u_h)\} \, dx \leq \limsup_h \int_{\Omega} \nabla_\xi L(x, u_h, \nabla z_h) \cdot \nabla z_h \exp\{\zeta(u_h)\} \, dx \leq \exp\{MR\} \int_{\Omega} \nabla_\xi L_\infty(x, \nabla z) \cdot \nabla z \, dx, \]

namely, we get

\[ \int_{\Omega} \nabla_\xi L(x, u_h, \nabla z_h) \cdot \nabla z \exp\{\zeta(u_h)\} \, dx \to \int_{\Omega} \exp\{MR\} \nabla_\xi L_\infty(x, \nabla z) \cdot \nabla z \, dx. \]

Therefore, since

\[ \nu \exp\{-MR\} |\nabla z_h|^p \leq \nabla_\xi L(x, u_h, \nabla z_h) \cdot \nabla z_h \exp\{\zeta(u_h)\}, \]

thanks to the generalized Lebesgue’s theorem, we conclude that

\[ \lim \int_{\Omega} |\nabla z_h|^p \, dx = \int_{\Omega} |\nabla z|^p \, dx, \]

and \( z_h \) converges to \( z \) in \( W_0^{1,p}(\Omega) \).

III) Let us consider the test functions \( v = u_h + \varphi \exp\{-\psi(u_h)\} \) with \( \varphi \in W_0^{1,p} \cap L^\infty(\Omega) \) and \( \varphi \geq 0 \). Taking such \( v \) in (4.64) and dividing by \( g^{p-1}_h \) we obtain

\[ \int_{\Omega} \nabla_\xi L(x, u_h, \nabla z_h) \cdot \nabla \varphi \exp\{-\psi(u_h)\} \, dx + \frac{1}{g^{p-1}_h} \int_{\Omega} \left[ D_s L(x, u_h, \nabla u_h) - \psi(u_h) \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \right] \varphi \exp\{-\psi(u_h)\} \, dx \geq \alpha \int_{\Omega} (z^+_h)^{p-1} \varphi \exp\{-\psi(u_h)\} \, dx + \int_{\Omega} \frac{g_0(x, u_h)}{g^{p-1}_h} \varphi \exp\{-\psi(u_h)\} \, dx + \int_{\Omega} \frac{\varphi^{p-1}}{g^{p-1}_h} \exp\{-\psi(u_h)\} \, dx + \frac{1}{g^{p-1}_h} \varphi_\psi(\varphi, \varphi \exp\{-\psi(u_h)\}). \]
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Note that, since by step II we have $z_h \to z$ in $W_0^{1,p}(\Omega)$, the term
\[ \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u_h, \nabla z_h) \cdot \nabla \varphi \exp \{-\psi(u_h)\} \, dx \]
passes to the limit, yielding
\[ \int_{\Omega} \nabla_{\xi} \mathcal{L}_\infty(x, \nabla z) \cdot \nabla \varphi \exp \{-\overline{\psi}\} \, dx. \]
By means of (4.63), we have
\[ D_s \mathcal{L}(x, u_h, \nabla u_h) - \psi'(u_h) \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \leq 0, \]
then passing to the limit as $h \to +\infty$, we obtain
\[ \int_{\Omega} \nabla_{\xi} \mathcal{L}_\infty(x, \nabla z) \cdot \nabla \varphi \exp \{-\overline{\psi}\} \, dx - \alpha \int_{\Omega} z^{p-1} \varphi \exp \{-\overline{\psi}\} \, dx \geq 0, \]
for each $\varphi \in W_0^{1,p} \cap L^\infty(\Omega)$ with $\varphi \geq 0$ which yields
\[ \int_{\Omega} \nabla_{\xi} \mathcal{L}_\infty(x, \nabla z) \cdot \nabla \varphi \, dx \geq \alpha \int_{\Omega} z^{p-1} \varphi \, dx \tag{4.68} \]
for each $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$.

In a similar fashion, considering in (4.64) the admissible test functions
\[ v = u_h - \left( \varphi \wedge \frac{z_h - \vartheta / \vartheta_h}{\exp(\psi)} \right) \exp(\psi(u_h)) \]
with $\varphi \in W_0^{1,p} \cap L^\infty(\Omega)$ and $\varphi \geq 0$ and dividing by $\vartheta_h^{p-1}$, recalling that $z_h \to z$ strongly, we get
\[ \int_{\Omega} \nabla_{\xi} \mathcal{L}_\infty(x, \nabla z) \cdot \nabla \left[ \varphi \wedge \frac{z}{\exp(\psi)} \right] \, dx \leq \alpha \int_{\Omega} z^{p-1} \left[ \varphi \wedge \frac{z}{\exp(\psi)} \right] \, dx, \]
for each $\varphi \in W_0^{1,p} \cap L^\infty(\Omega)$ with $\varphi \geq 0$. Actually this holds for any $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$.

By substituting $\varphi$ with $t \varphi$ with $t > 0$ we obtain
\[ \int_{\Omega} \nabla_{\xi} \mathcal{L}_\infty(x, \nabla z) \cdot \nabla \left[ \varphi \wedge \frac{z}{t \exp(\psi)} \right] \, dx \leq \alpha \int_{\Omega} z^{p-1} \left[ \varphi \wedge \frac{z}{t \exp(\psi)} \right] \, dx. \]
Letting $t \to +\infty$, and taking into account (4.68), it results
\[ \int_{\Omega} \nabla_{\xi} \mathcal{L}_\infty(x, \nabla z) \cdot \nabla \varphi \, dx = \alpha \int_{\Omega} z^{p-1} \varphi \, dx \tag{4.69} \]
for each $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$. Clearly (4.69) holds for any $\varphi \in W_0^{1,p}(\Omega)$, so that $z$ is a positive eigenfunction related to $\alpha$. This is a contradiction by [74, Remark 1, pp. 161].

**Theorem 4.2.9.** Let $c \in \mathbb{R}$, $\alpha > \lambda_1$ and $t > 0$. Then $f_t$ satisfies the $(PS)c-$condition.

**Proof.** Since $f_t'(u) = \frac{f(tu)}{t}$, it is sufficient to combine Theorem 4.2.8, Theorem 4.2.6 and Proposition 4.2.4.
4.2.5 Min–Max estimates

Let us first introduce the “asymptotic functional” \( f_\infty : W^{1,p}_0(\Omega) \to \mathbb{R} \cup \{+\infty\} \) by setting

\[
f_\infty(u) = \begin{cases} 
\int_\Omega \mathcal{L}_\infty(x, \nabla u) \, dx - \frac{\alpha}{p} \int_\Omega u^p \, dx + \int_\Omega \phi_1^{p-1} u \, dx & \text{if } u \in K_\infty \\
+\infty & \text{if } u \notin K_\infty
\end{cases}
\]

where

\[
K_\infty = \left\{ u \in W^{1,p}_0(\Omega) : u \geq 0 \text{ a.e. in } \Omega \right\}.
\]

**Proposition 4.2.10.** There exist \( r > 0, \sigma > 0 \) such that

(a) for every \( u \in W^{1,p}_0(\Omega) \) with \( 0 < \|u\|_{1,p} \leq r \) then \( f_\infty(u) > 0 \);

(b) for every \( u \in W^{1,p}_0(\Omega) \) with \( \|u\|_{1,p} = r \) then \( f_\infty(u) \geq \sigma > 0 \).

**Proof.** Let us consider the weakly closed set

\[
K^* = \left\{ u \in K_\infty : \int_\Omega \mathcal{L}_\infty(x, \nabla u) \, dx - \frac{\alpha}{p} \int_\Omega u^p \, dx \leq \frac{1}{2} \int_\Omega \mathcal{L}_\infty(x, \nabla u) \, dx \right\}.
\]

In \( K_\infty \setminus K^* \) the statements are evident. On the other hand, it is easy to see that

\[
\inf \left\{ \int_\Omega v \phi_1^{p-1} \, dx : v \in K^*, \|v\|_{1,p} = 1 \right\} = \varepsilon > 0
\]

arguing by contradiction. Therefore for each \( u \in K^* \) we have

\[
f_\infty(u) = \int_\Omega \mathcal{L}_\infty(x, \nabla u) \, dx - \frac{\alpha}{p} \int_\Omega u^p \, dx + \int_\Omega \phi_1^{p-1} u \, dx \geq c\|u\|_{1,p}^p + \varepsilon \|u\|_{1,p}
\]

where \( c \in \mathbb{R} \) is a suitable constant. Thus the statements follow.

**Proposition 4.2.11.** Let \( r > 0 \) be as in the Proposition 4.2.10. Then there exist \( \bar{t} > 0, \sigma' > 0 \) such that for every \( t \geq \bar{t} \) and for every \( u \in W^{1,p}_0(\Omega) \) with \( \|u\|_{1,p} = r \) then \( f_t(u) \geq \sigma' \).

**Proof.** By contradiction, we can find two sequences \( (t_h) \subseteq \mathbb{R} \) and \( (u_h) \subseteq W^{1,p}_0(\Omega) \) such that \( t_h \geq h \) for each \( h \in \mathbb{N} \), \( \|u_h\|_{1,p} = r \) and \( f_{t_h}(u_h) < \frac{1}{h} \). Up to a subsequence, \( (u_h) \) weakly converges in \( W^{1,p}_0(\Omega) \) to some \( u \in K_\infty \). Using (b) of [107, Theorem 5], it follows that

\[
f_\infty(u) \leq \liminf_h f_{t_h}(u_h) \leq 0.
\]

By (a) of Proposition 4.2.10, we have \( u = 0 \). On the other hand, since

\[
\limsup_h f_{t_h}(u_h) \leq 0 = f_\infty(u),
\]

using (c) of [107, Theorem 5] we deduce that \( (u_h) \) strongly converges to \( u \) in \( W^{1,p}_0(\Omega) \), namely \( \|u\|_{1,p} = r \). This is impossible. \( \square \)
Chapter 4. Problems of jumping type

Proposition 4.2.12. Let \( \sigma', \tilde{t} \) as in Proposition 4.2.11. Then there exists \( \tilde{t} \geq \tilde{t} \) such that for every \( t \geq \tilde{t} \) there exist \( v_t, w_t \in W_0^{1,p}(\Omega) \) such that \( \|v_t\|_{1,p} < r, \|w_t\|_{1,p} > r \), \( f_t(v_t) \leq \frac{\sigma'}{2} \) and \( f_t(w_t) \leq \frac{\sigma'}{2} \). Moreover we have

\[
\sup \{ f_t((1-s)v_t + sw_t) : 0 \leq s \leq 1 \} < +\infty.
\]

Proof. We argue by contradiction. We set \( \tilde{t} = \tilde{t} + h \) and suppose that there exists \( (t_h) \) such that \( t_h \geq h + \tilde{t} \) and such that for every \( v_{t_h}, w_{t_h} \) in \( W_0^{1,p}(\Omega) \) with \( \|v_{t_h}\|_{1,p} < r, \|w_{t_h}\|_{1,p} > r \) it results \( f_{t_h}(v_{t_h}) > \frac{\sigma'}{2} \) and \( f_{t_h}(w_{t_h}) > \frac{\sigma'}{2} \). It is easy to prove that there exists a sequence \( (u_h) \) in \( K_{t_h} \) which strongly converges to 0 in \( W_0^{1,p}(\Omega) \) and therefore \( \|u_h\|_{1,p} < r \) and \( f_{t_h}(u_{t_h}) \leq \frac{\sigma'}{2} \) eventually as \( h \to +\infty \). This contradicts our assumptions. In a similar way one can prove the statement for \( w_t \), while the last statement is straightforward.

\[ \square \]

4.2.6 Proof of the main result

Proof of Theorem 4.2.1. By combining Theorem 4.2.9, propositions 4.2.11 and 4.2.12 we can apply Theorem 1.1.9 and deduce the assertion.

\[ \square \]
Chapter 5

Problems with loss of compactness

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We refer the reader to [96, 99, 106, 110]. Some parts of these publications have been
slightly modified to give this collection a more uniform appearance.

5.1 Positive entire solutions for fully nonlinear problems

5.1.1 Introduction

In the last few years there has been a growing interest in the study of positive solutions
to variational quasilinear equations in unbounded domains of $\mathbb{R}^n$, since these problems are
involved in various branches of mathematical physics (see [21]).
Since 1988, quasilinear elliptic equations of the form
\[-\text{div} (\varphi(\nabla u)) = g(x, u) \text{ in } \mathbb{R}^n, \quad (5.1)\]
have been extensively treated, among the others, in [15, 43, 57, 73, 122] by means of a combination of topological and variational techniques.

Moreover, existence of a positive solution \( u \in H^1(\mathbb{R}^n) \) for the more general equation
\[- \sum_{i,j=1}^n D_j(a_{ij}(x, u)D_i u) + \frac{1}{2} \sum_{i,j=1}^n D_s a_{ij}(x, u)D_i uD_j u + b(x)u = g(x, u) \text{ in } \mathbb{R}^n, \]
behaving asymptotically (\(|x| \to +\infty\)) like the problem
\[-\Delta u + \lambda u = u^{q-1} \text{ in } \mathbb{R}^n, \]
for some suitable \( \lambda > 0 \) and \( q > 2 \), has been firstly studied in 1996 in [46] via techniques of non-smooth critical point theory.

On the other hand, more recently, in a bounded domain \( \Omega \) of \( \mathbb{R}^n \) some existence results for fully nonlinear problems of the type
\[
\begin{aligned}
-\text{div} \left( \nabla \mathcal{L}(x, u, \nabla u) \right) + D_s \mathcal{L}(x, u, \nabla u) &= g(x, u) \quad \text{in } \Omega \\
\lambda u + b(x)|u|^{p-2}u &= g(x, u) \quad \text{on } \partial \Omega,
\end{aligned}
\quad (5.2)
\]
have been established in [7, 88, 100].

The goal of this section is to prove existence of a nontrivial positive solution in \( W^{1,p}(\mathbb{R}^n) \) for the nonlinear elliptic equation
\[-\text{div} \left( \nabla \mathcal{L}(x, u, \nabla u) \right) + D_s \mathcal{L}(x, u, \nabla u) + b(x)|u|^{p-2}u = g(x, u) \text{ in } \mathbb{R}^n, \quad (5.3)\]
behaving asymptotically like the \( p \)-Laplacian problem
\[-\text{div} \left( |\nabla u|^{p-2}\nabla u \right) + \lambda |u|^{p-2}u = u^{q-1} \text{ in } \mathbb{R}^n, \]
for some suitable \( \lambda > 0 \) and \( q > p \). In other words, equation (5.3) tends to regularize as \(|x| \to +\infty\) together with its associated functional \( f : W^{1,p}(\mathbb{R}^n) \to \mathbb{R} \)
\[
f(u) = \int_{\mathbb{R}^n} \mathcal{L}(x, u, \nabla u) \, dx + \frac{1}{p} \int_{\mathbb{R}^n} b(x)|u|^p \, dx - \int_{\mathbb{R}^n} G(x, u) \, dx. \quad (5.4)
\]

Since in general \( f \) is continuous but not even locally Lipschitzian, unless \( \mathcal{L} \) does not depend on \( u \) or the growth conditions on \( \mathcal{L} \) are very restrictive, we shall refer to the non-smooth critical point theory developed in [37, 48, 54, 68, 69] and we shall follow the approach of [46].

We assume that \( 1 < p < n \), the function \( \mathcal{L} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) is measurable in \( x \) for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\), of class \( C^1 \) in \((s, \xi)\) for a.e. \( x \in \mathbb{R}^n \) and \( \mathcal{L}(x, s, \cdot) \) is strictly convex and homogeneous of degree \( p \). Take \( b \in L^\infty(\mathbb{R}^n) \) with \( \underline{b} \leq b(x) \leq \overline{b} \) for a.e. \( x \in \mathbb{R}^n \) for some \( \underline{b}, \overline{b} > 0 \). Morover, we shall assume that:
5.1. Positive entire solutions for fully nonlinear problems

- there exists $\nu > 0$ such that:
  \[
  \nu |\xi|^p \leq \mathcal{L}(x, s, \xi) \leq \frac{1}{p} |\xi|^p,
  \]
  for a.e. $x \in \mathbb{R}^n$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$;

- there exists $c_1 > 0$ such that:
  \[
  |D_s \mathcal{L}(x, s, \xi)| \leq c_1 |\xi|^p,
  \]
  for a.e. $x \in \mathbb{R}^n$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

Moreover, there exist $c_2 > 0$ and $a \in L^p(\mathbb{R}^n)$ such that:
\[
|\nabla \mathcal{L}(x, s, \xi)| \leq a(x) + c_2|s|^\frac{p}{p^*} + c_2|\xi|^{p-1},
\]
for a.e. $x \in \mathbb{R}^n$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$;

- there exists $R > 0$ such that:
  \[
  s \geq R \implies D_s \mathcal{L}(x, s, \xi) s \geq 0,
  \]
  for a.e. $x \in \mathbb{R}^n$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

- uniformly in $s \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^n$ with $|\xi| \leq 1$ and $|\eta| \leq 1$
  \[
  \lim_{|x| \to +\infty} \nabla \mathcal{L}(x, s, \xi) \cdot \eta = |\xi|^{p-2} \xi \cdot \eta,
  \]
  \[
  \lim_{|x| \to +\infty} D_s \mathcal{L}(x, s, \xi) s = 0,
  \]
  \[
  \lim_{|x| \to +\infty} b(x) = \lambda,
  \]
  for some $\lambda > 0$ and with $b(x) \leq \lambda$ for a.e. $x \in \mathbb{R}^n$.

- $G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $G(x, s) = \int_0^s g(x, t) \, dt$ and there exist $\beta > 0$ and $q > p$ such that:
  \[
  s > 0 \implies 0 < qG(x, s) \leq g(x, s) s,
  \]
  \[
  (q - p) \mathcal{L}(x, s, \xi) - D_s \mathcal{L}(x, s, \xi) s \geq \beta |\xi|^p,
  \]
  for a.e. $x \in \mathbb{R}^n$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

Moreover there exist $\sigma \in ]p, p^*[$ and $c > 0$ such that:
\[
|g(x, s)| \leq d(x) + c|s|^\sigma - 1,
\]
for a.e. $x \in \mathbb{R}^n$ and all $s > 0$, where $d \in L^r(\mathbb{R}^n)$ with $r \in \left[ \frac{np}{n + pr}, p^* \right]$.

- we assume that:
  \[
  \lim_{|x| \to +\infty} \frac{g(x, s)}{s^{q-1}} = 1,
  \]
uniformly in \( s > 0 \) and
\[
\lim_{|s| \to 0} \frac{G(x, s)}{|s|^p} = 0,
\]
uniformly in \( x \in \mathbb{R}^n \) and \( g(x, s) \geq s^{q-1} \) for each \( s > 0 \).

Under the previous assumptions, the following is our main result:

**Theorem 5.1.1.** The Euler’s equation of \( f \)
\[
-\text{div} (\nabla_x \mathcal{L}(x, u, \nabla u)) + D_s \mathcal{L}(x, u, \nabla u) + b|u|^{p-2} u = g(x, u) \quad \text{in} \quad \mathbb{R}^n
\]
admits at least one nontrivial positive solution \( u \in W^{1,p}(\mathbb{R}^n) \).

This result extends to a more general setting Theorem 2 of [46] dealing with the case:
\[
\mathcal{L}(x, s, \xi) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x, s) \xi_i \xi_j,
\]
and Theorem 2.1 of [43] involving integrands of the type:
\[
\mathcal{L}(x, \xi) = \frac{1}{p} a(x) |\xi|^p,
\]
where \( a \in L^\infty(\mathbb{R}) \) and \( 1 < p < n \). Let us remark that we assume (5.8) for large values of \( s \), while in [46] it was supposed that for a.e. \( x \in \mathbb{R}^n \) and all \( \xi \in \mathbb{R}^n \)
\[
\forall s \in \mathbb{R} : \sum_{i,j=1}^{n} sD_s a_{ij}(x, s) \xi_i \xi_j \geq 0.
\]
This assumption has been widely considered in literature, not only in studying existence but also to ensure local boundedness of weak solutions (see e.g. [7]).

Condition (5.13) has been already used in [7, 88, 100] and seems to be a natural extension of what happens in the quasilinear case [37].

We point out that in a bounded domain, conditions (5.12) and (5.13) may be assumed for large values of \( s \) (see e.g. [100]). Finally (5.9), (5.10), (5.11) and (5.15) fix the asymptotic behaviour of (5.3). By (5.9) and (5.10) there exist two maps \( \varepsilon_1 : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) and \( \varepsilon_2 : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) such that:
\[
\nabla_x \mathcal{L}(x, s, \xi) \cdot \eta = |\xi|^{p-2} \xi \cdot \eta + \varepsilon_1(x, s, \xi, \eta)|\xi|^{p-1} |\eta|
\]
\[
D_s \mathcal{L}(x, s, \xi) s = \varepsilon_2(x, s, \xi)|\xi|^p
\]
where \( \varepsilon_1(x, s, \xi, \eta) \to 0 \) and \( \varepsilon_2(x, s, \xi) \to 0 \) as \( |x| \to +\infty \) uniformly in \( s \in \mathbb{R} \) and \( \xi, \eta \in \mathbb{R}^n \).
5.1.2 The concrete Palais–Smale condition

Let us now set for a.e. \( x \in \mathbb{R}^n \) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n:\)

\[
\mathcal{L}(x, s, \xi) = \begin{cases} 
\mathcal{L}(x, s, \xi) & \text{if } s \geq 0 \\
\mathcal{L}(x, 0, \xi) & \text{if } s < 0 
\end{cases} \quad \bar{g}(x, s) = \begin{cases} 
g(x, s) & \text{if } s \geq 0 \\
0 & \text{if } s < 0. \end{cases} \tag{5.20}
\]

We define a modified functional \( \bar{f} : W^{1,p}(\mathbb{R}^n) \to \mathbb{R} \) by setting

\[
\bar{f}(u) = \int_{\mathbb{R}^n} \mathcal{L}(x, u, \nabla u) \, dx + \frac{1}{p} \int_{\mathbb{R}^n} b(x)|u|^p \, dx - \int_{\mathbb{R}^n} \bar{G}(x, u) \, dx. \tag{5.21}
\]

Then the Euler’s equation of \( \bar{f} \) is given by:

\[
-\text{div} \left( \nabla_x \mathcal{L}(x, u, \nabla u) \right) + D_s \mathcal{L}(x, u, \nabla u) + b(x)|u|^{p-2}u = \bar{g}(x, u) \quad \text{in } \mathbb{R}^n. \tag{5.22}
\]

**Lemma 5.1.2.** If \( u \in W^{1,p}(\mathbb{R}^n) \) is a solution of (5.22), then \( u \) is a positive solution of (5.17).

**Proof.** Let \( Q : \mathbb{R} \to \mathbb{R} \) the Lipschitz map defined by:

\[
Q(s) = \begin{cases} 
0 & \text{if } s \geq 0 \\
s & \text{if } -1 \leq s \leq 0 \\
-1 & \text{if } s \leq -1.
\end{cases}
\]

Testing \( \bar{f}(u) \) with \( Q(u) \in W^{1,p} \cap L^\infty(\mathbb{R}^n) \) and taking into account (5.20) we have:

\[
0 = \bar{f}'(u)(Q(u)) = \int_{\mathbb{R}^n} \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla Q(u) \, dx + \\
+ \int_{\mathbb{R}^n} D_s \mathcal{L}(x, u, \nabla u)Q(u) \, dx + \int_{\mathbb{R}^n} b(x)|u|^{p-2}uQ(u) \, dx - \int_{\mathbb{R}^n} \bar{g}(x, u)Q(u) \, dx = \\
= \int_{\{1 < u < 0\}} \nabla_x \mathcal{L}(x, 0, \nabla u) \cdot \nabla u \, dx + \int_{\{u < 0\}} D_s \mathcal{L}(x, u, \nabla u)Q(u) \, dx + \\
+ \int_{\mathbb{R}^n} b(x)|u|^{p-2}uQ(u) \, dx - \int_{\{u < 0\}} \bar{g}(x, u)Q(u) \, dx = \\
= \int_{\{1 < u < 0\}} p\mathcal{L}(x, 0, \nabla u) \, dx + \int_{\{u < 0\}} b(x)|u|^{p-2}uQ(u) \, dx \geq \\
\geq b \int_{\mathbb{R}^n} |u|^{p-2}uQ(u) \, dx \geq 0.
\]

In particular, it results \( Q(u) = 0 \), namely \( u \geq 0 \). \( \square \)

Therefore, without loss of generality we shall suppose that:

\[
\forall s \leq 0 : \quad g(x, s) = 0, \quad \mathcal{L}(x, s, \xi) = \mathcal{L}(x, 0, \xi)
\]

for a.e. \( x \in \mathbb{R}^n \) and all \( \xi \in \mathbb{R}^n \).
Lemma 5.1.3. Let $c \in \mathbb{R}$. Then each $(CPS)_c$–sequence for $f$ is bounded in $W^{1,p}(\mathbb{R}^n)$.

Proof. If $(u_h)$ is a $(CPS)_c$–sequence for $f$, arguing as in [46, Lemma 2], since

$$f(u_h) - \frac{1}{q}f'(u_h)(u_h) = c + o(1)$$

as $h \to +\infty$, by (5.12) and (5.13) we get:

$$\beta \int_{\mathbb{R}^n} |\nabla u_h|^p dx + \frac{q-p}{p} \int_{\mathbb{R}^n} |u_h|^p dx \leq C$$

for some $C > 0$, hence the assertion. ∎

Let us note that there exists $M > 0$ such that:

$$|D_s \mathcal{L}(x,s,\xi)| \leq M \nabla_x \mathcal{L}(x,s,\xi) \cdot \xi$$

(5.23)

for a.e. $x \in \mathbb{R}^n$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$.

We now prove a local compactness property for $(CPS)_c$–sequences. In the following, $\Omega \subset \mathbb{R}^n$ will always denote an open and bounded subset of $\mathbb{R}^n$.

Theorem 5.1.4. Let $(u_h)$ be a bounded sequence in $W^{1,p}(\mathbb{R}^n)$ and for each $v \in C_c^\infty(\mathbb{R}^n)$ set

$$\langle w_h, v \rangle = \int_{\mathbb{R}^n} \nabla_x \mathcal{L}(x,u_h,\nabla u_h) \cdot \nabla v dx + \int_{\mathbb{R}^n} D_s \mathcal{L}(x,u_h,\nabla u_h)v dx.$$ (5.24)

If $(w_h)$ is strongly convergent to some $w$ in $W^{-1,p'}(\Omega)$ for each $\Omega \subset \mathbb{R}^n$, then $(u_h)$ admits a strongly convergent subsequence in $W^{1,p}(\Omega)$ for each $\Omega \subset \mathbb{R}^n$.

Proof. Since $(u_h)$ is bounded in $W^{1,p}(\mathbb{R}^n)$, we find a $u$ in $W^{1,p}(\mathbb{R}^n)$ such that, up to a subsequence, $u_h \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^n)$. Moreover, for each $\Omega \subset \mathbb{R}^n$ we have:

$$u_h \rightarrow u \text{ in } L^p(\Omega), \quad u_h(x) \rightarrow u(x) \text{ for a.e. } x \in \mathbb{R}^n.$$ 

By a natural extension of [23, Theorem 2.1] to unbounded domains, we have

$$\nabla u_h(x) \rightarrow \nabla u(x) \text{ for a.e. } x \in \mathbb{R}^n.$$ 

Then, following the blueprint of [100, Theorem 3.2] we obtain for each $v \in C_c^\infty(\mathbb{R}^n)$

$$\langle w, v \rangle = \int_{\mathbb{R}^n} \nabla_x \mathcal{L}(x,u,\nabla u) \cdot \nabla v dx + \int_{\mathbb{R}^n} D_s \mathcal{L}(x,u,\nabla u)v dx.$$ (5.25)

Choose now $\Omega \subset \mathbb{R}^n$ and fix a positive smooth cut–off function $\eta$ on $\mathbb{R}^n$ with $\eta = 1$ on $\Omega$. Moreover, let $\vartheta : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by:

$$\vartheta(s) = \begin{cases} M s & \text{if } 0 < s < R \\ M R & \text{if } s \geq R \\ -M s & \text{if } -R < s < 0 \\ M R & \text{if } s \leq -R, \end{cases}$$ (5.26)
where $M$ is as in (5.23). Since by [100, Proposition 3.1] $v_h = \eta u_h \exp{\vartheta(u_h)}$ are admissible test functions for (5.24), we get:

$$\int_{\mathbb{R}^n} \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \eta \exp{\vartheta(u_h)} \, dx - \langle w_h, \eta u_h \exp{\vartheta(u_h)} \rangle +$$

$$+ \int_{\mathbb{R}^n} \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \eta u_h \exp{\vartheta(u_h)} \, dx +$$

$$+ \int_{\mathbb{R}^n} \left[ D_s \mathcal{L}(x, u_h, \nabla u_h) + \vartheta'(u_h) \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \right] \eta u_h \exp{\vartheta(u_h)} \, dx = 0.$$  

Let us observe that:

$$\nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \to \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla u \quad \text{for a.e. } x \in \mathbb{R}^n.$$  

Since for each $h \in \mathbb{N}$ we have

$$\left[ -D_s \mathcal{L}(x, u_h, \nabla u_h) - \vartheta'(u_h) \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \right] \eta u_h \exp{\vartheta(u_h)} \leq 0,$$

Fatou’s Lemma yields:

$$\limsup_{h} \int_{\mathbb{R}^n} \left[ -D_s \mathcal{L}(x, u_h, \nabla u_h) - \vartheta'(u_h) \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \right] \eta u_h \exp{\vartheta(u_h)} \, dx \leq$$

$$\leq \int_{\mathbb{R}^n} \left[ -D_s \mathcal{L}(x, u, \nabla u) - \vartheta'(u) \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla u \right] \eta u \exp{\vartheta(u)} \, dx.$$  

Therefore, we conclude that:

$$\limsup_{h} \int_{\mathbb{R}^n} \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \eta \exp{\vartheta(u_h)} \, dx =$$

$$\limsup_{h} \left\{ \int_{\mathbb{R}^n} \left[ -D_s \mathcal{L}(x, u_h, \nabla u_h) - \vartheta'(u_h) \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \right] \eta u_h \exp{\vartheta(u_h)} \, dx \right.$$  

$$+ \langle w_h, \eta u_h \exp{\vartheta(u_h)} \rangle - \int_{\mathbb{R}^n} \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \eta u_h \exp{\vartheta(u_h)} \, dx \right\} \leq$$

$$\leq \left\{ \int_{\mathbb{R}^n} \left[ -D_s \mathcal{L}(x, u, \nabla u) - \vartheta'(u) \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla u \right] \eta u \exp{\vartheta(u)} \, dx +$$

$$+ \langle w, \eta u \exp{\vartheta(u)} \rangle - \int_{\mathbb{R}^n} \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla \eta u \exp{\vartheta(u)} \, dx \right\} =$$

$$= \int_{\mathbb{R}^n} \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla u \eta \exp{\vartheta(u)} \, dx,$$
where we used (5.25) with \( v = \eta u \exp\{\vartheta(u)\} \). In particular, we have

\[
\int_{\mathbb{R}^n} \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \eta \exp\{\vartheta(u)\} \, dx \leq \tag{5.27}
\]

\[
\leq \liminf_{h} \int_{\mathbb{R}^n} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \eta \exp\{\vartheta(u_h)\} \, dx \leq
\]

\[
\leq \limsup_{h} \int_{\mathbb{R}^n} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \eta \exp\{\vartheta(u_h)\} \, dx \leq
\]

\[
\leq \int_{\mathbb{R}^n} \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \eta \exp\{\vartheta(u)\} \, dx , \tag{5.28}
\]

defined.

Since \( \mathcal{L}(x, s, \cdot) \) is \( p \)-homogeneous, by (5.5) for each \( h \in \mathbb{N} \) we have

\[\nu \eta |\nabla u_h|^p \leq \eta \exp\{\vartheta(u_h)\} |\nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h|,\]

by the generalized Lebesgue’s theorem we deduce that:

\[
\lim_{h} \int_{\mathbb{R}^n} \eta |\nabla u_h|^p \, dx = \int_{\mathbb{R}^n} \eta |\nabla u|^p \, dx .
\]

Up to substituting \( \eta \) with \( \eta^p \), we get:

\[
\lim_{h} \int_{\mathbb{R}^n} \eta |\nabla u_h|^p \, dx = \int_{\mathbb{R}^n} \eta |\nabla u|^p \, dx ,
\]

which implies that

\[\eta \nabla u_h \rightarrow \eta \nabla u \text{ in } L^p(\mathbb{R}^n) ,\]

namely \( \nabla u_h \rightarrow \nabla u \) in \( L^p(\Omega) \).

Let us remark that, in general, since the imbedding

\[W^{1,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)\]

is not compact, we cannot have strong convergence of \((CPS)_c\) sequences on unbounded domains of \( \mathbb{R}^n \). Nevertheless, we have the following result:

**Lemma 5.1.5.** Let \( (u_h) \) be a \((CPS)_c\)–sequence for \( f \). Then there exists \( u \) in \( W^{1,p}(\mathbb{R}^n) \) such that, up to a subsequence, the following facts hold:

(a) \( (u_h) \) converges to \( u \) weakly in \( W^{1,p}(\mathbb{R}^n) \);

(b) \( (u_h) \) converges to \( u \) strongly in \( W^{1,p}(\Omega) \) for each \( \Omega \subset \mathbb{R}^n \);
Proof. Since by Lemma 5.1.3 the sequence \((u_h)\) is bounded in \(W^{1,p}(\mathbb{R}^n)\), of course (a) holds. Now, fixed \(\Omega \subseteq \mathbb{R}^n\), if we set \(w_h = \gamma_h + g(x, u_h) - b|u_h|^{p-2}u_h \in W^{-1,p'}(\Omega), \quad \gamma_h \to 0 \text{ in } W^{-1,p'}(\Omega),\)
(b) follows by Theorem 5.1.4 with \(w = g(x, u) - b|u|^{p-2}u\). Finally by Lemma 5.1.2, (c) is a consequence of equation (5.25).

Let us now prove a technical Lemma that we shall use later.

**Lemma 5.1.6.** Let \(c \in \mathbb{R}\) and \((u_h)\) be a bounded \((CPS)_c\)-sequence for \(f\). Then for each \(\varepsilon > 0\) there exists \(\varrho > 0\) such that
\[
\int_{\{|u_h| \leq \varrho\}} |\nabla u_h|^p \, dx \leq \varepsilon
\]
for each \(h \in \mathbb{N}\).

**Proof.** Let \(\varepsilon, \varrho > 0\) and define for \(\delta \in [0, 1]\) the function \(\vartheta_\delta : \mathbb{R} \to \mathbb{R}\) by setting
\[
\vartheta_\delta(s) = \begin{cases} 
  s & \text{if } |s| \leq \varrho \\
  \varrho + \delta \varrho - \delta s & \text{if } \varrho < s < \varrho + \frac{\varrho}{3} \\
  -\varrho - \delta \varrho - \delta s & \text{if } -\varrho - \frac{\varrho}{3} < s < -\varrho \\
  0 & \text{if } |s| \geq \varrho + \frac{\varrho}{3}.
\end{cases}
\tag{5.29}
\]

Since \(\vartheta_\delta(u_h) \in W^{1,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\), we get:
\[
\langle w_h, \vartheta_\delta(u_h) \rangle = \int_{\mathbb{R}^n} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \vartheta_\delta(u_h) \, dx + \int_{\mathbb{R}^n} D_s \mathcal{L}(x, u_h, \nabla u_h) \vartheta_\delta(u_h) \, dx + \int_{\mathbb{R}^n} b|u_h|^{p-2}u_h \vartheta_\delta(u_h) + \int_{\mathbb{R}^n} g(x, u_h) \vartheta_\delta(u_h) \, dx.
\]

Then condition (5.8), \(b(x) > 0\) and \(|\vartheta_\delta(u_h)| \leq \varrho\) yield:
\[
\int_{\mathbb{R}^n} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \vartheta_\delta(u_h) \, dx \leq \int_{\mathbb{R}^n} g(x, u_h) \vartheta_\delta(u_h) \, dx + \varrho \|u_h\|_{1,p}^p + \frac{1}{p'p' \delta \varrho} \|w_h\|_{p'-1,p'}^{p'} + \delta \|u_h\|_{1,p}^p.
\]

Since \((u_h)\) is bounded in \(W^{1,p}(\mathbb{R}^n)\), there exists \(\delta > 0\) such that
\[
\delta \|u_h\|_{1,p}^p \leq \frac{\varepsilon \varrho}{8}.
\]
and
\begin{equation}
\delta \int_{\mathbb{R}^n} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leq \frac{\varepsilon \nu}{2},
\end{equation}
uniformly with \( h \in \mathbb{N} \) so large that \( \frac{1}{p' \rho - \frac{\delta}{\rho}} \| w_h \|_{p' \rho - \frac{\delta}{\rho}} \leq \frac{\varepsilon \nu}{8} \). Now, since
\begin{equation}
\int_{\mathbb{R}^n} g(x, u_h) \partial \delta(u_h) \, dx \leq \int \left\{ |u_h| \leq \theta + \frac{\delta}{2} \right\} g(x, u_h) u_h \, dx \leq \left( \int \left\{ |u_h| \leq \theta + \frac{\delta}{2} \right\} |u_h|^{p'} \, dx \right)^{\frac{1}{p'}} + c \left( \int \left\{ |u_h| \leq \theta + \frac{\delta}{2} \right\} |u|^{\sigma} \, dx, \right.
\end{equation}
we can find \( \varrho > 0 \) such that:
\begin{equation}
\int_{\mathbb{R}^n} g(x, u_h) \partial \delta(u_h) \, dx \leq \frac{\varepsilon \nu}{8}
\end{equation}
and \( \varrho \| u_h \|_{1,p} \leq \frac{\varepsilon \nu}{8} \). Therefore we obtain
\begin{equation}
\int \left\{ |u_h| \leq \theta + \frac{\delta}{2} \right\} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \partial \delta(u_h) \, dx \leq \frac{\varepsilon \nu}{2},
\end{equation}
namely, taking into account (5.30)
\begin{equation}
\int \left\{ |u_h| \leq \theta \right\} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leq \varepsilon \nu.
\end{equation}
By (5.5) the proof is complete. \( \square \)

Let us now introduce the “asymptotic functional” \( f_\infty : W^{1,p}(\mathbb{R}^n) \to \mathbb{R} \) by setting
\begin{equation}
f_\infty(u) = \frac{1}{p} \int_{\mathbb{R}^n} |\nabla u|^p \, dx + \frac{\lambda}{p} \int_{\mathbb{R}^n} |u|^p \, dx - \frac{1}{q} \int_{\mathbb{R}^n} |u^+|^q \, dx
\end{equation}
and consider the associated \( p \)--Laplacian problem
\begin{equation}
-\text{div} (|\nabla u|^{p-2} \nabla u) + \lambda |u|^{p-2} u = u^{q-1} \text{ in } \mathbb{R}^n.
\end{equation}
(See [43] for the case \( p > 2 \) and [20] for the case \( p = 2 \).

We now investigate the behaviour of the functional \( f \) over its \((CPS)_c\)--sequences.

**Lemma 5.1.7.** Let \( (u_h) \) be a \((CPS)_c\)--sequence for \( f \) and \( u \) its weak limit. Then
\begin{equation}
f(u_h) \approx f(u) + f_\infty(u_h - u),
\end{equation}
\begin{equation}
f'(u_h)(u_h) \approx f'(u)(u) + f'_\infty(u_h - u)(u_h - u)
\end{equation}
as \( h \to +\infty \), where the notation \( A_h \approx B_h \) means \( A_h - B_h \to 0 \).
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Proof. By [38, Lemma 2.2] we have the splitting:

$$
\int_{\mathbb{R}^n} G(x, u_h) \, dx - \int_{\mathbb{R}^n} G(x, u) \, dx - \frac{1}{q} \int_{\mathbb{R}^n} |(u_h - u)^+|^q \, dx = o(1),
$$
as $h \to +\infty$. Moreover, we easily get:

$$
\int_{\mathbb{R}^n} b|u_h|^p \, dx - \int_{\mathbb{R}^n} b|u|^p \, dx - \lambda \int_{\mathbb{R}^n} |u_h - u|^p \, dx = o(1),
$$
as $h \to +\infty$. Observe now that thanks to (5.18) we have

$$
\int_{\{x > \rho\}} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx - \int_{\{x > \rho\}} |\nabla u_h|^p \, dx \to 0, \quad \text{as } \rho \to +\infty,
$$

uniformly in $h \in \mathbb{N}$ and

$$
\int_{\{x > \rho\}} \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \, dx - \int_{\{x > \rho\}} |\nabla u|^p \, dx \to 0, \quad \text{as } \rho \to +\infty.
$$

Therefore, taking into account that for each $\sigma > 0$ there exists $c_\sigma > 0$ with

$$
|\nabla u_h|^p \leq c_\sigma |\nabla u|^p + (1 + \sigma)|\nabla u_h - \nabla u|^p,
$$

we deduce that for each $\varepsilon > 0$ there exists $\rho > 0$ such that for each $h \in \mathbb{N}$

$$
\int_{\{x > \rho\}} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx - \int_{\{x > \rho\}} \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \, dx +
$$

$$
\quad - \int_{\{x > \rho\}} |\nabla (u_h - u)|^p \, dx < \tilde{c} \varepsilon,
$$

for some $\tilde{c} > 0$. On the other hand, since by Lemma 5.1.5 we have

$$
\nabla u_h \to \nabla u \text{ in } L^p(B(0, \rho), \mathbb{R}^n),
$$

we deduce

$$
\int_{\{x \leq \rho\}} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx = \int_{\{x \leq \rho\}} \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \, dx + o(1),
$$
as $h \to +\infty$. Then, for each $\varepsilon > 0$ there exists $\rho \in \mathbb{N}$ such that

$$
\int_{\{x \leq \rho\}} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx - \int_{\{x \leq \rho\}} \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \, dx +
$$

$$
\quad - \int_{\{x \leq \rho\}} |\nabla (u_h - u)|^p \, dx < \tilde{c} \varepsilon,
$$
for each $h \geq \hat{h}$, for some $\hat{c} > 0$. Putting the previous inequalities together, we have
\[
\int_{\mathbb{R}^n} \nabla_x \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx = \int_{\mathbb{R}^n} \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla u \, dx + \int_{\mathbb{R}^n} |(u_h - u)|^p \, dx + o(1)
\]
as $h \to +\infty$. Taking into account that $\mathcal{L}(x, s, \cdot)$ is homogeneous of degree $p$, (5.31) is proved.

To prove (5.32), by the previous step and condition (5.15), it suffices to show that
\[
\int_{\mathbb{R}^n} D_s \mathcal{L}(x, u_h, \nabla u_h) u_h \, dx = \int_{\mathbb{R}^n} D_s \mathcal{L}(x, u, \nabla u) u \, dx + o(1),
\]
as $h \to +\infty$. By (5.19), we find $b_1, b_2 > 0$ such that for each $\varepsilon > 0$ there exists $\rho > 0$ with
\[
\int_{\{|x| > \rho\}} D_s \mathcal{L}(x, u_h, \nabla u_h) u_h \, dx \leq b_1 \varepsilon, \quad \int_{\{|x| > \rho\}} D_s \mathcal{L}(x, u, \nabla u) u \, dx \leq b_2 \varepsilon,
\]
uniformly in $h \in \mathbb{N}$. On the other hand, combining (b) of Lemma 5.1.5 with (5.13), the generalized Lebesgue’s Theorem yields
\[
\int_{\{|x| \leq \rho\}} D_s \mathcal{L}(x, u_h, \nabla u_h) u_h \, dx = \int_{\{|x| \leq \rho\}} D_s \mathcal{L}(x, u, \nabla u) u \, dx + o(1),
\]
as $h \to +\infty$. Then, (5.32) follows by the arbitrariness of $\varepsilon$.

Let us recall from [77] the following result:

**Lemma 5.1.8.** Let $1 < p \leq \infty$ and $1 \leq q < \infty$ with $q \neq p^*$. Assume that $(u_h)$ is a bounded sequence in $L^q(\mathbb{R}^n)$ with $(\nabla u_h)$ bounded in $L^p(\mathbb{R}^n)$ and there exists $R > 0$ such that:
\[
\sup_{y \in \mathbb{R}^n} \int_{y + B_R} |u_h|^q \, dx = o(1),
\]
as $h \to +\infty$. Then $u_h \to 0$ in $L^\alpha(\mathbb{R}^n)$ for each $\alpha \in [q, p^*[$.

**Proof.** See [77, Lemma I.1].

Let $(u_h)$ denote a concrete Palais–Smale sequence for $f$ and let us assume that its weak limit $u$ is 0. If $\frac{np'}{n + p'} < r < p'$, recalling that by (5.33) it results
\[
\int_{\mathbb{R}^n} D_s \mathcal{L}(x, u_h, \nabla u_h) u_h \, dx = o(1),
\]
as $h \to +\infty$, we get:
\[
pc = pf(u_h) - f'(u_h)(u_h) + o(1) \leq \int_{\mathbb{R}^n} g(x, u_h) u_h \, dx+
\]
\[
+ o(1) \leq \|d\|_r \|u_h\|_r + c\|u_h\|_\sigma^r + o(1).
\]
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Hence, either \( \| u_h \|_{\sigma'} \) or \( \| u_h \|_{\sigma} \) does not converge strongly to 0. If we now apply Lemma 5.1.8 with \( p = q \) (note also that \( p < r', \sigma < p^* \)), taking into account that \( (u_h) \) is bounded in \( W^{1,p}(\mathbb{R}^n) \) we find \( C > 0 \) and a sequence \( (y_h) \subset \mathbb{R}^n \) with \( |y_h| \to +\infty \) such that:

\[
\int_{y_h + B_R} |u_h|^p \, dx \geq C,
\]

for some \( R > 0 \). In particular, if \( \tau_h u_h(x) = u_h(x - y_h) \), we have

\[
\int_{B_R} |\tau_h u_h|^p \, dx \geq C
\]

and there exists \( \varpi \neq 0 \) such that:

\[
\tau_h u_h \to \varpi \text{ in } W^{1,p}(\mathbb{R}^n). \quad (5.34)
\]

If \( r = \frac{np'}{n + p'} \), the same can be obtained in a similar fashion since for each \( \varepsilon > 0 \) there exist

\[
d_{1,\varepsilon} \in L^\ell(\mathbb{R}^n) \quad \ell \in \left[ \frac{np'}{n + p'}, p' \right], \quad d_{2,\varepsilon} \in L^\frac{np'}{n + p'}(\mathbb{R}^n)
\]

such that:

\[
d = d_{1,\varepsilon} + d_{2,\varepsilon}, \quad \| d_{2,\varepsilon} \|_{\frac{np'}{n + p'}} \leq \varepsilon.
\]

We now show that \( \varpi \) is a weak solution of:

\[
-\text{div}(|\nabla u|^{p-2} \nabla u) + \lambda |u|^{p-2} u = u^{q-1} \text{ in } \mathbb{R}^n. \quad (5.35)
\]

**Lemma 5.1.9.** Let \( (u_h) \) a \((CPS)_c\)-sequence for \( f \) with \( u_h \to 0 \). Then \( \varpi \) is a weak solution of (5.35). Moreover \( \varpi > 0 \).

**Proof.** For all \( \varphi \in C_c^\infty(\mathbb{R}^n) \) and \( h \in \mathbb{N} \) we set

\[
\forall x \in \mathbb{R}^n : (\tau^h \varphi)(x) := \varphi(x + y_h).
\]

Since \( (u_h) \) is a \((CPS)_c\)-sequence for \( f \), we have that

\[
\forall \varphi \in C_c^\infty(\mathbb{R}^n) : f'(u_h)(\tau^h \varphi) = o(1),
\]

namely, as \( h \to +\infty \)

\[
\int_{\mathbb{R}^n} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \tau^h \varphi \, dx + \int_{\mathbb{R}^n} D_n \mathcal{L}(x, u_h, \nabla u_h) \tau^h \varphi \, dx +
\]

\[
+ \int_{\mathbb{R}^n} b(x)|u_h|^{p-2} u_h \tau^h \varphi \, dx - \int_{\mathbb{R}^n} g(x, u_h) \tau^h \varphi \, dx = o(1).
\]

Of course, as \( h \to +\infty \) we have

\[
\int_{\mathbb{R}^n} b(x)|u_h|^{p-2} u_h \tau^h \varphi \, dx = \int_{\text{supp} \varphi} b(x - y_h)|\tau_h u_h|^{p-2} \tau_h u_h \varphi \, dx + \lambda \int_{\mathbb{R}^n} |\varpi|^{p-2} \varpi \varphi \, dx,
\]
\[
\int_{\mathbb{R}^n} g(x, u_h) \tau^h \varphi \, dx = \int_{\sup \varphi} g(x - y_h, \tau_h u_h) \varphi \, dx \to \int_{\mathbb{R}^n} |\overline{u}|^{q-1} \varphi \, dx.
\]

Next, we have
\[
\int_{\mathbb{R}^n} \nabla \xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \tau^h \varphi \, dx =
\]
\[
= \int_{\sup \varphi} \nabla \xi \mathcal{L}(x - y_h, \tau_h u_h, \nabla \tau_h u_h) \cdot \nabla \varphi \, dx \to \int_{\mathbb{R}^n} \left| \nabla \overline{u} \right|^{p-2} \nabla \overline{u} \cdot \nabla \varphi \, dx.
\]

Now, for each \( \varepsilon > 0 \), Lemma 5.1.6 gives a \( \rho > 0 \) such that
\[
\int_{\mathbb{R}^n} D_s \mathcal{L}(x, u_h, \nabla u_h) \tau^h \varphi \, dx \leq \varepsilon c + \int_{\{|u_h| > \rho\}} D_s \mathcal{L}(x, u_h, \nabla u_h) \tau^h \varphi \, dx.
\]
On the other hand, by (5.10) we have:
\[
\int_{\{|u_h| > \rho\}} D_s \mathcal{L}(x, u_h, \nabla u_h) \tau^h \varphi \, dx =
\]
\[
= \int_{\sup \varphi \cap \{|\tau_h u_h| > \rho\}} D_s \mathcal{L}(x - y_h, \tau_h u_h, \nabla \tau_h u_h) \varphi \, dx = o(1),
\]
as \( h \to +\infty \). By arbitrariness of \( \varepsilon \) we conclude the proof. Finally \( \overline{u} \geq 0 \) follows by Lemma 5.1.2 and \( \overline{u} > 0 \) follows by [120, Theorem 1.1].

Lemma 5.1.10. Let \((u_h)\) be a \((CPS)_{c}\)-sequence for \( f \) with \( u_h \rightharpoonup 0 \). Then
\[
f_{\infty}(\overline{u}) \leq \liminf_h f_{\infty}(\tau_h u_h).
\]

Proof. Since \((u_h)\) weakly goes to 0, Lemma 5.1.7 gives \( f'_{\infty}(u_h)(u_h) \to 0 \) as \( h \to +\infty \), so that
\[
f'_{\infty}(\tau_h u_h)(\tau_h u_h) \to 0 \quad \text{as} \quad h \to +\infty,
\]

namely
\[
\int_{\mathbb{R}^n} |\nabla \tau_h u_h|^p \, dx + \lambda \int_{\mathbb{R}^n} |\tau_h u_h|^p \, dx - \int_{\mathbb{R}^n} (\tau_h u_h^+)^q \, dx \to 0
\]
as \( h \to +\infty \). Therefore
\[
f_{\infty}(\tau_h u_h) - \left( \frac{1}{p} - \frac{1}{q} \right) \int_{\mathbb{R}^n} (\tau_h u_h^+)^q \, dx \to 0.
\]
Similarly, Lemma 5.1.9 yields
\[
f_{\infty}(\overline{u}) = \left( \frac{1}{p} - \frac{1}{q} \right) \int_{\mathbb{R}^n} |\overline{u}|^q \, dx,
\]
and the assertion follows by Fatou’s Lemma.

Lemma 5.1.11. If \((u_h)\) is a \((CPS)_{c}\)-sequence for \( f \) with \( u_h \rightharpoonup 0 \), then \( f_{\infty}(\overline{u}) \leq c \)
Proof. Since Lemma 5.1.7 yields
\[ f(u_h) \approx f_\infty (\tau_h u_h), \quad \text{as } h \to +\infty, \]
by the previous Lemma we conclude the proof.

We finally come to the proof of the main result of this section:

Proof of Theorem 5.1.1. Since \( G \) is superlinear at \(+\infty\) (5.12), we have
\[
\forall u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\} : \quad u \geq 0 \implies \lim_{t \to +\infty} f(tu) = -\infty.
\]
Let \( v \in C_c^\infty(\mathbb{R}^n) \) positive be such that
\[
\forall t > 1 : \quad f(tv) < 0,
\]
and define the minimax class
\[
\Gamma = \left\{ \gamma \in C([0,1], W^{1,p}(\mathbb{R}^n)) : \quad \gamma(0) = 0, \quad \gamma(1) = v \right\},
\]
and the minimax value
\[
c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)).
\]
Let us remark that for each \( u \in W^{1,p}(\mathbb{R}^n) \)
\[
f(u) \geq \nu \|\nabla u\|_p^p + \frac{b}{p} \|u\|_p^p - \int_{\mathbb{R}^n} G(x,u) \, dx.
\]
Then, since by (5.16) it results
\[
\lim_h \frac{\int_{\mathbb{R}^n} G(x,w_h)}{\|w_h\|_{1,p}^p} = 0
\]
for each \((w_h)\) that goes to 0 in \(W^{1,p}(\mathbb{R}^n)\), \( f \) has a mountain pass geometry, and by the deformation Lemma of [37] there exists a \((CPS)_c\)–sequence \((u_h) \subset W^{1,p}(\mathbb{R}^n)\) for \( f \). By Lemma 5.1.5 it results that \((u_h)\) converges weakly to a positive weak solution \( u \) of (5.3). Therefore, if \( u \neq 0 \), we are done. On the other hand, if \( u = 0 \) let us consider \( \overline{v} \). We now prove that \( \overline{v} \) is a weak solution to our problem. Since we have for each \( u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\} \)
\[
0 \geq 0 \implies \lim_{t \to +\infty} f_\infty(tu) = -\infty,
\]
we find \( R > 0 \) so large that
\[
\forall a, b \geq 0 : \quad a + b = R \implies f_\infty(a\overline{v} + bv) < 0.
\]
Define the path \( \gamma : [0,1] \to W^{1,p}(\mathbb{R}^n) \) by
\[
\gamma(t) = \begin{cases} 
3Rt\overline{v} & \text{if } t \in [0,\frac{1}{3}], \\
(3t - 1)Rv + (2 - 3t)R\overline{v} & \text{if } t \in [\frac{1}{3},\frac{2}{3}], \\
(3R + 3t - 3Rt - 2)v & \text{if } t \in [\frac{2}{3},1].
\end{cases}
\]
Of course we have $\gamma \in \Gamma$, $f_\infty(\gamma(t)) < 0$ for each $t \in [1, 1/3]$ and by [43, Lemma 2.4]

$$\max_{t \in [0, 1/3]} f_\infty(\gamma(t)) = f_\infty(\bar{\gamma}).$$

Hence, by Lemma (5.1.11) and the assumptions on $\mathcal{L}$ and $g$, we have

$$c \leq \max_{t \in [0,1]} f(\gamma(t)) \leq \max_{t \in [0,1]} f_\infty(\gamma(t)) = f_\infty(\bar{\gamma}) \leq c.$$

Therefore, since $\gamma$ is an optimal path in $\Gamma$, by the non-smooth deformation Lemma of [37], there exists $\bar{t} \in [0,1]$ such that $\gamma(\bar{t})$ is a critical point of $f$ at level $c$. Moreover $\gamma(\bar{t}) = \pi$, otherwise

$$f(\gamma(\bar{t})) \leq f_\infty(\gamma(\bar{t})) < f_\infty(\pi) = c,$$

in contradiction with $f(\gamma(\bar{t})) = c$. Then $\bar{\gamma}$ is a positive solution to (5.3).

**Remark 5.1.12.** Let $1 < p < n$, $q > p$ and $\lambda > 0$. As a by-product of Theorem 5.1.1, taking

$$\mathcal{L}(x,s,\xi) = \frac{1}{p} |\xi|^p + \lambda \frac{1}{q} |s|^q,$$

we deduce that the problem

$$-\text{div} (|\nabla u|^{p-2}\nabla u) + \lambda |u|^{q-2} u = |u|^{q-2}u \quad \text{in } \mathbb{R}^n,$$

has at least one nontrivial positive solution $u \in W^{1,p}(\mathbb{R}^n)$. (see also [43, 122]).

In some sense, Theorem 5.1.1 implies that the $\varepsilon$-perturbed problem

$$-\text{div} ((1 + \varepsilon(x,u,\nabla u))|\nabla u|^{p-2}\nabla u) + \lambda |u|^{q-2} u = |u|^{q-2}u \quad \text{in } \mathbb{R}^n,$$

has at least one nontrivial positive solution $u \in W^{1,p}(\mathbb{R}^n)$.

**Remark 5.1.13.** By [7, Lemma 1.4] we have a local boundedness property for solutions of problem (5.3), namely for each $\Omega \subset \mathbb{R}^n$ each weak solution $u \in W^{1,p}(\Omega)$ of (5.3) belongs to $L^\infty(\Omega)$ provided that in (5.14) is $d \in L^s(\Omega)$ for a sufficiently large $s$. (see [7, 37]).

### 5.2 Fully nonlinear problems at critical growth

#### 5.2.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $1 < p < n$ and $p < q < p^* = \frac{np}{n-p}$. In this section we are concerned with the existence of two nontrivial solutions in $W^{1,p}_0(\Omega)$ of the following problem

$$\left\{ \begin{array}{ll} -\text{div} (\nabla_x \mathcal{L}(x,u,\nabla u)) + D_u \mathcal{L}(x,u,\nabla u) = |u|^{p^*-2}u + \lambda |u|^{q-2}u + \varepsilon h & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{array} \right.$$ 

with $h \in L^{p'}(\Omega)$, $h \neq 0$, provided that $\varepsilon > 0$ is small and $\lambda > 0$ is large.
Motivations for investigating problems as \((\mathcal{P}_{\varepsilon,\lambda})\) come from various situations in geometry and physics which present lack of compactness (see e.g. [29]). A typical example is Yamabe’s problem, i.e. find \(u > 0\) such that
\[
-4 \frac{n-1}{n-2} \Delta_M u = R' u^{(n+2)/(n-2)} - R(x)u \quad \text{on } M,
\]
for some constant \(R'\), where \(M\) is an \(n\)-dimensional Riemannian manifold, \(R(x)\) its scalar curvature and \(-\Delta_M\) is the Laplace–Beltrami operator on \(M\). Since \(p^*\) is the critical Sobolev exponent for which the embedding \(W^{1,p}_0(\Omega) \hookrightarrow L^{p^*}(\Omega)\) fails to be compact, as known, one encounters serious difficulties in applying variational methods to \((\mathcal{P}_{\varepsilon,\lambda})\).

As known, in general, if \(h = 0\) and \(\lambda = 0\), to obtain a solution of
\[
\begin{align*}
-\Delta_p u &= |u|^{p^* - 2} u \quad \text{in } \Omega \\
0 &= u \quad \text{on } \partial \Omega,
\end{align*}
\]
one has to consider in detail the geometry of \(\Omega\) (see e.g. [17]) or has to replace the critical term \(u^{p^* - 1}\) with \(u^{p^* - 1 - \varepsilon}\) and then investigate the limits of \(u_\varepsilon\) as \(\varepsilon \to 0\) (nearly critical growth, see [61] and references therein).

Let us now assume that \(h = 0\) and \(\lambda \neq 0\). As we showed in Corollary 6.2.7 by the general Pohozaev identity of Pucci and Serrin [91], if
\[
p^* \nabla x \mathcal{L}(x, s, \xi) \cdot x - n D_s \mathcal{L}(x, s, \xi) s \geq 0,
\]
a.e. in \(\Omega\) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\), then \((\mathcal{P}_{\varepsilon,\lambda})\) admits no nontrivial smooth solution for each \(\lambda \leq 0\) when the domain \(\Omega\) is star–shaped and \(\mathcal{L}\) is sufficiently smooth. Therefore, in this case we are reduced to consider positive \(\lambda\).

Let us briefly recall the historical background of existence results for problems at critical growth with lower–order perturbations. In 1983, in a pioneering paper [29], Brézis and Nirenberg proved that the problem:
\[
\begin{align*}
-\Delta u &= u^{(n+2)/(n-2)} + \lambda u \quad \text{in } \Omega \\
u > 0 &= \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]
has at least one solution \(u \in H^1_0(\Omega)\) provided that:

- \(\lambda \in (0, \lambda_1)\) if \(n \geq 4\),
- \(\lambda \in (\frac{2n}{n+2}, \lambda_1)\) if \(n = 3\) and \(\Omega = B(0, R)\),

where \(\lambda_1\) is the first eigenvalue of \(-\Delta\) in \(\Omega\). The extension to the \(p\)–Laplacian was achieved by Garcia Azorero and Peral Alonso in [59, 60] (see also [13]) . Namely, they proved the existence of a nontrivial solution of:
\[
\begin{align*}
-\Delta_p u &= |u|^{p^* - 2} u + \lambda |u|^{q^* - 2} u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]
provided that:

- \( \lambda \in (0, \lambda_1) \) if \( 1 < p = q < p^* \) and \( p^2 \leq n \);
- \( \lambda \in (\lambda_0, +\infty) \) if \( 1 < p < q < p^* \) and \( p^2 > n \);
- \( \lambda \in (0, +\infty) \) if \( 1 < p < q < p^* \) and \( p^2 \leq n \);
- \( \lambda \in (0, +\infty) \) if \( \max\{p, p^* - \frac{n}{p-1}\} < q < p^* \),

where \( \lambda_1 \) is the first eigenvalue of \(-\Delta_p\) and \( \lambda_0 \) is a suitable positive real number. Finally, for bifurcation and multiplicity results in the semilinear case \( (p = 2) \), we refer to the paper of Cerami, Fortunato and Struwe [39].

Let us now assume \( h \neq 0 \). Then, a natural question is whether inhomogeneous problems like \((\mathcal{P}_{\varepsilon, \lambda})\) have more than one solution. For bounded domains one of the first answers was given in 1992 by Tarantello in [118], where it is shown that the problem:

\[
\begin{cases}
-\Delta u = |u|^{2^*-2}u + h(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]

admits two distinct solutions \( u_1, u_2 \in H^1_0(\Omega) \) if \( \|h\|_p \) is small. The existence of two nontrivial solutions for the \( p \)-Laplacian problem:

\[
\begin{cases}
-\Delta_p u = |u|^{p^*-2}u + \lambda |u|^{q-2}u + h(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]

for \( 1 < p < q < p^* \), \( \lambda \) large and \( \|h\|_p \) small enough, has been proven in 1995 by Chabrowski in [40]. This achievement has been recently extended by Zhou in [124] to the equation:

\[-\Delta_p u + c|u|^{p-2}u = |u|^{p^*-2}u + f(x, u) + h(x)\]

on the entire \( \mathbb{R}^n \), where \( f(x, u) \) is a lower-order perturbation of \( |u|^{p^*-2}u \). This case involves a double loss of compactness, one due to the unboundedness of the domain and the other due to the critical Sobolev exponent. Now, more recently, some results for the more general problem

\[
\begin{cases}
-\text{div} (\nabla \mathcal{L}(x, u, \nabla u)) + D_s \mathcal{L}(x, u, \nabla u) = g(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]

with \( g \) subcritical and superlinear have been considered in [7, 88] and [100]. It is therefore natural to see what happens when \( g \) has a critical growth.

A first answer was given in 1998 by Arioli and Gazzola in [12], where they proved the existence of a nontrivial solution \( u \in H^1_0(\Omega) \) for a class of quasilinear equations of the type:

\[
-\sum_{i,j=1}^n D_j(a_{ij}(x, u)D_iu) + \frac{1}{2} \sum_{i,j=1}^n D_s a_{ij}(x, u)D_iuD_ju = |u|^{2^*-2}u + \lambda u,
\]

where the coefficients \((a_{ij}(x, s))\) satisfy some suitable assumptions, including a semilinear asymptotic behaviour as \( s \to +\infty \) (see remark 5.2.2).
5.2. Fully nonlinear problems at critical growth

Now, in view of the above mentioned results for \(-\Delta, -\Delta_p\), we expect that problems \((P_{\epsilon, \lambda})\) admits at least two nontrivial solutions for \(\epsilon\) small and \(\lambda\) large. In order to prove this, we shall argue on the functional \(f_{\epsilon, \lambda} : W^{1,p}_0(\Omega) \to \mathbb{R}\) given by

\[
f_{\epsilon, \lambda}(u) = \int_{\Omega} \mathcal{L}(x, u, \nabla u) \, dx + \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx - \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx - \epsilon \int_{\Omega} hu \, dx,
\]

where \(W^{1,p}_0(\Omega)\) will be endowed with the norm \(\|u\|_{1,p} = (\int_{\Omega} |\nabla u|^p \, dx)^{1/p}\).

The first solution is obtained via a local minimization argument while the second solution will follow by the mountain pass theorem without Palais–Smale condition in its non–smooth version (see [37]).

In general, under reasonable assumptions on \(\mathcal{L}\), \(f_{\epsilon, \lambda}\) is continuous but not even locally Lipschitzian unless \(\mathcal{L}\) does not depend on \(u\) or is subjected to some very restrictive growth conditions. Then, we shall refer to the non–smooth critical point theory developed in [37, 48, 54].

We assume that \(\mathcal{L}(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}\) is measurable in \(x\) for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\), of class \(C^1\) in \(s\) and of class \(C^2\) in \(\xi\) and that \(\mathcal{L}(x, s, \cdot)\) is strictly convex and \(p\)–homogeneous with \(\mathcal{L}(x, s, 0) = 0\). Moreover, we shall assume that:

- there exists \(\nu > 0\) such that:
  \[\mathcal{L}(x, s, \xi) \geq \frac{\nu}{p} |\xi|^p\]
a.e. in \(\Omega\) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\);
- there exists \(c_1, c_2 \in \mathbb{R}\) such that:
  \[|D_s \mathcal{L}(x, s, \xi)| \leq c_1 |\xi|^p\]
a.e. in \(\Omega\) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\) and
- there exist \(R > 0\) and \(\gamma \in [0, q - p]\) such that:
  \[|s| \geq R \implies D_s \mathcal{L}(x, s, \xi)s \geq 0\]
a.e. in \(\Omega\) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\) and
- there exist \(R > 0\) and \(\gamma \in [0, q - p]\) such that:
  \[D_s \mathcal{L}(x, s, \xi)s \leq \gamma \mathcal{L}(x, s, \xi)\]
a.e. in \(\Omega\) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\).

Assumptions (5.41) and (5.42) have already been considered in literature (see [7, 88, 100]). Under the previous assumptions, the following is our main result:
Theorem 5.2.1. There exists \( \lambda_0 > 0 \) such that for all \( \lambda > \lambda_0 \) there exists \( \varepsilon_0 > 0 \) such that \( (\mathcal{P}_{\varepsilon, \lambda}) \) has at least two nontrivial solutions in \( W^{1,p}_0(\Omega) \) for each \( 0 < \varepsilon < \varepsilon_0 \).

This result extends the achievements of [40, 118] to a more general class of elliptic boundary value problems. We stress that, unlike in [40], we proved our result without any use of concentration–compactness techniques. Indeed, to prove the existence of the first solution as a local minimum of \( f_{\varepsilon, \lambda} \), we showed that our functional is weakly lower semicontinuous on small balls of \( W^{1,p}_0(\Omega) \). From this point of view, our approach seems to be simpler and more direct.

Furthermore, we gave in Theorem 5.2.12 a precise range of compactness for \( f_{\varepsilon, \lambda} \). This, to our knowledge, has not been previously stated for fully nonlinear elliptic problems and not even for the quasilinear elliptic equation (5.38). In fact, in [12] it was only found a “nontrivial energy range” for the functional, inside which weak limits of Palais–Smale sequences are nontrivial and are solutions of (5.38).

Remark 5.2.2. We stress that no asymptotic behaviour has been assumed on \( L(x,s,\xi) \) and \( \partial_s L(x,s,\xi)s \) when \( s \) goes to \( +\infty \), while in [12], to prove that problem (5.38) has a solution, it was assumed that

\[
\lim_{s \to +\infty} a_{ij}(x,s) = \delta_{ij}, \quad \lim_{s \to +\infty} sD_s a_{ij}(x,s) = 0, \quad (i,j = 1, \ldots, n)
\]

uniformly with respect to \( x \in \Omega \), namely problem (5.38) converges “in some sense” to the semilinear equation \(-\Delta u = |u|^{2^* - 2}u + \lambda u\).

Remark 5.2.3. We point out that we assumed (5.41) just for \( |s| \geq R \), while in [12], for problem (5.38), it was assumed that:

\[
\forall s \in \mathbb{R} : \sum_{i,j=1}^{n} sD_s a_{ij}(x,s)\xi_i \xi_j \geq 0
\]

for a.e. \( x \in \Omega \) and each \( \xi \in \mathbb{R}^n \).

5.2.2 The first solution

Let us note that by combining \( L(x,s,0) = 0 \) and (5.40), one finds \( b_1, b_2 > 0 \) such that:

\[
L(x,s,\xi) \leq b_1 |\xi|^p, \tag{5.43}
\]

for a.e. \( x \in \Omega \) and each \( (s,\xi) \in \mathbb{R} \times \mathbb{R}^n \) and

\[
|\nabla_\xi L(x,s,\xi)| \leq b_2 |\xi|^{p-1} \tag{5.44}
\]

for a.e. \( x \in \Omega \) and each \( (s,\xi) \in \mathbb{R} \times \mathbb{R}^n \).

We now prove a weakly lower semicontinuity property for \( f_{\varepsilon, \lambda} \).

Theorem 5.2.4. There exists \( \varrho > 0 \) such that the functional \( f_{\varepsilon, \lambda} \) is weakly lower semicontinuous on \( \{ u \in W^{1,p}_0(\Omega) : \|u\|_{1,p} \leq \varrho \} \), for each \( \lambda \in \mathbb{R} \) and \( \varepsilon > 0 \).
5.2. Fully nonlinear problems at critical growth

Proof. Let \((u_h) \subset W^{1,p}_0(\Omega)\) and \(u\) with \(u_h \to u\) in \(W^{1,p}_0(\Omega)\) and \(\|u_h\|_{1,p} \leq \rho\). Taking into account that up to a subsequence we have

\[ u_h \to u \quad \text{in} \quad L^p(\Omega), \quad \nabla u_h \to \nabla u \quad \text{in} \quad L^p(\Omega), \tag{5.45} \]

and \(u_h(x) \to u(x)\) for a.e. \(x \in \Omega\), by the growth condition (5.43), it results:

\[ \int_{\Omega} \mathcal{L}(x, u_h, \nabla u) \, dx = \int_{\Omega} \mathcal{L}(x, u, \nabla u) \, dx + o(1), \]

as \(h \to +\infty\). Moreover, note also that:

\[ \int_{\Omega} |u_h|^q \, dx = \int_{\Omega} |u|^q \, dx + o(1), \]

as \(h \to +\infty\) and

\[ \int_{\Omega} hu_h \, dx = \int_{\Omega} hu \, dx + o(1), \]

as \(h \to +\infty\). In particular, it suffices to show that for \(\rho\) small:

\[ \liminf_h \left\{ \int_{\Omega} \mathcal{L}(x, u_h, \nabla u_h) \, dx - \int_{\Omega} \mathcal{L}(x, u_h, \nabla u) \, dx + \frac{1}{p^*} \int_{\Omega} |u_h|^{p^*} \, dx + \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx \right\} \geq 0. \tag{5.46} \]

Let us now consider for each \(k \geq 1\) the function \(T_k : \mathbb{R} \to \mathbb{R}\) given by

\[ T_k(s) = \begin{cases} -k & \text{if } s \leq -k \\ s & \text{if } -k \leq s \leq k \\ k & \text{if } s \geq k \end{cases} \]

and let \(R_k : \mathbb{R} \to \mathbb{R}\) be the map defined by \(R_k = Id - T_k\), namely

\[ R_k(s) = \begin{cases} s + k & \text{if } s \leq -k \\ 0 & \text{if } -k \leq s \leq k \\ s - k & \text{if } s \geq k \end{cases} \]

It is easily seen that:

\[ \int_{\Omega} \mathcal{L}(x, u_h, \nabla u_h) \, dx = \int_{\Omega} \mathcal{L}(x, u_h, \nabla T_k(u_h)) \, dx + \int_{\Omega} \mathcal{L}(x, u_h, \nabla R_k(u_h)) \, dx, \tag{5.47} \]

for each \(k \in \mathbb{N}\). Of course, we also have:

\[ \int_{\Omega} \mathcal{L}(x, u_h, \nabla u) \, dx = \int_{\Omega} \mathcal{L}(x, u, \nabla T_k(u)) \, dx + \int_{\Omega} \mathcal{L}(x, u_h, \nabla T_k(u)) \, dx + \int_{\Omega} \mathcal{L}(x, u, \nabla R_k(u)) \, dx, \tag{5.48} \]
for each $k \in \mathbb{N}$. Now, taking into account that
\[
\int_{\Omega} |u|^{p^*-1} |u_h - u| \, dx = o(1)
\]
as $h \to +\infty$, and that for any $k \in \mathbb{N}$
\[
\int_{\Omega} |T_k(u_h) - T_k(u)|^{p^*} \, dx = o(1)
\]
as $h \to +\infty$, there exist $c_1, c_2, c_3 > 0$ such that:
\[
\frac{1}{p^*} \int_{\Omega} |u_h|^{p^*} \, dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx \leq c_1 \int_{\Omega} \left( |u_h|^{p^*-1} + |u|^{p^*-1} \right) |u_h - u| \, dx \leq c_2 \int_{\Omega} |u_h - u|^{p^*} \, dx + o(1) \leq c_3 \int_{\Omega} |T_k(u_h) - T_k(u)|^{p^*} \, dx + c_3 \int_{\Omega} |R_k(u_h) - R_k(u)|^{p^*} \, dx + o(1) = c_3 \int_{\Omega} |R_k(u_h) - R_k(u)|^{p^*} \, dx + o(1)
\]
for any $k$ fixed, as $h \to +\infty$. For each $h, k \in \mathbb{N}$ we have:
\[
\int_{\Omega} \mathcal{L}(x, u_h, \nabla R_k(u_h)) \, dx \geq \frac{\nu}{p} \int_{\Omega} |\nabla R_k(u_h)|^p \, dx.
\]
On the other hand, by the definition of $R_k$ we have:
\[
\int_{\Omega} \mathcal{L}(x, u_h, \nabla R_k(u)) \, dx \leq c_1 \int_{\Omega} |\nabla R_k(u)|^p \, dx \leq \frac{\nu}{p} \int_{\Omega} |\nabla R_k(u)|^p \, dx + o(1),
\]
as $k \to +\infty$, uniformly in $h \in \mathbb{N}$. In particular, since for each $k \in \mathbb{N}$ it holds
\[
\liminf_h \left\{ \int_{\Omega} \mathcal{L}(x, u_h, T_k(\nabla u_h)) \, dx - \int_{\Omega} \mathcal{L}(x, u_h, T_k(\nabla u)) \, dx \right\} \geq 0,
\]
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by (5.47), (5.48) and (5.49) there exists $c_p > 0$ such that:

$$\liminf_h \left\{ \int_{\Omega} \mathcal{L}(x, u_h, \nabla u_h) \, dx - \int_{\Omega} \mathcal{L}(x, u_h, \nabla u) \, dx + \right. $$

$$\left. - \frac{1}{p^*} \int_{\Omega} |u_h|^{p^*} \, dx + \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx \right\} \geq$$

$$\geq \liminf_h \left\{ \int_{\Omega} \mathcal{L}(x, u_h, \nabla R_k(u_h)) \, dx - \int_{\Omega} \mathcal{L}(x, u_h, \nabla R_k(u)) \, dx + \right. $$

$$\left. - c_3 \int_{\Omega} |R_k(u_h) - R_k(u)|^{p^*} \, dx \right\} \geq$$

$$\geq \liminf_h \left\{ \frac{\nu}{p} \int_{\Omega} |\nabla R_k(u_h)|^p \, dx - \frac{\nu}{p} \int_{\Omega} |\nabla R_k(u)|^p \, dx + \right. $$

$$\left. - c_3 \int_{\Omega} |R_k(u_h) - R_k(u)|^{p^*} \, dx \right\} - o(1) \geq$$

$$\geq \liminf_h \left\{ c_p \int_{\Omega} |\nabla R_k(u_h) - \nabla R_k(u)|^p \, dx + \right. $$

$$\left. - c_3 \int_{\Omega} |R_k(u_h) - R_k(u)|^{p^*} \, dx \right\} - o(1)$$

as $k \to +\infty$. Now, by Sobolev inequality, we find $b_1, b_2 > 0$ with

$$\liminf_h \left\{ c_p \int_{\Omega} |\nabla R_k(u_h) - \nabla R_k(u)|^p \, dx - c_3 \int_{\Omega} |R_k(u_h) - R_k(u)|^{p^*} \, dx \right\} \geq$$

$$\geq \liminf_h \|R_k(u_h) - R_k(u)\|_{p^*}^p \left\{ b_1 - b_2\|R_k(u_h) - R_k(u)\|_{p^*}^{p^* - p} \right\} \geq 0$$

provided that $\|u_h\|_{1,p} \leq \varrho$ with $\varrho$ sufficiently small and independent of $\varepsilon$ and $\lambda$. In particular, (5.46) follows by (5.50) by the arbitrariness of $k$.

\[ \Box \]

**Lemma 5.2.5.** For each $\lambda \in \mathbb{R}$ there exist $\varepsilon > 0$ and $\varrho, \eta > 0$ such that:

$$\forall u \in W_0^{1,p}(\Omega) : \|u\|_{1,p} = \varrho \implies f_{\varepsilon, \lambda}(u) > \eta.$$ 

**Proof.** Since

$$f_{\varepsilon, \lambda}(u) \geq \frac{\nu}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx - \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx - \varepsilon \int_{\Omega} hu \, dx,$$

arguing as in [40, Lemma 2], one gets

$$f_{\varepsilon, \lambda}(u) \geq \|u\|_{1,p} \left\{ \|u\|_{1,p}^{p^* - 1} \varphi_\lambda(\|u\|_{1,p}) - \varepsilon \|\varrho\|_{p'} \mathcal{L}^n(\Omega)^{\frac{p^* - p}{p^*}} \right\}$$

(5.51)

where $\varphi_\lambda : [0, +\infty] \to \mathbb{R}$ is given by

$$\varphi_\lambda(\tau) = \frac{\nu}{p} - \frac{S}{p^*} \tau^{p^* - p} - \frac{\lambda}{q} \varrho^q \mathcal{L}^n(\Omega)^{\frac{p^* - p}{p^*}} \tau^{q - p}$$

for some $c > 0$. The assertion now follows. \[ \Box \]
Proposition 5.2.6. For each \( \lambda \in \mathbb{R} \) there exists \( \varepsilon_0 > 0 \) such that \((\mathcal{P}_{\varepsilon, \lambda})\) admits at least one nontrivial solution \( u_1 \in W^{1,p}_0(\Omega) \) for each \( \varepsilon < \varepsilon_0 \). Moreover \( f_{\varepsilon, \lambda}(u_1) < 0 \).

Proof. Let us choose \( \phi \in W^{1,p}_0(\Omega) \) in such a way that:
\[
\int_{\Omega} h\phi \, dx > 0.
\]
Therefore, since for each \( t > 0 \) it results
\[
f_{\varepsilon, \lambda}(t\phi) = t^p \int_{\Omega} \mathcal{L}(x, t\phi, \nabla \phi) \, dx + 
\frac{t^{p^*}}{p^*} \int_{\Omega} |\phi|^{p^*} \, dx - \frac{\lambda t^q}{q} \int_{\Omega} |\phi|^q \, dx - \varepsilon t \int_{\Omega} h\phi \, dx,
\]
there exists \( t_{\varepsilon, \lambda} > 0 \) such that \( f_{\varepsilon, \lambda}(t\phi) < 0 \) for each \( t \in ]0, t_{\varepsilon, \lambda}[, \) In particular,
\[
\inf_{\|u\|_{1,p} \leq g} f_{\varepsilon, \lambda}(u) < 0,
\]
for each \( g > 0 \) sufficiently small. Now, by Theorem 5.2.4 there exist \( g > 0 \) and \( u_1 \in W^{1,p}_0(\Omega) \) with \( \|u_1\|_{1,p} \leq g \) such that:
\[
f_{\varepsilon, \lambda}(u_1) = \min_{\|u\|_{1,p} \leq g} f_{\varepsilon, \lambda}(u) < 0.
\]
Moreover, up to reducing \( g \), it has to be \( \|u_1\|_{1,p} < g \) if \( \varepsilon > 0 \) is small enough, otherwise by Lemma 5.2.5 we would get \( f_{\varepsilon, \lambda}(u_1) \geq 0 \). In particular, \( u_1 \) is a solution of \((\mathcal{P}_{\varepsilon, \lambda})\).

Remark 5.2.7. Note that by (5.51), one can get a weak solution of \((\mathcal{P}_{\varepsilon, \lambda})\) for each \( \varepsilon > 0 \) on domains \( \Omega \) with \( \mathcal{L}^n(\Omega) \) sufficiently small.

Remark 5.2.8. Following Lemmas 3 and 4 in [40], one obtains existence of a weak solution also in the case \( p \geq q \). On the other hand we remark that if \( p \geq q \) and \( \lambda > 0 \) one has to require that \( \mathcal{L}^n(\Omega) \) is sufficiently small.

5.2.3 The concrete Palais–Smale condition

In this section we prove that \( f_{\varepsilon, \lambda} \) satisfies the concrete Palais–Smale condition at levels \( c \) within a suitable range of values.

Lemma 5.2.9. Let \( c \in \mathbb{R} \). Then each \((\text{CPS})_c\)-sequence for \( f_{\varepsilon, \lambda} \) is bounded.

Proof. Let \( c \in \mathbb{R} \) and let \((u_h)\) be a \((\text{CPS})_c\)-sequence for \( f_{\varepsilon, \lambda} \). Set:
\[
\langle w_h, \varphi \rangle = \int_{\Omega} \nabla \varepsilon \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \varphi \, dx + \int_{\Omega} D_s \mathcal{L}(x, u_h, \nabla u_h) \varphi \, dx + 
\int_{\Omega} g_{\varepsilon, \lambda}(x, u_h) \varphi \, dx - \int_{\Omega} |u_h|^{p^*-2} u_h \varphi \, dx
\]
for all \( \varphi \in C^\infty_c(\Omega) \) where \( \|w_h\|_{-1,p'} \to 0 \) as \( h \to +\infty \) and
\[
g_{\varepsilon,\lambda}(x,s) = \lambda |s|^q s + \varepsilon h(x) .
\]
for a.e. \( x \in \Omega \) and all \( s \in \mathbb{R} \). It is easily verified that for each \( \alpha \in [p,p^*] \) there exists \( b_\alpha \in L^1(\Omega) \) such that:
\[
g_{\varepsilon,\lambda}(x,s) + |s|^{p^*} \geq \alpha \left\{ \frac{\lambda}{q} |s|^q + \frac{1}{p^*} |s|^{p^*} + \varepsilon h(x) s \right\} - b_\alpha(x)
\]
a.e. in \( \Omega \) and for each \( s \in \mathbb{R} \). Now, from \( \frac{f_{\varepsilon,\lambda}(w_h)}{\|w_h\|_{1,p}} = o(1) \) as \( h \to +\infty \), one deduces that:
\[
\int_\Omega p \mathcal{L}(x,u_h,\nabla u_h) \, dx + \int_\Omega D_s \mathcal{L}(x,u_h,\nabla u_h) u_h \, dx =
\]
\[
= \int_\Omega g_{\varepsilon,\lambda}(x,u_h) u_h \, dx + \int_\Omega |u_h|^{p^*} \, dx + \langle w_h, u_h \rangle \geq
\]
\[
\geq \alpha \left\{ \frac{\lambda}{q} \int_\Omega |u_h|^q \, dx + \frac{1}{p^*} \int_\Omega |u_h|^{p^*} \, dx + \varepsilon \int_\Omega h u_h \, dx \right\} +
\]
\[- \int_\Omega b_\alpha(x) \, dx + \langle w_h, u_h \rangle \geq \alpha \int_\Omega \mathcal{L}(x,u_h,\nabla u_h) \, dx +
\]
\[- \alpha f_{\varepsilon,\lambda}(w_h) - \int_\Omega b_\alpha(x) \, dx + \langle w_h, u_h \rangle .
\]
On the other hand, by (5.42) one obtains:
\[
\frac{\nu}{p} (\alpha - \gamma - p) \int_\Omega |\nabla u_h|^p \, dx \leq
\]
\[
\leq (\alpha - \gamma - p) \int_\Omega \mathcal{L}(x,u_h,\nabla u_h) \, dx \leq
\]
\[
\leq \alpha f_{\varepsilon,\lambda}(w_h) + \int_\Omega b_\alpha(x) \, dx + \|w_h\|_{-1,p'} \|u_h\|_{1,p} .
\]
Choosing now \( \alpha > p \) in such a way that \( \alpha - \gamma - p > 0 \), one obtains the assertion. \( \square \)

**Remark 5.2.10.** By exploiting the proof of Lemma 5.2.9 one notes that
\[
\sup \left\{ \left| \int_\Omega h u \, dx \right| : u \text{ is critical point of } f_{\varepsilon,\lambda} \text{ at level } c \in \mathbb{R} \right\} \leq \sigma
\]
for some \( \sigma > 0 \) independent on \( \varepsilon > 0 \) and \( \lambda > 0 \).

**Remark 5.2.11.** Let \( 1 \leq p < \infty \). It is readily seen that the following proposition holds: assume that \( u_h \to u \) strongly in \( L^p(\Omega) \) and \( v_h \to v \) weakly in \( L^{p'}(\Omega) \) and a.e. in \( \Omega \). Then \( u_h v_h \to u v \) strongly in \( L^1(\Omega) \).

Let now \( S \) denote the best Sobolev constant (cf. [116])
\[
S = \inf \left\{ \| \nabla u \|_{p'}^p : u \in W_0^{1,p}(\Omega), \| u \|_{p'} = 1 \right\} .
\]
The next result is the main technical tool of this section.
Chapter 5. Problems with loss of compactness

**Theorem 5.2.12.** There exist $K > 0$ and $\varepsilon_0 > 0$ such that $f_{\varepsilon, \lambda}$ satisfies $(CPS)_c$ with

$$0 < c < \frac{p^* - \gamma - p}{p^*(\gamma + p)} (\mu S)^{\frac{p}{p^*}} - K\varepsilon$$

for each $\varepsilon < \varepsilon_0$ and $\lambda > 0$.

**Proof.** Let $(u_h)$ be a concrete Palais–Smale sequence for $f_{\varepsilon, \lambda}$ at level $c$. Since $(u_h)$ is bounded in $W^{1,p}_0(\Omega)$ by Lemma 5.2.9, up to a subsequence we have:

$$u_h \rightarrow u \quad \text{in} \quad L^p(\Omega), \quad \nabla u_h \rightharpoondown \nabla u \quad \text{in} \quad L^p(\Omega).$$

Moreover, as shown in [23], we also have:

for a.e. $x \in \Omega$:

$$\nabla u_h(x) \rightharpoondown \nabla u(x).$$

Arguing as in [100, Theorem 3.2] we get

$$\langle w_{\varepsilon, \lambda}, u \rangle + \|u\|_{\mathcal{H}^r}^p = \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla u \, dx + \int_{\Omega} \mathcal{L}(x, u, \nabla u) u \, dx,$$

where $w_{\varepsilon, \lambda} \in W^{-1, p'}(\Omega)$ is defined by

$$\langle w_{\varepsilon, \lambda}, v \rangle = \lambda \int_{\Omega} |u|^{q-2} u v \, dx + \varepsilon \int_{\Omega} hv \, dx.$$

This, following again [100, Theorem 3.2], yields the existence of $d \in \mathbb{R}$ with

$$\limsup_{h} \left\{ \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h - \int_{\Omega} |u_h|^p \, dx \right\} \leq d \leq \left\{ \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla u - \int_{\Omega} |u|^p \, dx \right\}.$$

Of course, we have:

$$\left\{ \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) - \nabla_{\xi} \mathcal{L}(x, u_h, \nabla (u_h - u)) \right\} \rightharpoondown \nabla_{\xi} \mathcal{L}(x, u, \nabla u)$$

in $L^p(\Omega)$. Let us note that it actually holds the strong limit

$$\left\{ \nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) - \nabla_{\xi} \mathcal{L}(x, u_h, \nabla (u_h - u)) \right\} \rightarrow \nabla_{\xi} \mathcal{L}(x, u, \nabla u)$$

in $L^p(\Omega)$, since by (5.40) there exist $\tau \in [0, 1]$ and $c > 0$ with:

$$|\nabla_{\xi} \mathcal{L}(x, u_h, \nabla u_h) - \nabla_{\xi} \mathcal{L}(x, u_h, \nabla (u_h - u))| \leq c|\nabla u_h|^{p-2} |\nabla u| + c|\nabla u|^{p-1}.$$
Therefore, by Remark 5.2.11, we have
\[
\nabla_x \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h = \nabla_x \mathcal{L}(x, u_h, \nabla (u_h - u)) \cdot \nabla u_h + \\
\nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla u_h + o(1) = \nabla_x \mathcal{L}(x, u_h, \nabla (u_h - u)) \cdot \nabla (u_h - u) + \\
\n\nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla u + o(1) \quad \text{in } L^1(\Omega),
\]
as \(h \to +\infty\), namely
\[
\nabla_x \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h - \nabla_x \mathcal{L}(x, u, \nabla u) \cdot \nabla u = \\
= \nabla_x \mathcal{L}(x, u_h, \nabla (u_h - u)) \cdot \nabla (u_h - u) + o(1) \quad \text{in } L^1(\Omega), \quad (5.54)
\]
as \(h \to +\infty\). In a similar way, since there exists \(\tilde{c} > 0\) with
\[
\left| |u_h|^{p^*} - |u_h|^{p^*-p}|u_h - u|^p \right| \leq \tilde{c} \left[ |u_h|^{p^*-p}(|u_h|^{p-1} + |u|^p) \right] |u|,
\]
one obtains
\[
\left\{ |u_h|^{p^*} - |u_h|^{p^*-p}|u_h - u|^p \right\} \to |u|^{p^*} \text{ in } L^1(\Omega). \quad (5.55)
\]
In particular, by combining (5.53), (5.54) and (5.55), it results:
\[
\limsup_h \int_{\Omega} \left[ \nabla_x \mathcal{L}(x, u_h, \nabla (u_h - u)) \cdot \nabla (u_h - u) + \\
- |u_h|^{p^*-p}|u_h - u|^p \right] dx < 0. \quad (5.56)
\]
On the other hand, by Hölder and Sobolev inequalities, we get:
\[
\int_{\Omega} \left[ \nabla_x \mathcal{L}(x, u_h, \nabla (u_h - u)) \cdot \nabla (u_h - u) - |u_h|^{p^*-p}|u_h - u|^p \right] dx \geq \\
\geq \nu \|\nabla (u_h - u)\|_p^p - \frac{1}{S} \|u_h\|_p^{p^*-p} \|\nabla (u_h - u)\|_p^p = \\
= \left\{ \nu - \frac{1}{S} \|u_h\|_p^{p^*-p} \right\} \|\nabla (u_h - u)\|_p^p, \quad (5.57)
\]
which turns out to be coercive if:
\[
\limsup_h \|u_h\|_p^{p^*} < (\nu S)^{\frac{1}{p}}. \quad (5.58)
\]
Now, from \(f_{\varepsilon, \lambda}(u_h) \to c\) we deduce
\[
\int_{\Omega} \mathcal{L}(x, u_h, \nabla u_h) dx - \frac{1}{p^*} \|u_h\|_{p^*}^{p^*} = \frac{\lambda}{q} \|u\|_q^q + \varepsilon \int_{\Omega} hu dx + c + o(1), \quad (5.59)
\]
as \(h \to +\infty\). On the other hand, by using (5.42), from \(f'_{\varepsilon, \lambda}(u_h)(u_h) \to 0\) we obtain
\[
\frac{\gamma + p}{p} \int_{\Omega} \mathcal{L}(x, u_h, \nabla u_h) dx - \frac{1}{p} \|u_h\|_p^{p^*} \geq \frac{\lambda}{p} \|u\|_q^q + \frac{\varepsilon}{p} \int_{\Omega} hu dx + o(1), \quad (5.60)
\]
as $h \to +\infty$. Multiplying (5.59) by $\frac{\gamma + p}{p}$, we obtain
\[
\frac{\gamma + p}{p} \int_\Omega L(x, u_h, \nabla u_h) \, dx - \frac{\gamma + p}{p p^*} \| u_h \|_{p^*}^* = (5.61)
\]
\[
= \frac{\gamma + p}{pq} \chi \| u \|_q^q + \frac{\gamma + p}{p} \varepsilon \int_\Omega hu + \frac{\gamma + p}{p} c + o(1),
\]
as $h \to +\infty$. Therefore, by combining (5.61) with (5.60), one gets
\[
\frac{p^* - \gamma - p}{p p^*} \| u_h \|_{p^*}^* \leq - \frac{q - \gamma - p}{pq} \chi \| u \|_q^q + \frac{\gamma + p}{p} \varepsilon \int_\Omega hu + \frac{\gamma + p}{p} c + o(1) \leq \frac{c^*}{c} \int_\Omega hu + \frac{\gamma + p}{p} c + o(1),
\]
as $h \to +\infty$. Now, taking into account Remark 5.2.10, we deduce
\[
\| u_h \|_{p^*}^* \leq \frac{p^*(\gamma + p)}{p^* - \gamma - p} c + \tilde{K} \varepsilon + o(1),
\]
as $h \to +\infty$ for some $\tilde{K} > 0$. In particular, condition (5.58) is fulfilled if
\[
\frac{p^*(\gamma + p)}{p^* - \gamma - p} c + \tilde{K} \varepsilon < (\nu S)^{\frac{\gamma}{p}}
\]
which yields range (5.52) for $\varepsilon$ small and a suitable $K > 0$. By combining (5.56) and (5.57) we conclude that $u_h$ goes to $u$ strongly in $W_0^{1,p}(\Omega)$.
\[ \Box \]

**Remark 5.2.13.** We observe that for the equation
\[-\Delta_p u = |u|^{p^*-2} u + \lambda |u|^{q-2} u + \varepsilon h \quad \text{in } \Omega,\]
being $\gamma = 0$ and $\nu = 1$, our range of compactness (5.52) reduces to:
\[
0 < c < \frac{S^\frac{\gamma}{p}}{n} - K \varepsilon.
\]
See also the results of [40].

### 5.2.4 The second solution

Let us finally come to the proof of Theorem 5.2.1.

**Proof.** Let us choose $\phi \in W_0^{1,p} \cap L^\infty(\Omega)$ such that
\[
\| \phi \|_{p^*} = 1 \quad \text{and} \quad \int_\Omega h \phi \, dx < 0.
\]
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It is easily seen that
\[ \lim_{t \to +\infty} f_{\varepsilon, \lambda}(t \phi) = -\infty , \]
so that there exists \( t_{\lambda, \varepsilon} > 0 \) with
\[ f_{\varepsilon, \lambda}(t_{\lambda, \varepsilon} \phi) = \sup_{t \geq 0} f_{\varepsilon, \lambda}(t \phi) > 0. \]  
(5.63)

Taking into account (5.42), the value \( t_{\lambda, \varepsilon} \) must satisfy:
\[ \varepsilon \int_{\Omega} h \phi = t_{\lambda, \varepsilon}^{\theta - 1} \left\{ t_{\lambda, \varepsilon}^{p - q} \left[ \int_{\Omega} p \mathcal{L}(x, t_{\lambda, \varepsilon} \phi, \nabla \phi) \, dx + \right. ight. \\
+ \left. \int_{\Omega} D_s \mathcal{L}(x, t_{\lambda, \varepsilon} \phi, \nabla \phi) t_{\lambda, \varepsilon} \phi \, dx \right] - t_{\lambda, \varepsilon}^{p - q} - \lambda \int_{\Omega} |\phi|^q \, dx \right\} \leq \\
\leq t_{\lambda, \varepsilon}^{\theta - 1} \left\{ t_{\lambda, \varepsilon}^{p - q} M \int_{\Omega} |\nabla \phi|^p \, dx - t_{\lambda, \varepsilon}^{p - q} - \lambda \int_{\Omega} |\phi|^q \, dx \right\}, \]
for some \( M > 0 \). Now, being
\[ \lim_{\lambda \to +\infty} \left\{ t_{\lambda, \varepsilon}^{p - q} + \lambda \int_{\Omega} |\phi|^q \, dx \right\} = +\infty \]
it has to be \( t_{\lambda, \varepsilon} \to 0 \) as \( \lambda \to +\infty \). In particular, by (5.63) we obtain
\[ \lim_{\lambda \to +\infty} \sup_{t \geq 0} f_{\varepsilon, \lambda}(t \phi) = 0, \]
so that there exists \( \lambda_0 > 0 \) such that:
\[ 0 < \sup_{t \geq 0} f_{\varepsilon, \lambda}(t \phi) < \frac{p^* - \gamma - p}{p^*(\gamma + p)} (\nu S)^{\frac{q}{p}} - K \varepsilon \]  
(5.64)
for each \( \lambda \geq \lambda_0 \) and \( \varepsilon < \varepsilon_0 \). Let \( w = t \phi \) with \( t \) so large that \( f_{\varepsilon, \lambda}(w) < 0 \) and set
\[ \Phi = \left\{ \gamma \in C([0,1], W_0^1 p(\Omega)) : \gamma(0) = 0, \gamma(1) = w \right\} \]
and
\[ \beta_{\varepsilon, \lambda} = \inf_{\gamma \in \Phi} \max_{t \in [0,1]} f_{\varepsilon, \lambda}(\gamma(t)) \]

Taking into account Lemma 5.2.5, by Theorem 1.1.10 one finds \( (u_h) \subset W_0^1 p(\Omega) \) with:
\[ f_{\varepsilon, \lambda}(u_h) \to \beta_{\varepsilon, \lambda}, \quad |df_{\varepsilon, \lambda}|(u_h) \to 0, \]
\[ 0 < \eta \leq \beta_{\varepsilon, \lambda} = \inf_{\gamma \in \Phi} \max_{t \in [0,1]} f_{\varepsilon, \lambda}(\gamma(t)) \leq \sup_{t \geq 0} f_{\varepsilon, \lambda}(t \phi). \]  
(5.65)

By Theorem 5.2.12 \( f_{\varepsilon, \lambda} \) satisfies \((\text{CPS})_{\beta_{\varepsilon, \lambda}}\), since by (5.64) and (5.65)
\[ \lambda \geq \lambda_0 \implies 0 < \beta_{\varepsilon, \lambda} < \frac{p^* - \gamma - p}{p^*(\gamma + p)} (\nu S)^{\frac{q}{p}} - K \varepsilon \]
for each \( \varepsilon < \varepsilon_0 \). Therefore there exist a subsequence of \( (u_h) \subset W_0^1 p(\Omega) \) strongly convergent to some \( u_2 \) which solves \((\mathcal{P}_{\varepsilon, \lambda})\). Since \( f_{\varepsilon, \lambda}(u_1) < 0 \) and \( f_{\varepsilon, \lambda}(u_2) > 0 \), of course \( u_1 \neq u_2 \).
Remark 5.2.14. In the case \( 1 < q \leq p < p^* \), in general, our method is inconclusive since it may happen that:
\[
\lim_{\lambda \to +\infty} \sup_{t \geq 0} \lambda f_{\varepsilon, \lambda}(t) \phi(t) \neq 0.
\]
See section 4 of [40] where this is discussed for the \( p \)-Laplacian.

5.3 One solution for a more general nonlinearity

We assume that \( \mathcal{L}(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) is measurable in \( x \) for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \), of class \( C^1 \) in \( (s, \xi) \) and that \( \mathcal{L}(x, s, \cdot) \) is strictly convex and \( p \)-homogeneous with \( \mathcal{L}(x, s, 0) = 0 \). Moreover:

\( \mathcal{A}_1 \) there exist \( \nu > 0 \) and \( c_1, c_2 > 0 \) such that:
\[
\mathcal{L}(x, s, \xi) \geq \frac{\nu}{p}|\xi|^p, \quad |D_s \mathcal{L}(x, s, \xi)| \leq c_1|\xi|^p, \tag{5.66}
\]
a.e. in \( \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \) and
\[
|\nabla \mathcal{L}(x, s, \xi)| \leq c_2|\xi|^{p-1}, \tag{5.67}
\]
a.e. in \( \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \);

\( \mathcal{A}_2 \) there exist \( R, R' > 0 \) and \( \gamma \in (0, p^* - p) \) such that:
\[
|s| \geq R \implies D_s \mathcal{L}(x, s, \xi)s \geq 0, \tag{5.68}
\]
\[
|s| \geq R' \implies D_s \mathcal{L}(x, s, \xi)s \leq \gamma \mathcal{L}(x, s, \xi), \tag{5.69}
\]
a.e. in \( \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \);

\( \mathcal{A}_3 \) Let \( \lambda_1 \) be the first eigenvalue of \(-\Delta_p\) with homogeneous boundary conditions. Let \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function such that:
\[
\forall \varepsilon > 0 \quad \exists a_\varepsilon \in L^{\frac{np}{np + np - p}}(\Omega) : \quad |g(x, s)| \leq a_\varepsilon(x) + \varepsilon|s|^{p^* - 1}, \tag{5.70}
\]
\[
\limsup_{s \to 0} \frac{G(x, s)}{|s|^p} < \frac{\nu \lambda_1}{p}, \quad G(x, s) \geq 0, \tag{5.71}
\]
uniformly for a.e. \( x \in \Omega \) and each \( s \in \mathbb{R} \). Moreover, we assume that there exists a nonempty open set \( \Omega_0 \subset \Omega \) such that:

- if \( n < p^2 \) (critical dimensions):
\[
\lim_{s \to +\infty} \frac{G(x, s)}{s^{p(np + np - 2)/p(n - p)}} = +\infty, \tag{5.72}
\]
uniformly for a.e. \( x \in \Omega_0 \).

- if \( n = p^2 \): \( \exists \mu > 0, \exists a > 0 \):
\[
\forall s \in [0, a] : \quad G(x, s) \geq \mu|s|^p \quad \forall s \geq a : \quad G(x, s) \geq \mu(|s|^p - a^p), \tag{5.73}
\]
for a.e. $x \in \Omega_0$.

- if $n > p^2$ : $\exists \mu > 0$, $\exists b > a$ :

$$\forall s \in [a, b] : \ G(x, s) \geq \mu$$  (5.74)

for a.e. $x \in \Omega_0$.

Conditions (5.66), (5.67), (5.68) and (5.69) have already been considered in [7, 100], while assumptions (5.70), (5.71), (5.72), (5.73) and (5.74) can be found in [13]. Note that $g(x, u)$ is neither assumed to be positive nor homogeneous in $u$.

Under additional assumptions (5.77) and (5.78), that will be stated in the next sections, we have the following result.

**Theorem 5.3.1.** $\mathcal{C}_g$ admits at least one nontrivial solution.

This result extends the achievements of [12, 13] to a more general class of elliptic boundary value problems. We remark that we assume (5.68) and (5.69) for $|s| > R$, while in [12] these assumptions are requested for each $s \in \mathbb{R}$.

### 5.3.1 Existence of one nontrivial solution

Let us first prove that the concrete Palais–Smale sequences of

$$f(u) = \int_{\Omega} \mathcal{L}(x, u, \nabla u) \, dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx - \int_{\Omega} G(x, u) \, dx$$  (5.75)

are bounded. We will make a new choice of test function, which also removes some of the technicalities involved in [100].

**Lemma 5.3.2.** Let $c \in \mathbb{R}$. Then each $(CPS)_c$–sequence for $f$ is bounded.

**Proof.** Let $c \in \mathbb{R}$ and let $(u_h)$ be a $(CPS)_c$–sequence for $f$. In the usual notations, one has $\|w_h\|_{-1, p^*} \to 0$ as $h \to +\infty$. It is easily verified that for each $\alpha \in [p, p^*]$ there exists $b_\alpha \in L^1(\Omega)$ with:

$$g(x, s)s + |s|^{p^*} \geq \alpha \left\{ G(x, s) + \frac{1}{p^*} |s|^{p^*} \right\} - b_\alpha(x)$$

a.e. in $\Omega$ and for each $s \in \mathbb{R}$. Let now $M > 0$, $k \geq 1$ and $\vartheta_k : \mathbb{R} \to \mathbb{R}$,

$$\vartheta_k(s) = \begin{cases} 
  s & \text{if } s \geq kM \\
  \frac{M}{M-1}s - \frac{M}{M-1}k & \text{if } k \leq s \leq kM \\
  0 & \text{if } -k \leq s \leq k \\
  -\frac{M}{M-1}s + \frac{M}{M-1}k & \text{if } -kM \leq s \leq -k \\
  s & \text{if } s \leq -kM
\end{cases}$$
Since for each $k \in \mathbb{N}$ we have $f'(u_h)(\vartheta_k(u_h)) = o(1)$ as $h \to +\infty$, there exists $C_{k,M} > 0$ such that:

\[
\int_{\{u_h \geq kM\}} p\mathcal{L}(x, u_h, \nabla u_h) \, dx + \frac{M}{M-1} \int_{\{k \leq u_h \leq kM\}} p\mathcal{L}(x, u_h, \nabla u_h) \, dx + \\
+ \int_{\{u_h \geq kM\}} D_s\mathcal{L}(x, u_h, \nabla u_h) u_h \, dx + \frac{M}{M-1} \int_{\{k \leq u_h \leq kM\}} D_s\mathcal{L}(x, u_h, \nabla u_h)(u_h \pm k) \, dx =
\int_{\{u_h \geq kM\}} g(x, u_h) u_h \, dx + \frac{M}{M-1} \int_{\{k \leq u_h \leq kM\}} g(x, u_h)(u_h \pm k) \, dx + \\
+ \int_{\{u_h \geq kM\}} |u_h|^{p^*} \, dx + \frac{M}{M-1} \int_{\{k \leq u_h \leq kM\}} |u_h|^{p^*-2} u_h(u_h \pm k) \, dx + \langle w_h, \vartheta_k(u_h) \rangle \geq \\
\int_{\Omega} g(x, u_h) u_h \, dx - kM \int_{\{u_h \leq k\}} |g(x, u_h)| \, dx + \frac{M}{M-1} \int_{\{k \leq u_h \leq kM\}} g(x, u_h)(u_h \pm k) \, dx + \\
+ \int_{\Omega} |u_h|^{p^*} \, dx - kM \int_{\{u_h \leq k\}} |u_h|^{p^*-1} \, dx + \frac{M}{M-1} \int_{\{k \leq u_h \leq kM\}} |u_h|^{p^*-2} u_h(u_h \pm k) \, dx + \langle w_h, \vartheta_k(u_h) \rangle \geq \\
\alpha \int_{\Omega} \mathcal{L}(x, u_h, \nabla u_h) \, dx - \alpha f(u_h) - \int_{\Omega} b_\alpha(x) \, dx - C_{k,M} + \langle w_h, \vartheta_k(u_h) \rangle.
\]

On the other hand, by (5.69) and (5.68) one obtains

\[
\int_{\{u_h \geq \overline{K}\}} D_s\mathcal{L}(x, u_h, \nabla u_h) u_h \, dx \leq \gamma \int_{\{u_h \geq \overline{K}\}} \mathcal{L}(x, u_h, \nabla u_h) \, dx,
\]

and

\[
-\overline{k} \int_{\{u_h \leq \overline{K}\}} D_s\mathcal{L}(x, u_h, \nabla u_h) \, dx \leq 0,
\]

\[
\overline{k} \int_{\{u_h \leq \overline{K}M\}} D_s\mathcal{L}(x, u_h, \nabla u_h) \, dx \leq 0,
\]

for some $\overline{k} \geq 1$ so that $\overline{k} \geq \max\{R, R'\}$. Therefore, we find $\overline{C}_{\overline{k},M} > 0$ with

\[
\frac{\nu}{p} \left( \alpha - \frac{M}{M-1} \gamma - \frac{M}{M-1} p \right) \int_{\Omega} |\nabla u_h|^p \, dx \leq \\
\leq \left( \alpha - \frac{M}{M-1} \gamma - \frac{M}{M-1} p \right) \int_{\Omega} \mathcal{L}(x, u_h, \nabla u_h) \, dx \leq \\
\leq \alpha f(u_h) + \int_{\Omega} b_\alpha(x) \, dx + \overline{C}_{\overline{k},M} + \|w_h\|_{-1,p'} \|\vartheta_{\overline{k}}(u_h)\|_{1,p}.
\]

To conclude, choose $\alpha \in [p, p^*]$ and $M > 0$ so that $\alpha - \frac{M}{M-1} \gamma - \frac{M}{M-1} p > 0$. \hfill \Box
5.3. One solution for a more general nonlinearity

Remark 5.3.3. It has to be pointed out that with the choice of test function \( \vartheta_k \) there is no need of using Lemma 3.3 in [100], which involves lots of very technical computations.

Lemma 5.3.4. Let \( c \in \mathbb{R} \) and let \((u_h)\) be a \((CPS)_c\)-sequence for \( f \) such that \( u_h \rightharpoonup 0 \). Then for each \( \varepsilon > 0 \) and \( p > 0 \) we have
\[
\int_{\{\|u_h\| \leq \varepsilon\}} \mathcal{L}(x, u_h, \nabla u_h) \, dx \leq \varepsilon \int_{\{\|u_h\| > \varepsilon\}} \mathcal{L}(x, u_h, \nabla u_h) \, dx + o(1),
\]
uniformly as \( h \to +\infty \).

Proof. It is a consequence of [100, Lemma 3.3], taking into account that
\[
\int_{\Omega} (g(x, u_h) + |u_h|^{p-1})\vartheta_\delta(u_h) \, dx \to 0
\]
as \( h \to +\infty \) (where \( \vartheta_\delta \) is the bounded test function defined in the proof).

Assume now furthermore that (asymptotic behaviour):
\[
\lim_{s \to +\infty} \mathcal{L}(x, s, \xi) = \frac{1}{p} |\xi|^p,
\]
uniformly with respect to \( x \in \Omega \) and to \( \xi \in \mathbb{R}^n \) with \( |\xi| \leq 1 \). This means that there exist \( \varepsilon_1 : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) and \( \varepsilon_2 : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) such that
\[
\mathcal{L}(x, s, \xi) = \frac{1}{p} |\xi|^p + \varepsilon_1(x, s, \xi)|\xi|^p
\]
\[
D_s \mathcal{L}(x, s, \xi)s = \varepsilon_2(x, s, \xi)|\xi|^p
\]
where \( \varepsilon_{1,2}(x, s, \xi) \to 0 \) as \( s \to +\infty \) uniformly in \( x \in \Omega \) and \( \xi \in \mathbb{R}^n \).

Let \( S \) denote the best Sobolev constant
\[
S = \inf \left\{ \|\nabla u\|_p^p : u \in W_0^{1,p}(\Omega), \|u\|_{p^*} = 1 \right\}.
\]

Lemma 5.3.5. Let \((u_h) \subset W_0^{1,p}(\Omega)\) be a concrete Palais–Smale sequence for \( f \) at level \( c \) with
\[
0 < c < \frac{1}{n} S^{n/p}.
\]
Assume that \( u_h \rightharpoonup u \). Then \( u \neq 0 \).

Proof. Assume by contradiction that \( u = 0 \). In particular, \( u \to 0 \) in \( L^s(\Omega) \) for each \( 1 \leq s < p^* \). Therefore, taking into account (5.70) and the \( p \)–homogeneity of \( \mathcal{L} \) with respect to \( \xi \), from \( f'(u_h)(u_h) \to 0 \) we obtain
\[
\int_{\Omega} p \mathcal{L}(x, u_h, \nabla u_h) \, dx + \int_{\Omega} D_s \mathcal{L}(x, u_h, \nabla u_h)u_h \, dx + \int_{\Omega} |u_h|^{p^*} \, dx = o(1),
\]
(5.79)
as \( h \to +\infty \). Let us now prove that for each \( \varrho > 0 \)

\[
\lim_{h \to +\infty} \left| \int_{\{|u_h| \leq \varrho\}} D_s \mathcal{L}(x, u_h, \nabla u_h) u_h \, dx \right| \leq \frac{C''}{\varrho},
\]

(5.80)

for some \( C'' > 0 \). Indeed, since \( u_h \to 0 \), by Lemma 5.3.4 and (5.66), one has:

\[
\int_{\{|u_h| \leq \varrho\}} D_s \mathcal{L}(x, u_h, \nabla u_h) u_h \, dx \leq C \varrho \int_{\{|u_h| \leq \varrho\}} \mathcal{L}(x, u_h, \nabla u_h) \, dx \leq C \varrho \varepsilon \int_{\{|u_h| > \varrho\}} \mathcal{L}(x, u_h, \nabla u_h) \, dx + o(1) \leq C' \varrho \varepsilon \int_\Omega |\nabla u_h|^p \, dx + o(1) \leq C'' \varrho \varepsilon + o(1),
\]

for each \( \varrho > 0 \) and \( \varepsilon > 0 \) uniformly as \( h \to +\infty \). Then (5.80) follows by choosing \( \varepsilon = 1/\varrho^2 \).

In particular, since condition (5.78) yields

\[
\lim_{h \to +\infty} \int_{\{|u_h| \geq \varrho\}} D_s \mathcal{L}(x, u_h, \nabla u_h) u_h \, dx = 0,
\]

(5.81)

uniformly in \( h \in \mathbb{N} \), by combining (5.80) with (5.81), one gets

\[
\lim_{h \to +\infty} \int_\Omega D_s \mathcal{L}(x, u_h, \nabla u_h) u_h \, dx = 0.
\]

(5.82)

In a similar way, by (5.77), one shows that, as \( h \to +\infty \),

\[
\int_\Omega \mathcal{L}(x, u_h, \nabla u_h) \, dx = \frac{1}{p} \int_\Omega |\nabla u_h|^p \, dx + o(1).
\]

(5.83)

Therefore, by (5.79) one gets

\[
\|u_h\|_{1,p}^p - \|u_h\|_{p^*}^{p^*} = o(1),
\]

as \( h \to +\infty \). In particular, from the definition of \( S \), it holds

\[
\|u_h\|_{1,p}^p \left( 1 - S^{-p/p} \|u_h\|_{1,p}^{p-p} \right) \leq o(1),
\]

as \( h \to +\infty \). Since \( c > 0 \) it has to be

\[
\|u_h\|_{1,p}^p \geq S^{n/p} + o(1), \quad \|u_h\|_{p^*}^{p^*} \geq S^{n/p} + o(1),
\]

as \( h \to +\infty \). Hence, by (5.82) and (5.83) one deduces that

\[
f(u_h) = \frac{1}{n} \|u_h\|_{1,p}^p + \frac{1}{p} (\|u_h\|_{1,p}^p + \|u_h\|_{p^*}^{p^*}) + o(1) \geq \frac{1}{n} S^{n/p},
\]

contradicting the assumption. \( \square \)
5.3. One solution for a more general nonlinearity

Proof of Theorem 5.3.1 Let us consider the min–max class
\[ \Gamma = \left\{ \gamma \in C([0, 1], W^{1,p}_0(\Omega)) : \gamma(0) = 0, \gamma(1) = w \right\} \]
with \( f(tw) < 0 \) for \( t \) large and
\[ \beta = \inf_{\gamma \in \Phi} \max_{t \in [0, 1]} f(\gamma(t)) . \]
Then, by the mountain pass theorem in its nonsmooth version (see [37]), one finds a Palais–Smale sequence for \( f \) at level \( \beta \). We have to prove that
\[ 0 < \beta < \frac{1}{n} S^{n/p} . \]
Consider the family of maps on \( \mathbb{R}^n \)
\[ T_{\delta,x_0}(x) = \frac{c_n \delta^{n-p}}{\left( \delta^{\frac{p}{p-1}} + |x-x_0|^{\frac{p}{p-1}} \right)^{\frac{n-p}{p}}} \]
with \( \delta > 0 \) and \( x_0 \in \mathbb{R}^n \). \( T_{\delta,x_0} \) is a solution of \(-\Delta_p u = u^{p-1} \) on \( \mathbb{R}^n \). Taking a function \( \phi \in C_c^\infty(\Omega) \) with \( 0 \leq \phi \leq 1 \) and \( \phi = 1 \) in a neighbourhood of \( x_0 \) and setting \( v_\delta = \phi T_{\delta,x_0} \), it results
\[ \|v_\delta\|_{1,p}^p = S^{n/p} + o\left( \epsilon^{(n-p)/(p-1)} \right), \quad \|v_\delta^*\|_{p^*}^p = S^{n/p} + o\left( \epsilon^{n/(p-1)} \right) \]
as \( \delta \to 0 \), so that, as \( \delta \to 0 \),
\[ \frac{\partial}{\partial \delta} \|v_\delta\|_{1,p}^p - \frac{\partial}{\partial \delta} \|v_\delta^*\|_{p^*}^p \leq \frac{1}{n} S^{n/p} + o\left( \epsilon^{(n-p)/(p-1)} \right) . \]
Assume by contradiction that for each \( \delta > 0 \) there exists \( t_\delta > 0 \) with
\[ f(t_\delta v_\delta) = \frac{\partial}{\partial \delta} \|v_\delta\|_{1,p}^p + \frac{\partial}{\partial \delta} \int_\Omega \left\{ \mathcal{L}(x, t_\delta v_\delta, \nabla v_\delta) - \frac{1}{p} |\nabla v_\delta|^p \right\} \, dx + \]
\[ - \int_\Omega G(x, t_\delta v_\delta) \, dx - \frac{\partial}{\partial \delta} \|v_\delta^*\|_{p^*}^p \geq \frac{1}{n} S^{n/p} \]
In particular, there exist \( M_1, M_2 > 0 \) with \( M_1 \leq t_\delta \leq M_2 \). Moreover, as proved in [13, Lemma 5], there exists \( \tau : [0, 1] \to \mathbb{R} \) with \( \tau(\epsilon) \to +\infty \) and
\[ \int_\Omega G(x, t_\delta v_\delta) \, dx \geq \tau(\epsilon) \epsilon^{(n-p)/(p-1)} . \]
as \( \epsilon \to 0 \). By (5.68) and (5.77) one also has
\[ \int_\Omega \left\{ \mathcal{L}(x, t_\delta v_\delta, \nabla v_\delta) - \frac{1}{p} |\nabla v_\delta|^p \right\} \, dx \leq 0 \]
for each \( \delta > 0 \). By putting together (5.85), (5.86), (5.87), (5.88), one concludes
\[ f(t_\delta v_\delta) \leq \frac{1}{n} S^{n/p} + (C - \tau(\epsilon)) \epsilon^{(n-p)/(p-1)} \]
which contradict (5.86) for \( \epsilon \) sufficiently small.
Chapter 5. Problems with loss of compactness

5.4 Problems with nearly critical growth

5.4.1 Introduction

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \), \( 1 < p < n \) and \( p^* = \frac{np}{n-p} \). In 1989 Guedda and Veron [65] proved that the \( p \)-Laplacian problem at critical growth

\[
\begin{align*}
-\Delta_p u &= u^{p^*-1} \quad \text{in } \Omega \\
u &> 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

has no non-trivial solution \( u \in W^{1,p}_0(\Omega) \) if the domain \( \Omega \) is star-shaped. As known, this non-existence result is due to the failure of compactness for the critical Sobolev embedding \( W^{1,p}_0(\Omega) \hookrightarrow L^{p^*}(\Omega) \), which causes a loss of global Palais–Smale condition for the functional associated with (*). On the other hand, if for instance one considers annular domains

\[
\Omega_{r_1,r_2} = \{ x \in \mathbb{R}^n : 0 < r_1 < |x| < r_2 \},
\]

then the radial embedding

\[
W^{1,p}_{0,rad}(\Omega_{r_1,r_2}) \hookrightarrow L^q(\Omega_{r_1,r_2})
\]

is compact for each \( q < +\infty \) and one can find a non-trivial radial solution of (*) (see [70]). Therefore, we see how the existence of non-trivial solutions of (*) is related to the shape of the domain and not just to the topology. In the case \( p = 2 \), the problem

\[
\begin{align*}
-\Delta u &= u^{(n+2)/(n-2)} \quad \text{in } \Omega \\
u &> 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

has been deeply studied and existence results have been obtained provided that \( \Omega \) satisfies suitable assumptions. In a striking paper [17], Bahri and Coron have proved that if \( \Omega \) has a non-trivial topology, i.e. if \( \Omega \) has a non-trivial homology in some positive dimension, then (**) always admits a non-trivial solution. Moreover, Dancer [52] constructed for each \( n \geq 3 \) a contractible domain \( \Omega_n \), homeomorphic to a ball, for which (**) has a non-trivial solution. See also [87] and references therein for more recent existence and multiplicity results.

We remark that, to our knowledge, this type of achievements are not known when \( p \neq 2 \). In our opinion, one of the main difficulties is the fact, that differently from the case \( p = 2 \), it is not proven that all positive solutions of \(-\Delta_p u = u^{p^*-1} \) in \( \mathbb{R}^n \) are Talenti’s radial functions, which attain the best Sobolev constant (see Proposition 5.4.5).

Now, there is a second approach in the study of problem (*), which in general does not require any geometrical or topological assumption on \( \Omega \), namely to investigate the asymptotic behaviour of solutions \( u_\varepsilon \) of problems with nearly critical growth

\[
\begin{align*}
-\Delta_p u &= |u|^{p^*-2-\varepsilon}u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]  (* ***)
5.4. Problems with nearly critical growth

as \( \varepsilon \) goes to 0. If \( \Omega \) is a ball and \( p = 2 \), Atkinson and Peletier [14] showed in 1987 the blow-up of a sequence of radial solutions. The extension to the case \( p \neq 2 \) was achieved by Knaap and Peletier [71] in 1989. On a general bounded domain, instead, the study of limits of solutions of \((***)\) was performed by Garcia Azorero and Peral Alonso [61] around 1992.

Let now \( \varepsilon > 0 \) and consider the following general class of Euler–Lagrange equations with nearly critical growth

\[
\begin{cases}
-\text{div} (\nabla \mathcal{L}(x,u,\nabla u)) + D_s \mathcal{L}(x,u,\nabla u) = |u|^{p^*-2-\varepsilon}u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

\((\mathcal{P}_\varepsilon)\)

associated with the functional \( f_\varepsilon : W^{1,p}_0(\Omega) \to \mathbb{R} \) given by

\[
f_\varepsilon(u) = \int_\Omega \mathcal{L}(x,u,\nabla u) \, dx - \frac{1}{p^* - \varepsilon} \int_\Omega |u|^{p^* - \varepsilon} \, dx.
\]  

\((5.89)\)

As noted in [100], in general these functionals are not even locally Lipschitzian under natural growth assumptions. Nevertheless, via techniques of non–smooth critical point theory (see [100] and references therein) it can be shown that \((\mathcal{P}_\varepsilon)\) admits a non–trivial solution \( u_\varepsilon \in W^{1,p}_0(\Omega) \).

Let \((u_\varepsilon)_{\varepsilon>0}\) denote a sequence of solutions of \((\mathcal{P}_\varepsilon)\). The main goal of this section is to prove that if the weak limit of \((|\nabla u_\varepsilon|^p)_{\varepsilon>0}\) has no blow–up points in \( \Omega \), then the limit problem:

\[
\begin{cases}
-\text{div} (\nabla \mathcal{L}(x,u,\nabla u)) + D_s \mathcal{L}(x,u,\nabla u) = |u|^{p^*-2}u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

\((\mathcal{P}_0)\)

has a non–trivial solution (the weak limit of \((u_\varepsilon)_{\varepsilon>0}\)), provided that \( f_\varepsilon(u_\varepsilon) \to c \) with

\[
\frac{p^* - p - \gamma}{pp^*}(\nu S)^{\frac{n}{p}} < c < 2 \frac{p^* - p - \gamma}{pp^*}(\nu S)^{\frac{n}{p}},
\]  

\((5.90)\)

where \( \nu > 0 \) and \( \gamma \in (0, p^*-p) \) will be defined later. In our framework \((5.90)\) plays the role of a generalized second critical energy range (if \( \gamma = 0 \) and \( \nu = 1 \), one finds the usual range \( \frac{s_n}{p} < c < 2 \frac{s_n}{p} \) for problem \((***)\)).

The plan is as follows: in Section 5.4.2 we shall state our main results; in Section 5.4.3 we shall collect the main tools, namely the lower bounds on the non–vanishing Dirac masses and on the non–trivial weak limits; in Section 5.4.4 we shall prove our main results; finally, in Section 5.4.5 we shall see that at the mountain pass levels the sequence \((u_\varepsilon)_{\varepsilon>0}\) blows up. Moreover, we shall state a non–existence results obtained via the Pucci–Serrin variational identity.

In the following, we shall always consider the space \( W^{1,p}_0(\Omega) \) endowed with the standard norm \( \|u\|_1,p = \int_\Omega |\nabla u|^p \, dx \) and we shall denote by \( \| \cdot \|_p \) the usual norm of \( L^p(\Omega) \).

5.4.2 The main results

Let \( \Omega \) be any bounded domain of \( \mathbb{R}^n \) and assume that \( \mathcal{L} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) is measurable in \( x \) for all \((s,\xi) \in \mathbb{R} \times \mathbb{R}^n\), of class \( C^1 \) in \((s,\xi)\) a.e. in \( \Omega \), that \( \mathcal{L}(x,s,\cdot) \) is strictly convex and \( p \)–homogeneous with \( \mathcal{L}(x,s,0) = 0 \). Moreover:
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(\mathcal{A}_1) there exist \( b_0 > 0 \) and \( \nu > 0 \) such that
\[
\frac{1}{\nu} |\xi|^p \leq \mathcal{L}(x, s, \xi) \leq b_0 |s|^p + b_0 |\xi|^p
\]  
(5.91)
for a.e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \);

(\mathcal{A}_2) there exists \( b_1 > 0 \) such that for each \( \delta > 0 \) there exists \( a_\delta \in L^1(\Omega) \) with
\[
|D_s \mathcal{L}(x, s, \xi)| \leq a_\delta(x) + \delta |s|^p + b_1 |\xi|^p
\]  
(5.92)
for a.e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \), and
\[
|\nabla_\xi \mathcal{L}(x, s, \xi)| \leq a_1(x) + b_1 |s|^{\frac{p^*}{p}} + b_1 |\xi|^{p-1}
\]  
(5.93)
for a.e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \), where \( a_1 \in L^{p^*}(\Omega) \);

(\mathcal{A}_3) for a.e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \),
\[
D_s \mathcal{L}(x, s, \xi)s \geq 0
\]  
(5.94)
and there exists \( \gamma \in (0, p^* - p) \) such that:
\[
D_s \mathcal{L}(x, s, \xi)s \leq \gamma \mathcal{L}(x, s, \xi)
\]  
(5.95)
for a.e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \).

The growth conditions of (\mathcal{A}_1) and (\mathcal{A}_2) and the assumptions in (\mathcal{A}_3) are natural in the fully nonlinear setting and were considered in [100] and in a stronger form in [7].

We stress that although as noted in the introduction \( f_\varepsilon \) fails to be differentiable on \( W^{1,p}_0(\Omega) \), one may compute the derivatives along the \( L^\infty \)-directions, namely:
\[
\forall u \in W^{1,p}_0(\Omega), \forall \varphi \in W^{1,p}_0 \cap L^\infty(\Omega) : \quad f'_\varepsilon(u)(\varphi) = \int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi \, dx + \int_\Omega D_s \mathcal{L}(x, u, \nabla u) \varphi \, dx - \int_\Omega |u|^{p^*-2-\varepsilon} u \varphi \, dx.
\]

By combining the following proposition with (2.47), one can also compute \( f'_\varepsilon(u)(u) \).

**Proposition 5.4.1.** Let \( u, v \in W^{1,p}_0(\Omega) \) be such that \( D_s \mathcal{L}(x, u, \nabla u) v \geq 0 \) and
\[
\forall \varphi \in C_c^\infty(\Omega) : \quad \langle w, \varphi \rangle = \int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi \, dx + \int_\Omega D_s \mathcal{L}(x, u, \nabla u) \varphi \, dx
\]  
(5.96)
with \( w \in W^{-1,p^*}(\Omega) \). Then \( D_s \mathcal{L}(x, u, \nabla u) v \in L^1(\Omega) \) and one can take \( \varphi = v \) in (5.96).

**Proof.** See [100, Proposition 3.1]. \( \square \)

Under the preceding assumptions, by [100, Theorem 1.1], for each \( \varepsilon > 0 \) one deduces that (\( \mathcal{P}_\varepsilon \)) admits at least one non-trivial solution \( u_\varepsilon \in W^{1,p}_0(\Omega) \) (by solution we shall always mean weak solution, namely \( f'_\varepsilon(u_\varepsilon) = 0 \) in the sense of distributions). We point out that the technical aspects in the verification of the Palais–Smale condition are, in our opinion, interesting and not trivial.
As a starting point, let us show that \( (u_\varepsilon) \) is bounded in \( W^{1,p}_0(\Omega) \).
Lemma 5.4.2. Let \((u_\varepsilon)_{\varepsilon > 0} \subset W_0^{1,p}(\Omega)\) be a sequence of solutions of \((P_\varepsilon)\) such that
\[
\lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) < +\infty.
\]
Then \((u_\varepsilon)_{\varepsilon > 0}\) is bounded in \(W_0^{1,p}(\Omega)\).

Proof. If \(u_\varepsilon\) is a solution of \((P_\varepsilon)\), we have \(f'_\varepsilon(u_\varepsilon)(\varphi) = 0\) for each \(\varphi \in C_c^\infty(\Omega)\). On the other hand, taking into account (5.94), by Proposition 5.4.1 one can choose \(\varphi = u_\varepsilon\). Therefore, in view of (5.95) and the \(p\)-homogeneity of \(L(x,s,\cdot)\), one obtains:
\[
\lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) = \lim_{\varepsilon \to 0} \left( f_\varepsilon(u_\varepsilon) - \frac{1}{p^* - \varepsilon} f'_\varepsilon(u_\varepsilon)(u_\varepsilon) \right) =
\]
\[
= \lim_{\varepsilon \to 0} \left( \int_\Omega L(x,u_\varepsilon,\nabla u_\varepsilon) \, dx - \frac{p}{p^* - \varepsilon} \int_\Omega L(x,u_\varepsilon,\nabla u_\varepsilon) \, dx + \right.
\]
\[
- \frac{1}{p^* - \varepsilon} \int_\Omega D_sL(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon \, dx \right) 
\]
\[
\geq \lim_{\varepsilon \to 0} \frac{p^* - p - \varepsilon - \gamma}{p^* - \varepsilon} \int_\Omega L(x,u_\varepsilon,\nabla u_\varepsilon) \, dx 
\]
\[
\geq \frac{p^* - p - \gamma}{pp^*} \nu \lim_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon|^p \, dx.
\]
In particular, \((u_\varepsilon)_{\varepsilon > 0}\) is bounded in \(W_0^{1,p}(\Omega)\). \(\square\)

As a consequence, one may apply P.L. Lions’ concentration-compactness principle (see [75, 76]) and obtain a subsequence of \((u_\varepsilon)_{\varepsilon > 0}, u \in W_0^{1,p}(\Omega)\) and two bounded positive measures \(\mu\) and \(\sigma\) such that:
\[
u u_\varepsilon \rightharpoonup u \quad \text{in} \quad W_0^{1,p}(\Omega),
\]
\[
u |\nabla u_\varepsilon|^p \rightharpoonup \mu, \quad |u_\varepsilon|^{p^*} \rightharpoonup \sigma \quad \text{(in the sense of measures),}
\]
\[
\mu \geq |\nabla u|^p + \sum_{j=1}^\infty \mu_j \delta_{x_j}, \quad \mu_j \geq 0,
\]
\[
\sigma = |u|^{p^*} + \sum_{j=1}^\infty \sigma_j \delta_{x_j}, \quad \sigma_j \geq 0,
\]
\[
\mu_j \geq S\sigma_j^{\frac{p}{p^*}},
\]
where \(\delta_{x_j}\) denotes the Dirac measure at \(x_j \in \Omega^\prime\) and \(S\) denotes the best Sobolev constant for the embedding \(W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)\) (see e.g. [116]).

Under assumptions \((\omega_1), (\omega_2)\) and \((\omega_3)\), the following is our main result.

Theorem 5.4.3. Let \((u_\varepsilon)_{\varepsilon > 0}\) be any sequence of solutions of \((P_\varepsilon)\) with \(f_\varepsilon(u_\varepsilon) \to c\) and
\[
\frac{p^* - p - \gamma}{pp^*} (\nu S)^\frac{p}{p^*} < c < 2 \frac{p^* - p - \gamma}{pp^*} (\nu S)^\frac{p}{p^*}.
\]
Then $\mu_j = 0$ for $j \geq 2$ and the following alternative holds:

(a) $\mu_1 = 0$ and $u$ is a non-trivial solution of $(\mathcal{P}_0)$;

(b) $\mu_1 \neq 0$ and $u = 0$.

This result extends [61, Theorem 9] to fully nonlinear elliptic problems.

**Theorem 5.4.4.** Let $(u_\varepsilon)_{\varepsilon > 0}$ be any sequence of solutions of $(\mathcal{P}_\varepsilon)$ with

$$\lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) = \frac{p^* - p - \gamma}{pp^*}(\nu S)^{\frac{\alpha}{p^*}}.$$  \hspace{1cm} (5.102)

Then $u = 0$.

As we shall see in section (5.4.5), this is also the behaviour when one considers critical levels of mountain–pass type.

### 5.4.3 The weak limit

Let us briefly summarize the main properties of the best Sobolev constant.

**Proposition 5.4.5.** Let $1 < p < n$ and $S$ be the best Sobolev constant, i.e.

$$S = \inf \left\{ \int_\Omega |\nabla u|^p \, dx : \ u \in W_0^{1,p}(\Omega), \ \int_\Omega |u|^{p^*} \, dx = 1 \right\}. \hspace{1cm} (5.102)$$

Then, the following facts hold:

(a) $S$ is independent on $\Omega \subset \mathbb{R}^n$; it depends only on the dimension $n$;

(b) the infimum (5.102) is never achieved on bounded domains $\Omega \subset \mathbb{R}^n$;

(c) the infimum (5.102) is achieved if $\Omega = \mathbb{R}^n$ by the family of functions on $\mathbb{R}^n$

$$T_{\delta,x_0}(x) = \left( n\delta \left( \frac{n-p}{p-1} \right)^{p-1} \right)^{\frac{n-p}{p^*}} \left( \delta + |x-x_0|^{\frac{p}{p^*-1}} \right)^{-\frac{n-p}{p}}$$ \hspace{1cm} (5.103)

with $\delta > 0$ and $x_0 \in \mathbb{R}^n$. Moreover for each $\delta > 0$ and $x_0 \in \mathbb{R}^n$, $T_{\delta,x_0}$ is a solution of the equation $-\Delta_p u = u^{p^*-1}$ on $\mathbb{R}^n$.

**Proof.** See [116]. \hfill $\square$

Next result establishes uniform lower bounds for the Dirac masses.

**Lemma 5.4.6.** If $\sigma_j \neq 0$, then $\sigma_j \geq \nu^n S^n$ and $\mu_j \geq \nu^n S^n$.

**Proof.** Let $x_j \in \overline{\Omega}$ the point which supports the Dirac measure of coefficient $\sigma_j \neq 0$. Denoting with $B(x_j, \delta)$ the open ball of center $x_j$ and radius $\delta > 0$, we can consider a function $\psi_\delta \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \psi_\delta \leq 1$, $|\nabla \psi_\delta| \leq \frac{2}{3}$, $\psi_\delta(x) = 1$ if $x \in B(x_j, \delta)$ and $\psi_\delta(x) = 0$ if $x \not\in B(x_j, 2\delta)$.
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By Proposition 5.4.1 and the $p$–homogeneity of $\mathcal{L}(x, s, \cdot)$, we have

$$0 = f'_\varepsilon(u_\varepsilon)(\psi_\delta u_\varepsilon) =$$

$$= \int_\Omega u_\varepsilon \nabla \xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi_\delta \, dx + p \int \psi_\delta \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \, dx +$$

$$+ \int \psi_\delta D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \, dx - \int |u_\varepsilon|^p \, \psi_\delta \, dx$$

Applying Hölder inequality and (5.93) to the first term of the decomposition and keeping into account that $(u_\varepsilon)_{\varepsilon > 0}$ is bounded in $W_0^{1,p}((\Omega)$ and $u_\varepsilon \to u$ in $L^q(\Omega)$ for every $q < p^*$, one find $c_1 > 0$ and $c_2 > 0$ with

$$\lim_{\varepsilon \to 0} \left| \int_\Omega u_\varepsilon \nabla \xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi_\delta \, dx \right| \leq$$

$$\leq \left( \int_{B(x_j, 2\delta)} |a_1|^{p \frac{p}{p-1}} \, dx \right)^{\frac{p}{p-1}} \left( \int_{B(x_j, 2\delta)} |u|^{p^*} \, dx \right)^{\frac{1}{p^*}} \left( \int_{B(x_j, 2\delta)} |\nabla \psi_\delta|^n \, dx \right)^{\frac{1}{n}} +$$

$$+ b_1 \left( \int_{B(x_j, 2\delta)} |u|^{p^*} \, dx \right)^{\frac{n-1}{n}} \left( \int_{B(x_j, 2\delta)} |\nabla \psi_\delta|^n \, dx \right)^{\frac{1}{n}} +$$

$$+ \tilde{b}_1 \left( \int_{B(x_j, 2\delta)} |u|^{p^*} \, dx \right)^{\frac{1}{p^*}} \left( \int_{B(x_j, 2\delta)} |\nabla \psi_\delta|^n \, dx \right)^{\frac{1}{n}} \leq$$

$$\leq c_1 \left( \int_{B(x_j, 2\delta)} |u|^{p^*} \, dx \right)^{\frac{1}{p^*}} + c_2 \left( \int_{B(x_j, 2\delta)} |u|^{p^*} \, dx \right)^{\frac{n-1}{n}} = \beta_\delta$$

with $\beta_\delta \to 0$ as $\delta \to 0$. Then, taking into account (5.94) and (5.91) one has

$$0 \geq -\beta_\delta + \lim_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon| |\psi_\delta| \, dx - \lim_{\varepsilon \to 0} \mathcal{L}^n(\Omega)^{\frac{p}{p^*-1}} \left( \int_\Omega |u_\varepsilon|^{p^*} \psi_\delta \, dx \right)^{\frac{p^*-1}{p^*}} \geq$$

$$\geq -\beta_\delta + \nu \int_\Omega \psi_\delta \, d\mu - \int_\Omega \psi_\delta \, d\sigma.$$ 

Letting $\delta \to 0$, it results $\nu \mu_j \leq \sigma_j$ . By means of (5.101) one concludes the proof.

Next result establishes uniform lower bounds for the non–zero weak limits.

**Lemma 5.4.7.** If $u \neq 0$, then $\int_\Omega |\nabla u|^p \, dx > \nu \frac{p^*}{p} S^{\frac{n}{p}}$ and $\int_\Omega |u|^{p^*} \, dx > \nu \frac{n}{p} S^{\frac{n}{p}}$.

**Proof.** By Lemma 5.4.6, we may assume that $\mu$ has at most $r$ Dirac masses $\mu_1, \ldots, \mu_r$ at $x_1, \ldots, x_r$. Let now $0 < \delta < \frac{1}{4} \min_{i \neq j} |x_i - x_j|$ and $\psi_\delta \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \leq \psi_\delta \leq 1$, $|\nabla \psi_\delta| \leq \frac{2}{\delta}$, $\psi_\delta(x) = 1$ if $x \in B(x_j, \delta)$ and $\psi_\delta(x) = 0$ if $x \notin B(x_j, 2\delta)$. Taking into account (5.94), for each $\varepsilon, \delta > 0$ we have

$$\int_\Omega D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon (1 - \psi_\delta) \, dx \geq 0.$$
Then, since one can choose \((1 - \psi_\delta) u_\varepsilon\) as test, one obtains
\[
0 = f'_\varepsilon((1 - \psi_\delta) u_\varepsilon) = \int_\Omega p \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon)(1 - \psi_\delta) \, dx + \int_\Omega \nabla \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi_\delta u_\varepsilon \, dx + \int_\Omega D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon(1 - \psi_\delta) \, dx + \int_\Omega |u_\varepsilon|^{p^* - \varepsilon} (1 - \psi_\delta) \, dx \geq \nu \int_\Omega |\nabla u_\varepsilon|^{p^*} (1 - \psi_\delta) \, dx + \int_\Omega |\nabla \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) - \mathcal{L}^n(\Omega))^{p^*} \left( \int_\Omega |u_\varepsilon|^{p^*} (1 - \psi_\delta) \, dx \right)^\frac{p^*}{p^* - \varepsilon} .
\]

On the other hand, arguing as for (5.105), one gets
\[
\lim_{\varepsilon \to 0} \left| \int_\Omega u_\varepsilon \nabla \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi_\delta \, dx \right| \leq \beta_\delta
\]
for each \(\delta > 0\). Now, it results
\[
\lim_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon|^{p^*} (1 - \psi_\delta) \, dx = \int_\Omega (1 - \psi_\delta) \, d\mu \geq \int_\Omega |\nabla u|^{p^*} (1 - \psi_\delta) \, dx + \sum_{j=1}^{r} \mu_j (1 - \psi_\delta(x_j)) = \int_\Omega |\nabla u|^{p^*} \, dx + o(1)
\]
as \(\delta \to 0\) and
\[
\lim_{\varepsilon \to 0} \int_\Omega |u_\varepsilon|^{p^*} (1 - \psi_\delta) \, dx = \int_\Omega (1 - \psi_\delta) \, d\sigma = \int_\Omega |u|^{p^*} (1 - \psi_\delta) \, dx + \sum_{j=1}^{r} \sigma_j (1 - \psi_\delta(x_j)) = \int_\Omega |u|^{p^*} \, dx + o(1)
\]
as \(\delta \to 0\). Therefore, in view of (5.107), (5.108) and (5.109), by letting \(\delta \to 0\) and \(\varepsilon \to 0\) in (5.106), one concludes that
\[
\nu \int_\Omega |\nabla u|^{p^*} \, dx \leq \int_\Omega |u|^{p^*} \, dx .
\]
As \(\Omega\) is bounded, by \((b)\) of Proposition 5.4.5 one has
\[
\int_\Omega |\nabla u|^{p^*} \, dx > S \left( \int_\Omega |u|^{p^*} \, dx \right)^\frac{p^*}{p^*} ,
\]
which combined with (5.110) yields the assertion.\(\square\)

In the next result we show that weak limits of \((u_\varepsilon)_{\varepsilon > 0}\) are indeed solutions of \((\mathcal{P}_0)\).

**Lemma 5.4.8.** Let \((u_\varepsilon)_{\varepsilon > 0} \subset W^{1,p}_0(\Omega)\) be a sequence of solutions of \((\mathcal{P}_\varepsilon)\) and let \(u\) be its weak limit. Then \(u\) is a solution of \((\mathcal{P}_0)\).
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Proof. For each \( \varepsilon > 0 \) one has for all \( \varphi \in C_c^\infty(\Omega) \):

\[
\int_\Omega \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \varphi \, dx + \int_\Omega D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \varphi \, dx = \int_\Omega |u_\varepsilon|^{p^*-2} u_\varepsilon \varphi \, dx. \tag{5.111}
\]

Since \((u_\varepsilon)_{\varepsilon>0}\) is bounded in \(W_0^{1,p}(\Omega)\), up to a subsequence, as \(\varepsilon \to 0\), \(u\) satisfies:

\[
\nabla u_\varepsilon \to \nabla u \text{ in } L^p(\Omega), \quad u_\varepsilon \to u \text{ in } L^p(\Omega), \quad u_\varepsilon(x) \to u(x) \text{ for a.e. } x \in \Omega.
\]

Moreover, by [23, Theorem 1], up to a subsequence, we have \(\nabla u_\varepsilon(x) \to \nabla u(x)\) for a.e. \(x \in \Omega\). Therefore, in view of (5.93) one deduces that:

\[
\nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \to \nabla_\xi \mathcal{L}(x, u, \nabla u) \text{ in } L^p(\Omega, \mathbb{R}^n). \tag{5.112}
\]

By (5.91) and (5.92) one finds \(M > 0\) such that for each \(\delta > 0\)

\[
|D_s \mathcal{L}(x, s, \xi)| \leq M \nabla_\xi \mathcal{L}(x, s, \xi) \cdot \xi + a_\delta(x) + \delta|s|^{p^*} \tag{5.113}
\]

for a.e. \(x \in \Omega\) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\). If we test equation (5.111) with the functions

\[
\varphi_\varepsilon = \varphi \exp\{-Mu_\varepsilon^+\}, \quad \varphi \in W_0^{1,p} \cap L^\infty(\Omega), \quad \varphi \geq 0
\]

for each \(\varepsilon > 0\) we obtain

\[
\int_\Omega \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \varphi \exp\{-Mu_\varepsilon^+\} \, dx - \int_\Omega |u_\varepsilon|^{p^*-2} u_\varepsilon \varphi \exp\{-Mu_\varepsilon^+\} \, dx + \int_\Omega \left[D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) - M \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon^+ \right] \varphi \exp\{-Mu_\varepsilon^+\} \, dx = 0.
\]

Since by inequalities (5.94) and (5.113) for each \(\varepsilon > 0\) and \(\delta > 0\) we have

\[
[D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) - M \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon^+] \varphi \exp\{-Mu_\varepsilon^+\} \leq a_\delta(x) + \delta|u_\varepsilon|^{p^*},
\]

arguing as in [100, Theorem 3.4], one obtains:

\[
\limsup_{\varepsilon \to 0} \int_\Omega [D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) - M \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon^+] \varphi \exp\{-Mu_\varepsilon^+\} \, dx \\
\leq \int_\Omega \left[D_s \mathcal{L}(x, u, \nabla u) - M \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u^+ \right] \varphi \exp\{-Mu^+\} \, dx.
\]

Therefore, taking into account (5.112) and since as \(\varepsilon \to 0\)

\[
\int_\Omega |u_\varepsilon|^{p^*-2} u_\varepsilon \varphi \, dx \to \int_\Omega |u|^{p^*-2} u \varphi \, dx
\]

for each \(\varphi \in W_0^{1,p} \cap L^\infty(\Omega)\) positive, one may conclude that

\[
\int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi \exp\{-Mu^+\} \, dx - \int_\Omega |u|^{p^*-2} u \varphi \exp\{-Mu^+\} \, dx + \]
for each \( \varphi \in W^{1,p}_0 \cap L^\infty(\Omega) \) positive. Testing now (5.111) with

\[
\varphi_k = \varphi \vartheta \left( \frac{u}{k} \right) \exp\{ Mu^+ \}, \quad \varphi \in C_c^\infty(\Omega), \quad \varphi \geq 0,
\]

where \( \vartheta \) is smooth, \( \vartheta = 1 \) in \([-\frac{1}{2}, \frac{1}{2}]\) and \( \vartheta = 0 \) in \([ -\infty, -1 ] \cup [ 1, +\infty[\), it follows that

\[
\int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi_k \exp\{ -Mu^+ \} \, dx - \int_\Omega |u|^{p^* - 2} u \varphi \left( \frac{u}{k} \right) \, dx + \int_\Omega [ D_s \mathcal{L}(x, u, \nabla u) - M \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u^+ ] \varphi \left( \frac{u}{k} \right) \, dx \geq 0.
\]

which, arguing again as [100, Theorem 3.4], yields as \( k \to +\infty \)

\[
\int_\Omega \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi \, dx + \int_\Omega D_s \mathcal{L}(x, u, \nabla u) \varphi \, dx \geq \int_\Omega |u|^{p^* - 2} u \varphi \, dx.
\]

for each \( \varphi \in C_c^\infty(\Omega) \) positive. Working analogously with \( \varphi_k = \varphi \exp\{-Mu^-\} \), one obtains the opposite inequality, i.e. \( u \) is a solution of \((\mathcal{P}_0)\). \( \square \)

### 5.4.4 Proof of the main results

Let us now consider a sequence \((u_\varepsilon)_{\varepsilon > 0}\) of solutions of \((\mathcal{P}_\varepsilon)\) with \( f_\varepsilon(u_\varepsilon) \to c \) and

\[
\frac{p^* - p - \gamma}{pp^*} (\nu S)^\frac{\alpha}{p} < c < 2 \frac{p^* - p - \gamma}{pp^*} (\nu S)^\frac{\alpha}{p}.
\] (5.114)

Then, there exist a subsequence of \((u_\varepsilon)_{\varepsilon > 0}\) and two bounded positive measures \( \mu \) and \( \sigma \) verifying (5.97), (5.98), (5.99), (5.100) and (5.101).

**Proof of Theorem 5.4.3.** Let us first show that there exists at most one \( j \) such that \( \mu_j \neq 0 \).

Suppose that \( \mu_j \neq 0 \) for every \( j = 1, \ldots, r \); in view of Lemma 5.4.6 one has that \( \mu_j \geq \nu^\frac{p^*}{pp^*} S^\frac{\alpha}{p} \).

Following the proof of Lemma 5.4.2, we obtain

\[
c = \lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) = p^* - p - \gamma \nu \lim_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon|^p \, dx \\
\geq p^* - p - \gamma \nu \int_\Omega d\mu \\
\geq p^* - p - \gamma \nu \sum_{j=1}^r \mu_j \\
\geq r p^* - p - \gamma (\nu S)^\frac{\alpha}{p}.
\]

Taking into account (5.114) one has

\[
2 \frac{p^* - p - \gamma}{pp^*} (\nu S)^\frac{\alpha}{p} > c \geq r \frac{p^* - p - \gamma}{pp^*} (\nu S)^\frac{\alpha}{p},
\]
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hence \( r \leq 1 \). Now, arguing as in Lemma 5.4.2 one obtains

\[
2 \frac{p^* - p - \gamma}{pp^*} (\nu S)^{\frac{\alpha}{p}} > c = \lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) = \lim_{\varepsilon \to 0} \left[ f_\varepsilon(u_\varepsilon) - \frac{1}{p^* - \varepsilon} f_\varepsilon'(u_\varepsilon)(u_\varepsilon) \right] \geq
\]

\[
\geq \frac{p^* - p - \gamma}{pp^*} \nu \lim_{\varepsilon \to 0} \int_{\Omega} |\nabla u_\varepsilon|^p \, dx \geq \frac{p^* - p - \gamma}{pp^*} \left( \nu \int_{\Omega} |\nabla u|^p \, dx + \nu \mu_1 \right).
\]

If both summands were non-zero, by Lemma 5.4.6 and Lemma 5.4.7 we would obtain

\[
\nu \int_{\Omega} |\nabla u|^p \, dx > (\nu S)^{\frac{\alpha}{p}}, \quad \nu \mu_1 \geq (\nu S)^{\frac{\alpha}{p}}
\]

and thus a contradiction. Vice versa, let us assume that \( u = 0 \) and \( \mu_1 = 0 \). Let \( \psi \in C_c^1(\Omega) \) with \( \psi \geq 0 \). By testing our equation with \( \psi u_\varepsilon \) and using Hölder inequality, one gets

\[
\int_{\Omega} u_\varepsilon \nabla_{\xi} \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi \, dx + \nu \int_{\Omega} \psi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \, dx + \int_{\Omega} \mathcal{D}_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \psi u_\varepsilon \, dx = \int_{\Omega} |u_\varepsilon|^{p^* - \varepsilon} \psi \, dx \leq
\]

\[
\leq \left( \int_{\Omega} |u_\varepsilon|^{p^* - \varepsilon} \psi \, dx \right)^{\frac{p^* - \varepsilon}{p^*}} L^p(\Omega)^{\frac{\varepsilon}{p^*}}.
\]

Since \((u_\varepsilon)_{\varepsilon > 0}\) is bounded in \( W^{1,p}_0(\Omega) \), by (5.93) there exists \( C > 0 \) such that

\[
\left| \int_{\Omega} u_\varepsilon \nabla_{\xi} \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi \, dx \right| \leq C \|u_\varepsilon\|_p,
\]

which, by \( u_\varepsilon \to 0 \) in \( L^p(\Omega) \), yields

\[
\lim_{\varepsilon \to 0} \int_{\Omega} u_\varepsilon \nabla_{\xi} \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi \, dx = 0.
\]

Moreover, since by (5.94) we get

\[
\int_{\Omega} \mathcal{D}_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \psi u_\varepsilon \, dx \geq 0,
\]

taking into account (5.91) and passing to the limit in (5.115), we get

\[
\forall \psi \in C_c(\Omega) : \psi \geq 0 \implies \nu \int_{\Omega} \psi \, d\mu \leq \int_{\Omega} \psi \, d\sigma.
\]

On the other hand \( \mu_1 = 0 \) and \( u = 0 \) imply \( \sigma = 0 \). Then, since \( \mu \geq 0 \), by (5.116), we get \( \mu = 0 \). In particular, one gets

\[
c = \lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) = \lim_{\varepsilon \to 0} \left[ \frac{p^* - p - \varepsilon}{p^* - \varepsilon} \int_{\Omega} \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \, dx + \right.
\]

\[
- \frac{1}{p^* - \varepsilon} \int_{\Omega} \mathcal{D}_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \, dx \right] \leq
\]

\[
\leq \frac{pb_0}{n} \lim_{\varepsilon \to 0} \left( \int_{\Omega} |u_\varepsilon|^p \, dx + \int_{\Omega} |\nabla u_\varepsilon|^p \, dx \right) = \frac{pb_0}{n} \int_{\Omega} d\mu = 0
\]

which is not possible. Therefore, either \( \mu_1 = 0 \) and \( u \neq 0 \), or \( \mu_1 \neq 0 \) and \( u = 0 \).  \( \square \)
Remark 5.4.9. If \((5.114)\) is replaced by the \((k + 1)\)th critical energy range
\[
k^\frac{p^* - p - \gamma}{pp^*}(\nu S)^{\frac{n}{p}} < c < (k + 1)^{\frac{p^* - p - \gamma}{pp^*}(\nu S)^{\frac{n}{p}}}
\]
for \(k \in \mathbb{N}, k \geq 1\), in a similar way one can prove that \(\mu_j = 0\) for any \(j \geq k + 1\) and
(a) if \(\mu_j = 0\) for every \(j \geq 1\), then \(u\) is a non-trivial solution of \((\mathcal{P}_0)\);
(b) if \(\mu_j \neq 0\) for every \(1 \leq j \leq k\), then \(u = 0\).

Remark 5.4.10. Let \(f_0 : W_0^{1,p}(\Omega) \to \mathbb{R}\) be the functional associated with \((\mathcal{P}_0)\) and \(u \in W_0^{1,p}(\Omega), u \neq 0\), a solution of \((\mathcal{P}_0)\) (obtained as weak limit of \((u_\varepsilon)_{\varepsilon > 0}\)). Then
\[
f_0(u) > \frac{p^* - p - \gamma}{pp^*}(\nu S)^{\frac{n}{p}}. \tag{5.117}
\]
Indeed:
\[
f_0(u) = f_0(u) - \frac{1}{p^*}f_0'(u)(u) \\
\geq \frac{p^* - p - \gamma}{p^*} \int_\Omega \mathcal{L}(x, u, \nabla u) \, dx \\
\geq \frac{p^* - p - \gamma}{pp^*} \nu \int_\Omega |\nabla u|^p \, dx,
\]
which yields \((5.117)\) in view of Lemma 5.4.7. This, in some sense, explains why one chooses \(c\) greater than \(\frac{p^* - p - \gamma}{pp^*}(\nu S)^{\frac{n}{p}}\) in Theorem 5.4.3.

Let now \((u_\varepsilon)_{\varepsilon > 0}\) be a sequence of solutions of \((\mathcal{P}_\varepsilon)\) with \(f_\varepsilon(u_\varepsilon) \to c\) and
\[
\lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) = \frac{p^* - p - \gamma}{pp^*}(\nu S)^{\frac{n}{p}}.
\]

Proof of Theorem 5.4.4. Let us first note that
\[
f_0(u) \leq \lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) + \frac{1}{p^*} \sum_{j=1}^\infty \sigma_j. \tag{5.118}
\]
Indeed, taking into account that by [50, Theorem 3.4]
\[
\int_\Omega \mathcal{L}(x, u, \nabla u) \, dx \leq \lim_{\varepsilon \to 0} \int_\Omega \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \, dx,
\]
\((5.118)\) follows by combining H"older inequality with \((5.100)\).

Now assume by contradiction that \(u \neq 0\). Then, there exists \(j_0 \in \mathbb{N}\) such that \(\mu_{j_0} \neq 0\) and \(\sigma_{j_0} \neq 0\) otherwise, by Remark 5.4.10 and \((5.118)\) we would get
\[
\frac{p^* - p - \gamma}{pp^*}(\nu S)^{\frac{n}{p}} < f_0(u) \leq \lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) = \frac{p^* - p - \gamma}{pp^*}(\nu S)^{\frac{n}{p}}.
\]
Then, arguing as in Lemma 5.4.2 and applying Lemma 5.4.6, we obtain
\[
\frac{p^* - p - \gamma}{pp^*}(\nu S)^{\frac{n}{p}} = \lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) \geq \frac{p^* - p - \gamma}{pp^*} \left( \nu \int_\Omega |\nabla u|^p \, dx + \nu \mu_{j_0} \right) \\
\geq \frac{p^* - p - \gamma}{pp^*} \nu \int_\Omega |\nabla u|^p \, dx + \frac{p^* - p - \gamma}{pp^*}(\nu S)^{\frac{n}{p}},
\]
which implies \(u = 0\), a contradiction. \qed
5.4.5 Mountain–pass critical values

In this section, we shall investigate the asymptotics of \((u_\varepsilon)\) in the case of critical levels of min–max type. We assume that \(\mathcal{L}\) satisfies a stronger assumption, i.e.

\[
\mathcal{L}(x, s, \xi) \leq \frac{1}{p} |\xi|^p \tag{5.119}
\]

for a.e. \(x \in \Omega\) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\). In particular, it results that \(\nu \leq 1\). Let \(u_\varepsilon\) be a critical point of \(f_\varepsilon\) associated with the mountain pass level

\[
c_\varepsilon = \inf_{\eta \in \mathcal{C}_\varepsilon} \max_{t \in [0,1]} f_\varepsilon(\eta(t)),
\]

where

\[
\mathcal{C}_\varepsilon = \{ \eta \in C([0,1], W^{1,p}_0(\Omega)) : \eta(0) = 0, \eta(1) = w_\varepsilon \}
\]

and \(w_\varepsilon \in W^{1,p}_0(\Omega)\) is chosen in such a way that \(f_\varepsilon(w_\varepsilon) < 0\). If \(u\) is the weak limit of \((u_\varepsilon)\) as \(\varepsilon \to 0\), as before one can apply P.L. Lions’ concentration–compactness principle.

Lemma 5.4.11. \(\lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) \leq \frac{1}{n} S^n_{p^*}\).

Proof. Let \(x_0 \in \Omega\) and \(\delta > 0\) and consider the functions \(T_{\delta,x_0}\) as in (5.103). By (c) of Proposition 5.4.5, one has:

\[
\|\nabla T_{\delta,x_0}\|_{p, \mathbb{R}^n}^p = \|T_{\delta,x_0}\|_{p^*, \mathbb{R}^n}^{p^*} = S^n_{p^*}.
\]

Moreover, taking a function \(\phi \in C^\infty_c(\Omega)\) with \(0 \leq \phi \leq 1\) and \(\phi = 1\) in a neighbourhood of \(x_0\) and setting \(v_\delta = \phi T_{\delta,x_0}\), it results

\[
\|\nabla v_\delta\|_p^p = S^n_{p^*} + o(1), \quad \|v_\delta\|_{p^*}^{p^*} = S^n_{p^*} + o(1), \tag{5.121}
\]

as \(\delta \to 0\) (see [65, Lemma 3.2]).

We want to prove that, for any \(t \geq 0\),

\[
\lim_{\varepsilon \to 0} f_\varepsilon(t v_\delta) \leq \frac{1}{n} S^n_{p^*} + o(1)
\]

as \(\delta \to 0\). By (5.119) one has

\[
\lim_{\varepsilon \to 0} f_\varepsilon(t v_\delta) = t^p \int_{\Omega} \mathcal{L}(x, t v_\delta, \nabla v_\delta) \, dx - \lim_{\varepsilon \to 0} \frac{t^{p^* - \varepsilon}}{p - \varepsilon} \int_{\Omega} |v_\delta|^{p^* - \varepsilon} \, dx \leq \frac{t^p}{p} \int_{\Omega} |\nabla v_\delta|^p \, dx - \frac{t^{p^*}}{p^*} \int_{\Omega} |v_\delta|^{p^*} \, dx.
\]

Keeping into account (5.121) and the fact that \(\frac{t^p}{p} - \frac{t^{p^*}}{p^*} \leq \frac{1}{n}\) for every \(t \geq 0\), one gets

\[
\lim_{\varepsilon \to 0} f_\varepsilon(t v_\delta) \leq \frac{t^p}{p} S^n_{p^*} - \frac{t^{p^*}}{p^*} S^n_{p^*} + o(1) \leq \frac{1}{n} S^n_{p^*} + o(1)
\]

as \(\delta \to 0\). Now choose \(t_0 > 0\) such that \(f_\varepsilon(t_0 v_\delta) < 0\); by (5.120) we have that

\[
\lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) \leq \lim_{\varepsilon \to 0} \max_{s \in [0,1]} f_\varepsilon(s t_0 v_\delta) \leq \frac{1}{n} S^n_{p^*} + o(1)
\]

and conclude the proof letting \(\delta \to 0\). \(\square\)
Theorem 5.4.12. Suppose that the number of non-zero Dirac masses is

\[
\left[ \frac{pp^*}{(p^* - p - \gamma)n\nu_\nu} \right]
\]

where \([x]\) denotes the integer part of \(x\). Then \(u = 0\).

Proof. Keeping into account the previous lemma and arguing as in Lemma 5.4.2, one obtains

\[
\frac{1}{n} S^n_{\nu} \geq \lim_{\varepsilon \to 0} f\varepsilon(u_\varepsilon) \geq
\]

\[
\geq \frac{p^* - p - \gamma}{pp^*} \nu \left( \int_{\Omega} |\nabla u|^p dx + \sum_{j=1}^r \mu_j \right) \geq
\]

\[
\geq \frac{p^* - p - \gamma}{pp^*} \nu \int_{\Omega} |\nabla u|^p dx + r \frac{p^* - p - \gamma}{pp^*} \nu \frac{n}{n} S^n_{\nu} = p^* - p - \gamma \frac{n}{n} S^n_{\nu},
\]

where \(r\) denotes the number of non-vanishing masses. Hence it must be

\[
0 \leq r \leq \left[ \frac{pp^*}{(p^* - p - \gamma)n\nu_\nu} \right].
\]

In particular, if \(r\) is maximum and \(u \neq 0\), by virtue of Lemma 5.4.7 one obtains

\[
\frac{p^* - p - \gamma}{pp^*} \nu S^n_{\nu} > \frac{p^* - p - \gamma}{pp^*} \nu \int_{\Omega} |\nabla u|^p dx > \frac{p^* - p - \gamma}{pp^*} \nu S^n_{\nu},
\]

which is a contradiction. \(\square\)
Chapter 6

Non–existence results

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We refer the reader to [105]. Some parts of this publication has been slightly modified to give this collection a more uniform appearance.

6.1 A general Pucci–Serrin type identity

In 1965 Pohozaev discovered a very important identity for solutions of the problem

\[
\begin{aligned}
\Delta u + g(u) &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

This variational identity enabled him to show that the above problem has no nontrivial solution provided that $\Omega$ is a bounded star–shaped domain of $\mathbb{R}^n$ and $g$ satisfies

\[
\forall s \in \mathbb{R} : \quad s \neq 0 \implies (n - 2)sg(s) - 2ng(s) > 0
\]

where $G$ is the primitive of $g$ with $G(0) = 0$.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and outer normal $\nu$. Assume that $\mathcal{L}$ is a function of class $C^1$ on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ with $\mathcal{L}(x, 0, 0) = 0$ and that the vector valued function

\[
\nabla \xi \mathcal{L}(x, s, \xi) = \left( \frac{\partial \mathcal{L}}{\partial \xi_1}(x, s, \xi), \ldots, \frac{\partial \mathcal{L}}{\partial \xi_n}(x, s, \xi) \right)
\]

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is of class $C^1$ in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. Moreover, let $\mathcal{G}$ be a continuous function in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. Consider the problem

$$
\begin{cases}
-\text{div} \left( \nabla \xi \mathcal{L}(x, u, \nabla u) \right) + D_s \mathcal{L}(x, u, \nabla u) = \mathcal{G}(x, u, \nabla u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(\mathcal{P})

Let us recall the celebrated identity proved by Pucci and Serrin [91].

Theorem 6.1.1. Assume that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of (\mathcal{P}). Then

$$
\int_{\partial \Omega} \left[ \mathcal{L}(x, 0, \nabla u) - \nabla \xi \mathcal{L}(x, 0, \nabla u) \cdot \nabla u \right] h \cdot \nu \, d\mathcal{H}^{n-1} \\
= \int_{\Omega} \left[ \mathcal{L}(x, u, \nabla u) \text{div} h + h \cdot \nabla_x \mathcal{L}(x, u, \nabla u) \right] dx \\
- \sum_{i,j=1}^{n} \int_{\Omega} \left[ D_j u D_i h_j + u D_i u \right] D_{\xi_i} \mathcal{L}(x, u, \nabla u) dx \\
- \int_{\Omega} a \left[ \nabla \xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u + u D_s \mathcal{L}(x, u, \nabla u) \right] dx \\
+ \int_{\Omega} [h \cdot \nabla u + au] \mathcal{G}(x, u, \nabla u) dx
$$

(6.1)

for each $a \in C^1(\overline{\Omega})$ and $h \in C^1(\overline{\Omega}, \mathbb{R}^n)$.

Remark 6.1.2. Identity (6.1) follows by testing the equation with $h \cdot \nabla u + au$. More generally, it is satisfied by solutions $u \in C^1(\overline{\Omega}) \cap W^{2,2}_{loc}(\Omega)$.

Theorem 6.1.1 generalizes a well–known identity of Pohozaev [90] which has turned out to be a powerful tool in proving non–existence of solutions of problem (\mathcal{P}). On the other hand, in some cases the requirement that $u$ is of class $C^2(\Omega)$ seems too restrictive, while $C^1(\overline{\Omega})$ is not (cf. [119]). See e.g. problems in which the $p$–Laplacian operator is involved [65].

The aim of this section is to remove the $C^2(\Omega)$ assumption on $u$, by imposing the strict convexity of $\mathcal{L}(x, s, \cdot)$. The main result is the following:

Theorem 6.1.3. Assume that $u \in C^1(\overline{\Omega})$ is a solution of (\mathcal{P}) and that the map

$$
\left\{ \xi \mapsto \mathcal{L}(x, s, \xi) \right\}
$$

is strictly convex for each $(x, s) \in \overline{\Omega} \times \mathbb{R}$. Then (6.1) holds for all $a \in C^1(\overline{\Omega})$, $h \in C^1(\overline{\Omega}, \mathbb{R}^n)$.

Let us observe that the strict convexity of $\mathcal{L}(x, s, \cdot)$ is indeed usually assumed in the applications and it is also natural if one expects the solution $u$ to be of class $C^1(\overline{\Omega})$. In some particular situations (see Section 6.1.2), we are also able to assume only the convexity of $\mathcal{L}(x, s, \cdot)$. This is the case, for instance, if one takes

$$
\mathcal{L}(x, s, \xi) = \alpha(x, s) \beta(\xi) + \gamma(x, s).
$$
Note that if the test functions \(a\) and \(h\) have compact support in \(\Omega\), we obtain the variational identity also when \(u\) is only locally Lipschitz in \(\Omega\). This seems to be useful in particular when \(\mathcal{L}(x, s, \cdot)\) is merely convex, as a \(C^1\) regularity of \(u\) cannot be expected.

Finally, we refer the reader to [91] for various applications of the previous result to non-existence theorems.

### 6.1.1 The approximation argument

Let \(\Omega\) be an open subset of \(\mathbb{R}^n\), not necessarily bounded. Assume that

\[
\left\{ \xi \mapsto \mathcal{L}(x, s, \xi) \right\}
\]

is strictly convex for each \((x, s) \in \overline{\Omega} \times \mathbb{R}\).

**Lemma 6.1.4.** Let \(u : \Omega \to \mathbb{R}\) be a locally Lipschitz solution of

\[
- \text{div} \left( \nabla_x \mathcal{L}(x, u, \nabla u) \right) + D_s \mathcal{L}(x, u, \nabla u) = \mathcal{G}(x, u, \nabla u) \quad \text{in } \Omega.
\]

Then

\[
\int_{\Omega} \left( \mathcal{L}(x, u, \nabla u) \text{div } h + h \cdot \nabla_x \mathcal{L}(x, u, \nabla u) \right) dx = \sum_{i,j=1}^{n} \int_{\Omega} D_i h_j D_{\xi_i} \mathcal{L}(x, u, \nabla u) D_j u \, dx - \int_{\Omega} \mathcal{G}(x, u, \nabla u) h \cdot \nabla u \, dx
\]

for every \(h \in C^1_c(\Omega, \mathbb{R}^n)\).

**Proof.** Since \(h\) has compact support, without loss of generality we may assume that \(\Omega\) is bounded. Let \(R > 0\) with \(|\nabla u(x)| \leq R\) for every \(x \in \text{supp } h\) and let \(\vartheta \in C^\infty(\mathbb{R})\) be such that \(\vartheta = 1\) on \([-R, R]\) and \(\vartheta = 0\) outside \([-R - 1, R + 1]\). Define now \(\overline{\mathcal{L}}(x, s, \xi)\) by

\[
\overline{\mathcal{L}}(x, s, \xi) = \mathcal{L}(x, s, \vartheta(|\xi|)\xi)
\]

for each \((x, s, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n\). Then there exists \(\omega > 0\) such that

\[
\sum_{i,j=1}^{n} \nabla^2_{\xi_i \xi_j} \overline{\mathcal{L}}(x, s, \xi) \eta_i \eta_j > -\omega |\eta|^2
\]

for each \((x, s, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n\) and \(\eta \in \mathbb{R}^n\). Let us now introduce \(A \in C^1([0, +\infty[)\) by

\[
A(\tau) = \begin{cases} 
0 & \text{if } 0 \leq \tau \leq R \\
\omega'(\tau - R)^2 & \text{if } \tau \geq R,
\end{cases}
\]

where \(\omega' > \omega\). Moreover, let \(\widetilde{\mathcal{L}} : \Omega \times \mathbb{R}^n \to \mathbb{R}\) be given by

\[
\widetilde{\mathcal{L}}(x, \xi) = \overline{\mathcal{L}}(x, u(x), \xi) + A(|\xi|)
\]

(6.4)
for each \((x, \xi) \in \Omega \times \mathbb{R}^n\). Then \(\tilde{\mathcal{L}}(x, \cdot)\) is strictly convex and there are \(\nu, c > 0\) with
\[
\tilde{\mathcal{L}}(x, \xi) \geq \nu |\xi|^2 - c
\]
for each \((x, \xi) \in \Omega \times \mathbb{R}^n\). In particular, since \(u\) solves (6.2), then it is the unique minimum of the functional \(f : H^1_0(\Omega) \to \mathbb{R}\) given by
\[
f(w) = \int_{\Omega} \tilde{\mathcal{L}}(x, \nabla w) \, dx + \int_{\Omega} \left[ D_s \mathcal{L}(x, u, \nabla u) - \mathcal{G}(x, u, \nabla u) \right] w \, dx.
\]
On the other hand, if \(u_k \in H^1_0(\Omega)\) denotes the minimum of the modified functional
\[
f_k(w) = f(w) + \frac{1}{k} \int_{\Omega} |\nabla w|^2 \, dx,
\]
by standard regularity arguments, \(u_k \in C^1(\overline{\Omega}) \cap W^{2,2}_{\text{loc}}(\Omega)\). Since \(f(u_k) \to f(u)\) as \(k \to +\infty\), we get \(u_k \to u\) in \(H^1_0(\Omega)\) and
\[
\int_{\Omega} \tilde{\mathcal{L}}(x, \nabla u_k) \, dx \to \int_{\Omega} \tilde{\mathcal{L}}(x, \nabla u) \, dx,
\]
which by [123, Theorem 3] implies \(u_k \to u\) in \(H^1_0(\Omega)\). In particular, \(\nabla u_k(x) \to \nabla u(x)\) a.e. in \(\Omega\), up to a subsequence. Put now
\[
\tilde{\mathcal{L}}(x, \xi) = \tilde{\mathcal{L}}(x, \xi) + \frac{1}{k} |\xi|^2.
\]
Since \(u_k\) satisfies the Euler’s equation of \(f_k\)
\[
\text{div}(\nabla \tilde{\mathcal{L}}(x, \nabla u_k)) = D_s \mathcal{L}(x, u, \nabla u) - \mathcal{G}(x, u, \nabla u),
\]
by (6.1) it results
\[
\int_{\Omega} \left( \mathcal{L}(x, \nabla u_k) \, \text{div} h + h \cdot \nabla_x \tilde{\mathcal{L}}(x, \nabla u_k) \right) \, dx
= \sum_{i,j=1}^n \int_{\Omega} D_i h_j D_{\xi, i} \tilde{\mathcal{L}}(x, \nabla u_k) D_j u_k \, dx
+ \int_{\Omega} \left[ D_s \mathcal{L}(x, u, \nabla u) - \mathcal{G}(x, u, \nabla u) \right] h \cdot \nabla u_k \, dx,
\]
namely
\[
\int_{\Omega} \mathcal{L}(x, u(x), \nabla u_k) \, \text{div} h \, dx + \int_{\Omega} \mathcal{A}(|\nabla u_k|) \, \text{div} h \, dx
+ \frac{1}{k} \int_{\Omega} |\nabla u_k|^2 \, \text{div} h \, dx + \int_{\Omega} h \cdot \nabla_x \tilde{\mathcal{L}}(x, \nabla u_k) \, dx
- \sum_{i,j=1}^n \int_{\Omega} D_i h_j D_{\xi, i} \tilde{\mathcal{L}}(x, u(x), \nabla u_k) D_j u_k \, dx
- \sum_{i,j=1}^n \int_{\Omega} D_i h_j D_{\xi, i} \mathcal{A}(|\nabla u_k|) D_j u_k \, dx
- \frac{2}{k} \sum_{i,j=1}^n \int_{\Omega} D_i h_j D_{\xi, i} D_j u_k D_j u_k \, dx - \int_{\Omega} \left[ D_s \mathcal{L}(x, u, \nabla u) - \mathcal{G}(x, u, \nabla u) \right] h \cdot \nabla u_k \, dx = 0.
\]
Notice that
\[
\frac{1}{k} \int_{\Omega} |\nabla u_k|^2 \text{ div } h \, dx \to 0, \quad \frac{2}{k} \sum_{i,j=1}^{n} \int_{\Omega} D_i h_j D_i u_k D_j u_k \, dx \to 0
\]
as \( k \to +\infty \). Moreover, since \( D_j u_k \to D_j u \) and \( D_{\xi_i} A(|\nabla u_k|) \to D_{\xi_i} A(|\nabla u|) \) in \( L^2(\Omega) \),
\[
\sum_{i,j=1}^{n} \int_{\Omega} D_i h_j D_{\xi_i} A(|\nabla u_k|) D_j u_k \, dx \to \sum_{i,j=1}^{n} \int_{\Omega} D_i h_j D_{\xi_i} A(|\nabla u|) D_j u \, dx = 0
\]
as \( k \to +\infty \) and
\[
\int_{\Omega} A(|\nabla u_k|) \text{ div } h \, dx \to \int_{\Omega} A(|\nabla u|) \text{ div } h \, dx = 0
\]
as \( k \to +\infty \). Since
\[
\int_{\Omega} h \cdot \nabla_x \overline{L}(x, \nabla u_k) \, dx = \int_{\Omega} h \cdot \nabla_x L(x, u(x), \vartheta(|\nabla u_k|) \nabla u_k) \, dx
\]
\[
+ \int_{\Omega} D_s L(x, u(x), \vartheta(|\nabla u_k|) \nabla u_k) h \cdot \nabla u \, dx
\]
and being
\[
|\nabla_x L(x, u(x), \vartheta(|\nabla u_k|) \nabla u_k)| \leq c_1, \quad |D_s L(x, u(x), \vartheta(|\nabla u_k|) \nabla u_k)| \leq c_2
\]
for some \( c_1, c_2 > 0 \), one obtains
\[
\int_{\Omega} h \cdot \nabla_x \overline{L}(x, \nabla u_k) \, dx \to \int_{\Omega} h \cdot \nabla_x L(x, u, \nabla u) \, dx + \int_{\Omega} D_s L(x, u, \nabla u) h \cdot \nabla u \, dx.
\]
Furthermore, since there exists \( c_3 > 0 \) with \( |L(x, u(x), \vartheta(|\nabla u_k|) \nabla u_k)| \leq c_3 \), one gets
\[
\int_{\Omega} L(x, u, \vartheta(|\nabla u_k|) \nabla u_k) \text{ div } h \, dx \to \int_{\Omega} L(x, u, \nabla u) \text{ div } h \, dx.
\]
Taking into account that there exists \( c_4 > 0 \) with
\[
\left| D_{\xi_i} L(x, u(x), \vartheta(|\nabla u_k|) \nabla u_k) \left[ \vartheta’(|\nabla u_k|) \frac{D_i u_k \nabla u_k}{|\nabla u_k|} + \vartheta(|\nabla u_k|) e_i \right] \right| \leq c_4
\]
and that \( D_j u_k \to D_j u \) in \( L^2(\Omega) \), one deduces
\[
\sum_{i,j=1}^{n} \int_{\Omega} D_i h_j D_{\xi_i} L(x, u, \nabla u_k) D_j u_k \, dx \to \sum_{i,j=1}^{n} \int_{\Omega} D_i h_j D_{\xi_i} L(x, u, \nabla u) D_j u \, dx
\]
as \( k \to +\infty \). Noting that, of course
\[
\int_{\Omega} [D_s L(x, u, \nabla u) - G(x, u, \nabla u)] h \cdot \nabla u_k \, dx \to
\]
\[
\int_{\Omega} [D_s L(x, u, \nabla u) - G(x, u, \nabla u)] h \cdot \nabla u \, dx
\]
as \( k \to +\infty \), the proof is complete. \( \square \)
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Remark 6.1.5. Let us observe that a (different) approximation technique was also used by Guedda and Véron [65] to deal with the particular case \( L(x,s) = \frac{1}{p} |\xi|^p \).

Let us now assume that \( \Omega \) is bounded with Lipschitz boundary and let \( \nu(x) \) denote the outer normal to \( \partial \Omega \) at \( x \) (which exists for \( \mathcal{H}^{n-1} \)-a.e. \( x \in \partial \Omega \)).

Lemma 6.1.6. Let \( u \in C^1(\overline{\Omega}) \) be a weak solution of \( (P) \). Then it holds

\[
\int_{\Omega} (\mathcal{L}(x,u,\nabla u) \div h + h \cdot \nabla_x \mathcal{L}(x,u,\nabla u)) \, dx = \sum_{i,j=1}^{n} \int_{\Omega} D_i h_j D_{\xi_i} \mathcal{L}(x,u,\nabla u) D_j u \, dx - \int_{\Omega} \mathcal{G}(x,u,\nabla u) h \cdot \nabla u \, dx + \int_{\partial \Omega} [\mathcal{L}(x,0,\nabla u) - \nabla_x \mathcal{L}(x,0,\nabla u) \cdot \nabla u] (h \cdot \nu) \, d\mathcal{H}^{n-1}
\]

for every \( h \in C^1(\overline{\Omega},\mathbb{R}^n) \).

Proof. Let \( k \geq 1 \) and \( \varphi_k : \mathbb{R} \rightarrow [0,1] \) be given by

\[
\varphi_k(s) = \begin{cases} 
0 & \text{if } s \leq \frac{1}{k} \\
ks - 1 & \text{if } \frac{1}{k} \leq s \leq \frac{2}{k} \\
1 & \text{if } s \geq \frac{2}{k},
\end{cases}
\]

and define the Lipschitz map \( \psi_k : \mathbb{R}^n \rightarrow [0,1] \) by setting

\[ \psi_k(x) = \varphi_k(d(x,\mathbb{R}^n \setminus \Omega)). \]

By applying Lemma 6.1.4 on \( \mathbb{R}^n \) with \( \psi_k h \) in place of \( h \), one deduces

\[
\int_{\mathbb{R}^n} \psi_k \mathcal{L}(x,u,\nabla u) \div h \, dx + \int_{\mathbb{R}^n} \mathcal{L}(x,u,\nabla u) \nabla \psi_k : h \, dx \\
+ \int_{\mathbb{R}^n} \psi_k h \cdot \nabla_x \mathcal{L}(x,u,\nabla u) \, dx = \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} h_j D_i \psi_k D_{\xi_i} \mathcal{L}(x,u,\nabla u) D_j u \, dx \\
+ \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \psi_k D_i h_j D_{\xi_i} \mathcal{L}(x,u,\nabla u) D_j u \, dx - \int_{\mathbb{R}^n} \mathcal{G}(x,u,\nabla u) \psi_k h \cdot \nabla u \, dx.
\]

Taking into account that \( (\psi_k) \) is bounded in \( BV(\mathbb{R}^n) \) and

\[ \forall \eta \in C(\mathbb{R}^n,\mathbb{R}^n) : \int_{\mathbb{R}^n} \nabla \psi_k \cdot \eta \, dx \rightarrow - \int_{\partial \Omega} \eta \cdot \nu \, d\mathcal{H}^{n-1}, \]

one has

\[ \int_{\mathbb{R}^n} \mathcal{L}(x,u,\nabla u) \nabla \psi_k : h \, dx \rightarrow - \int_{\partial \Omega} \mathcal{L}(x,0,\nabla u)(h \cdot \nu) \, d\mathcal{H}^{n-1} \]

as \( k \rightarrow +\infty \) and

\[ \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} h_j D_i \psi_k D_{\xi_i} \mathcal{L}(x,u,\nabla u) D_j u \, dx \rightarrow - \sum_{i,j=1}^{n} \int_{\partial \Omega} \nu_i h_j D_{\xi_i} \mathcal{L}(x,0,\nabla u) D_j u \, dx. \]
As observed in [91], clearly one has
\[ \sum_{i,j=1}^{n} \nu_{ij} D_{\xi i} \mathcal{L}(x,0,\nabla u) D_{\xi j} u = \nabla_{\xi} \mathcal{L}(x,0,\nabla u) \cdot \nabla (h \cdot u) \quad \text{on } \partial \Omega. \]
Since of course \( \psi_k(x) \to \chi_{\Omega}(x) \) for each \( x \in \mathbb{R}^n \), the proof is complete. \( \square \)

**Proof of Theorem 6.1.3.** Clearly, if \( u \in C^1(\Omega) \) is a solution of \( (\mathcal{P}) \) one has
\[ \int_{\Omega} a \left[ \nabla_{\xi} \mathcal{L}(x,u,\nabla u) \cdot \nabla u + u D_{s_k} \mathcal{L}(x,u,\nabla u) - u \mathcal{G}(x,u,\nabla u) \right] \, dx \\
+ \int_{\Omega} u \nabla a \cdot \nabla_{\xi} \mathcal{L}(x,u,\nabla u) \, dx = 0 \]
for each \( a \in C^1(\Omega) \). The assertion follows by combining (6.6) with Lemma 6.1.6. \( \square \)

**Remark 6.1.7.** Let \( N > 2 \). It is easily seen that Theorem 6.1.3 has a vectorial counterpart for solutions \( u \in C^1(\Omega, \mathbb{R}^N) \) of the system
\[ \begin{cases} 
- \text{div} (\nabla_{\xi_k} \mathcal{L}(x,u,\nabla u)) + D_{s_k} \mathcal{L}(x,u,\nabla u) = \mathcal{G}_k(x,u,\nabla u) & \text{in } \Omega \\
\quad u = 0 & \text{on } \partial \Omega \\
\quad k = 1, \ldots, N.
\end{cases} \]
See also Proposition 3 of [91].

### 6.1.2 Nonstrict convexity in some particular cases

In this section we will see that, in some particular cases, the assumption of strict convexity of \( \mathcal{L}(x,s,\cdot) \) can be relaxed to the weaker assumption of convexity. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) with Lipschitz boundary.

**Lemma 6.1.8.** Let \( \mathcal{F} : \Omega \times \mathbb{R}^n \to \mathbb{R} \) be a function with \( \mathcal{F}(x,\cdot) \) convex and \( C^1 \) and \( \mathcal{F}(\cdot,\xi) \) measurable. Assume that there exist \( a_0 \in L^1(\Omega), a_1 \in L^{p'}(\Omega), 1 < p < +\infty, \) and \( b, d > 0 \) with
\[ |\nabla_{\xi} \mathcal{F}(x,\xi)| \leq a_1(x) + b|\xi|^{p-1}, \]
\[ \mathcal{F}(x,\xi) \geq d|\xi|^p - a_0(x) \]
for a.e. \( x \in \Omega \) and all \( \xi \in \mathbb{R}^n \). Let \( (w_k) \subset L^p(\Omega, \mathbb{R}^n) \) and \( w \) be such that
\[ w_k \to w \quad \text{in } L^p(\Omega, \mathbb{R}^n), \quad \int_{\Omega} \mathcal{F}(x,w_k) \, dx \to \int_{\Omega} \mathcal{F}(x,w) \, dx \]
as \( k \to +\infty \). Then
\[ \mathcal{F}(x,w_k) \to \mathcal{F}(x,w) \quad \text{in } L^1(\Omega), \]
\[ \nabla_{\xi} \mathcal{F}(x,w_k) \to \nabla_{\xi} \mathcal{F}(x,w) \quad \text{in } L^{p'}(\Omega) \]
as \( k \to +\infty \). Moreover, up to a subsequence, \( |w_k|^p \leq \psi \) for some \( \psi \in L^1(\Omega) \).
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Proof. Let us define $\tilde{F}: \Omega \times \mathbb{R}^n \to \mathbb{R}$ by setting

$$\tilde{F}(x, \xi) = F(x, w(x) + \xi) - F(x, w(x)) - \nabla_{\xi} F(x, w(x)) : \xi.$$ 

Note that $\tilde{F} \geq 0$, $\tilde{F}(x, 0) = 0$, $\nabla_{\xi} \tilde{F}(x, 0) = 0$ and

$$\int_{\Omega} \tilde{F}(x, w_k - w) \, dx \to 0 \quad \text{as } k \to +\infty. \quad (6.11)$$

Therefore, since for each $\varphi \in L^\infty(\Omega)$

$$\int_{\Omega} \varphi \nabla_{\xi} F(x, w) \cdot (w_k - w) \, dx \to 0 \quad \text{as } k \to +\infty,$$

one has

$$\int_{\Omega} \varphi [F(x, w_k) - F(x, w)] \, dx \to 0 \quad \text{as } k \to +\infty,$$

which proves (6.9).

Note that, in view of (6.11), up to a subsequence one has $\tilde{F}(x, w_k(x) - w(x)) \to 0$ for a.e. $x \in \Omega$. Fix now such an $x$; then by (6.8) up to a subsequence $w_k(x) \to y$ for some $y \in \mathbb{R}^n$, which yields $\tilde{F}(x, y - w(x)) = 0$. In particular, $y - w(x)$ is a local minimum for $\tilde{F}(x, \cdot)$, so that $\nabla_{\xi} \tilde{F}(x, y - w(x)) = 0$. Hence we conclude

$$\nabla_{\xi} F(x, w_k(x)) \to \nabla_{\xi} F(x, w(x)). \quad (6.12)$$

Now, since by (6.11) there exists $\tilde{\psi} \in L^1(\Omega)$ such that

$$\mathcal{F}(x, w_k) - \mathcal{F}(x, w) - \nabla_{\xi} \mathcal{F}(x, w) \cdot (w_k - w) \leq \tilde{\psi},$$

by (6.8) and Young’s inequality one finds $c_1, c_2 > 0$ such that

$$c_1 |w_k|^p \leq a_0 + \mathcal{F}(x, w) - \nabla_{\xi} \mathcal{F}(x, w) \cdot w + \tilde{\psi} + c_2 |\nabla_{\xi} \mathcal{F}(x, w)|^{p'}$$

In particular, in view of (6.7) one deduces $|\nabla_{\xi} \mathcal{F}(x, w_k)| \leq g$ for some $g \in L^{p'}(\Omega)$, which combined with (6.12) yields the second assertion. \qed

6.1.3 The splitting case

In this subsection we shall deal with the case when $\mathcal{L}(x, s, \xi)$ is of the form $\alpha(x, s)\beta(\xi) + \gamma(x, s)$.

Lemma 6.1.9. Let $\alpha, \gamma \in W^{1,\infty}(\Omega)$ with $\alpha \geq 0$ and $\beta \in C^1(\mathbb{R}^n)$ convex such that

$$\mathcal{F}(x, \xi) = \alpha(x)\beta(\xi) + \gamma(x)$$

with $d|\xi|^p - b \leq \beta(\xi) \leq b(1 + |\xi|^p)$, $1 < p < +\infty$, for some $b, d > 0$. Let $(w_k)$ and $w$ with

$$w_k \to w \quad \text{in } L^p(\Omega, \mathbb{R}^n), \quad \int_{\Omega} \mathcal{F}(x, w_k) \, dx \to \int_{\Omega} \mathcal{F}(x, w) \, dx$$

as $k \to +\infty$. Then

$$\beta(w_k)\nabla \alpha(x) \to \beta(w)\nabla \alpha(x) \quad \text{in } L^1(\Omega), \quad (6.13)$$

as $k \to +\infty$. 

6.1. A general Pucci–Serrin type identity

Proof. If \( \Omega_0 \) denotes the set where \( \alpha = 0 \), one may argue on
\[
\Omega \setminus \Omega_0 = \bigcup_{h=1}^{+\infty} \Omega_h, \quad \Omega_h = \left\{ x \in \Omega : \alpha(x) > \frac{1}{h} \right\}.
\]
By Lemma 6.1.8 there exists \( \psi \in L^1(\Omega) \) such that
\[
\chi_{\Omega \setminus \Omega_h}(x)\beta(w_k(x))\nabla \alpha(x) \leq \psi(x)
\]
up to a subsequence; hence for each \( \varepsilon > 0 \) one finds \( h_0 \geq 1 \) such that
\[
\int_{\Omega \setminus \Omega_{h_0}} \beta(w_k(x))\nabla \alpha(x) \, dx < \varepsilon
\]
uniformly with respect to \( k \). On the other hand, again by Lemma 6.1.8 one knows that
\[
\mathcal{F}(x,w_k) \rightarrow \mathcal{F}(x,w) \quad \text{in} \quad L^1(\Omega_{h_0})
\]
as \( k \rightarrow +\infty \), which implies
\[
\alpha(x)\beta(w_k) \rightarrow \alpha(x)\beta(w) \quad \text{in} \quad L^1(\Omega_{h_0}).
\]
Then since \( 1/\alpha \in L^\infty(\Omega_{h_0}) \) one gets \( \beta(w_k) \rightharpoonup \beta(w) \) in \( L^1(\Omega_{h_0}) \), which yields (6.13). \( \square \)

**Theorem 6.1.10.** Let \( u : \Omega \rightarrow \mathbb{R} \) be a locally Lipschitz solution of (6.2). Assume that there exist \( \alpha, \gamma \in C^1(\Omega \times \mathbb{R}) \) and \( \beta \in C^2(\mathbb{R}^n) \) convex such that \( \alpha \geq 0, \beta(0) = 0 \) and
\[
\mathcal{L}(x,s,\xi) = \alpha(x,s)\beta(\xi) + \gamma(x,s).
\]
Then
\[
\sum_{i,j=1}^{n} \int_{\Omega} \left[ D_j u D_i h_j + u D_i u \right] D_{\xi_i} \mathcal{L}(x,u,\nabla u) \, dx
+ \int_{\Omega} a \left[ \nabla \xi \mathcal{L}(x,u,\nabla u) \cdot \nabla u + u D_s \mathcal{L}(x,u,\nabla u) \right] \, dx
= \int_{\Omega} \left[ \mathcal{L}(x,u,\nabla u) \text{div} h + h \cdot \nabla \mathcal{L}(x,u,\nabla u) \right] \, dx
= \int_{\Omega} \left[ h \cdot \nabla u + au \right] \mathcal{G}(x,u,\nabla u) \, dx
\]
holds for each \( a \in C^1_c(\Omega) \) and \( h \in C^1_c(\Omega,\mathbb{R}^n) \).

Proof. Let \( \theta, A, \tilde{\mathcal{L}} \) and \( (u_k) \subset H^1_0(\Omega) \) be as in Lemma 6.1.4. We apply Lemma 6.1.8 choosing
\[
w_k = \nabla u_k, \quad \mathcal{F}(x,\xi) = A(|\xi|) \quad \text{or} \quad \mathcal{F}(x,\xi) = \beta(\theta(|\xi|)|\xi|).
\]
By (6.10) one has
\[
\sum_{i,j=1}^{n} \int_{\Omega} D_i h_j D_{\xi_i} A(|\nabla u_k|) D_j u_k \, dx \rightarrow \sum_{i,j=1}^{n} \int_{\Omega} D_i h_j D_{\xi_i} A(|\nabla u|) D_j u \, dx = 0,
\]
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and the term
\[ \sum_{i,j=1}^{n} \int_{\Omega} D_i h_j \alpha(x,u) D_{\xi_i} \beta(\nabla u) \nabla u \, dx \]
goes to
\[ \sum_{i,j=1}^{n} \int_{\Omega} D_i h_j \alpha(x,u) D_{\xi_i} \beta(\nabla u) \nabla u \, dx \]
as \( k \to +\infty \). Moreover, by (6.9) one obtains
\[ \int_{\Omega} \Lambda(|\nabla u_k|) \nabla u \, dx \to \int_{\Omega} \Lambda(|\nabla u|) \nabla u \, dx = 0, \]
\[ \int_{\Omega} \alpha(x,u) \beta(\nabla u) \nabla u \, dx \to \int_{\Omega} \alpha(x,u) \beta(\nabla u) \nabla u \, dx \]
as \( k \to +\infty \). Finally, by (6.13) of Lemma 6.1.9 one gets
\[ \int_{\Omega} (h \cdot \nabla \mathcal{L}(x, \nabla u) \, dx \to \int_{\Omega} [h \cdot \nabla \mathcal{L}(x, \nabla u) + D_s \alpha(x,u) h \cdot \nabla u] \beta(\nabla u) \, dx \]
\[ + \int_{\Omega} \nabla \gamma(x,u) h \cdot \nabla u \, dx \]
as \( k \to +\infty \). Then (6.14) follows by exploiting the proof of Lemma 6.1.4.

At this point, arguing as in Lemma 6.1.6 and taking into account (6.6), we obtain the following result.

**Theorem 6.1.11.** Let \( u \in C^1(\overline{\Omega}) \) be a weak solution of \( (\mathcal{P}) \). Let \( \alpha, \gamma \in C^1(\overline{\Omega} \times \mathbb{R}) \) and \( \beta \in C^2(\mathbb{R}^n) \) convex such that \( \alpha \geq 0, \beta(0) = 0 \) and
\[ \mathcal{L}(x,s,\xi) = \alpha(x,s) \beta(\xi) + \gamma(x,s). \]

Then (6.1) holds for each \( a \in C^1(\overline{\Omega}) \) and \( h \in C^1(\overline{\Omega}, \mathbb{R}^n) \).

### 6.1.4 The one–dimensional case

In this subsection we assume that \( \Omega \) is an interval in \( \mathbb{R} \) and \( \mathcal{L}: \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is of class \( C^1 \) with \( \mathcal{L}(x,s,\cdot) \) convex and \( D_{\xi} \mathcal{L} \) of class \( C^1 \).

**Theorem 6.1.12.** Let \( u: \Omega \to \mathbb{R} \) be a locally Lipschitz solution of (6.2). Then (6.14) holds for each \( a \in C^1_c(\Omega) \) and \( h \in C^1_c(\Omega) \).

**Theorem 6.1.13.** Let \( u \in C^1(\overline{\Omega}) \) be a weak solution of \( (\mathcal{P}) \). Then (6.1) holds for each \( a \in C^1(\overline{\Omega}) \) and \( h \in C^1(\overline{\Omega}) \).

Taking into account next result, the above theorems follow arguing as in the proof of Lemmas 6.1.4 and 6.1.6.
Lemma 6.1.14. Let \( \mathcal{F} : \Omega \times \mathbb{R} \to \mathbb{R} \) be a \( C^1 \) function with \( \mathcal{F}(x, \cdot) \) convex. Let \((w_k) \subset L^p(\Omega)\) and \(w\) be such that
\[
w_k \to w \quad \text{in } L^p(\Omega), \quad \int_{\Omega} \mathcal{F}(x, w_k) \, dx \to \int_{\Omega} \mathcal{F}(x, w) \, dx
\]
as \( k \to +\infty \) and assume that \((D_x \mathcal{F}(x, w_k))\) is bounded in \( L^q \) for some \( q > 1 \). Then
\[
D_x \mathcal{F}(x, w_k) \to D_x \mathcal{F}(x, w) \quad \text{in } L^1(\Omega)
\]
as \( k \to +\infty \).

Proof. Let us set for each \( x \in \Omega \)
\[
y_b(x) = \liminf_k w_k(x), \quad y^*_b(x) = \limsup_k w_k(x).
\]
Notice that one has
\[
y_b(x) \leq w(x) \leq y^*_b(x) \quad (6.16)
\]
for a.e. \( x \in \Omega \). Without loss of generality, one can replace \( w_k(x) \) by its projection onto \([y_b(x), y^*_b(x)]\); in particular
\[
y_b(x) \leq w_k(x) \leq y^*_b(x) \quad (6.17)
\]
for a.e. \( x \in \Omega \). Arguing as in the proof of Lemma 6.1.8 one obtains
\[
\widetilde{\mathcal{F}}(x, y_b(x) - w(x)) = 0, \quad \widetilde{\mathcal{F}}(x, y^*_b(x) - w(x)) = 0
\]
for a.e. \( x \in \Omega \). Then, by \( \widetilde{\mathcal{F}} \geq 0 \) and the convexity of \( \widetilde{\mathcal{F}}(x, \cdot) \) one has
\[
\widetilde{\mathcal{F}}(x, (1 - \theta)y_b(x) + \theta y^*_b(x) - w(x)) = 0
\]
for every \( \theta \in [0, 1] \) and a.e. \( x \in \Omega \). This yields
\[
\mathcal{F}(x, (1 - \theta)y_b(x) + \theta y^*_b(x)) = (1 - \theta)\mathcal{F}(x, y_b(x)) + \theta \mathcal{F}(x, y^*_b(x)) \quad (6.18)
\]
for a.e. \( x \in \Omega \). For each \( m \geq 1 \) let us set
\[
\Omega_m = \left\{ x \in \Omega : y_b(x) - y_b(x) \geq \frac{1}{m} \right\}.
\]
By Lusin’s theorem, for each \( \varepsilon > 0 \) there exists a closed subset \( C_{m, \varepsilon} \subset \Omega_m \) such that
\[
y_b|_{C_{m, \varepsilon}}, \quad y^*_b\big|_{C_{m, \varepsilon}} \quad \text{are continuous}, \quad \mathcal{L}^1(\Omega_m \setminus C_{m, \varepsilon}) < \varepsilon,
\]
where \( \mathcal{L}^1 \) denotes the one–dimensional Lebesgue measure. We also cut off from \( C_{m, \varepsilon} \) the negligible set of isolated points. Let us now take \( x \in C_{m, \varepsilon} \) and \((x_k) \subset C_{m, \varepsilon}\) with \( x_k \to x \). If \( \delta > 0 \) is sufficiently small, by continuity one has
\[
y_b(x_k) \leq y_b(x) + \delta < y^*_b(x) - \delta \leq y^*_b(x_k) \quad (6.19)
\]
for each $k \in \mathbb{N}$ large enough. By (6.18), for each $\vartheta \in [0, 1]$ one obtains

$$\mathcal{F}(x, (1 - \vartheta)(y_b(x) + \delta) + \vartheta(y^*_b(x) - \delta)) = (1 - \vartheta)\mathcal{F}(x, y_b(x) + \delta) + \vartheta\mathcal{F}(x, y^*_b(x) - \delta).$$

Moreover, (6.19) implies

$$\mathcal{F}(x_k, (1 - \vartheta)(y_b(x) + \delta) + \vartheta(y^*_b(x) - \delta)) = (1 - \vartheta)\mathcal{F}(x_k, y_b(x) + \delta) + \vartheta\mathcal{F}(x_k, y^*_b(x) - \delta).$$

Therefore, combining the previous identities yields

$$D_x\mathcal{F}(x, (1 - \vartheta)(y_b(x) + \delta) + \vartheta(y^*_b(x) - \delta)) = (1 - \vartheta)D_x\mathcal{F}(x, y_b(x) + \delta) + \vartheta D_x\mathcal{F}(x, y^*_b(x) - \delta)$$

for each $\vartheta \in [0, 1]$. Letting $\delta \to 0$ one obtains

$$D_x\mathcal{F}(x, (1 - \vartheta)y_b(x) + \vartheta y^*_b(x)) = (1 - \vartheta)D_x\mathcal{F}(x, y_b(x)) + \vartheta D_x\mathcal{F}(x, y^*_b(x))$$

for each $\vartheta \in [0, 1]$. By (6.16) and (6.17) we can choose

$$\bar{\vartheta} = \frac{w(x) - y_b(x)}{y^*_b(x) - y_b(x)}, \quad \bar{\vartheta}_k = \frac{w_k(x) - y_b(x)}{y^*_b(x) - y_b(x)}.$$

Then one gets

$$D_x\mathcal{F}(x, w(x)) = \frac{y_b(x) - w(x)}{y^*_b(x) - y_b(x)} D_x\mathcal{F}(x, y_b(x)) + \frac{w(x) - y_b(x)}{y^*_b(x) - y_b(x)} D_x\mathcal{F}(x, y^*_b(x))$$

and

$$D_x\mathcal{F}(x, w_k(x)) = \frac{y_b(x) - w_k(x)}{y^*_b(x) - y_b(x)} D_x\mathcal{F}(x, y_b(x)) + \frac{w_k(x) - y_b(x)}{y^*_b(x) - y_b(x)} D_x\mathcal{F}(x, y^*_b(x)).$$

In particular, one concludes

$$D_x\mathcal{F}(x, w_k(x)) = D_x\mathcal{F}(x, w(x)) + \frac{(w_k(x) - w(x)) D_x\mathcal{F}(x, y^*_b(x)) - D_x\mathcal{F}(x, y_b(x))}{y^*_b(x) - y_b(x)}$$

for all $x \in C_{m, \varepsilon}$, which implies that

$$\forall \varphi \in L^\infty(C_{m, \varepsilon}) : \int_{C_{m, \varepsilon}} D_x\mathcal{F}(x, w_k(x)) \varphi \, dx \to \int_{C_{m, \varepsilon}} D_x\mathcal{F}(x, w) \varphi \, dx$$

as $k \to +\infty$. On the other hand, since $(D_x\mathcal{F}(x, w_k(x)))$ is bounded in $L^q(\Omega)$, for any $\varphi \in L^\infty(C_{m, \varepsilon})$ there exists $c > 0$ such that

$$\left| \int_{\Omega \setminus C_{m, \varepsilon}} D_x\mathcal{F}(x, w_k(x)) \varphi \, dx \right| \leq c L^1(\Omega \setminus C_{m, \varepsilon}) < c \varepsilon.$$
6.2 Applications to non-existence results

Letting \( \varepsilon \to 0 \), one gets
\[
\forall \varphi \in L^\infty(\Omega_m) : \int_{\Omega_m} D_x \mathcal{F}(x, w_k) \varphi \, dx \to \int_{\Omega_m} D_x \mathcal{F}(x, w) \varphi \, dx
\]
for each \( m \geq 1 \). Moreover, since on the set
\[
\Omega_\infty = \{ x \in \Omega : \ y_k(x) = y_k(x) \}
\]
one has \( w_k \to w \) pointwise, then
\[
\forall \varphi \in L^\infty(\Omega_\infty) : \int_{\Omega_\infty} D_x \mathcal{F}(x, w_k) \varphi \, dx \to \int_{\Omega_\infty} D_x \mathcal{F}(x, w) \varphi \, dx
\]
which concludes the proof.

6.2 Applications to non-existence results

In the following we want to recall from [91] a general variational identity that holds both for scalar-valued and vector-valued extremals of multiple integrals of calculus of variations that will allow us to get non-existence results for various classes of problems.

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \), \( n \geq 3 \) and \( k \geq 1 \). For each \( \alpha \in \mathbb{N}^n \) we set
\[
\xi^\alpha := \xi_1 \cdots \xi_k, \quad C_\alpha := \frac{|\alpha|!}{\alpha_1! \cdots \alpha_n!}, \quad \mathcal{L}^\alpha := C_\alpha D_{\xi_1 \cdots \xi_k} \mathcal{L}.
\]

Let now \( f : W^{k,p}_0(\Omega) \to \mathbb{R} \) be the \( k \)-th order functional of calculus of variations
\[
f(u) = \int_\Omega \mathcal{L}(x, u, \nabla u, \ldots, \nabla^k u) \, dx.
\]

By direct calculation, the Euler–Lagrange’s equation of \( f \) is given by
\[
\sum_{|\alpha| = 0}^k (-1)^{|\alpha|} D^\alpha \mathcal{L}_{\xi^\alpha}(x, u, \ldots, \nabla^k u) = 0 \text{ in } \Omega. \tag{6.20}
\]

If \( u \in W^{k,p}_0(\Omega) \) is a weak solution to \( (6.20) \) and \( \lambda \in C^k(\Omega), \ v \in C^k(\Omega, \mathbb{R}^n) \), we set
\[
\vartheta := v \cdot \nabla u + \lambda u, \quad \xi^\gamma := \sum_{i=1}^n \xi^{\alpha+\beta} \xi_i, \quad \partial_{\lambda, \nu}^\alpha := [D^\alpha(v \cdot D + \lambda) - (v \cdot D)D^\alpha].
\]

We now recall the following Pohozaev–type identity for general lagrangians.

**Proposition 6.2.1.** Assume that \( u \in C^k(\Omega) \) is a weak solution to \( (6.20) \). Then
\[
\text{div} \left\{ v \mathcal{L}(x, u, \ldots, \nabla^k u) - \sum_{|\alpha+\beta| = 0}^{k-1} (-1)^{|\beta|} C_\alpha C_\beta \frac{\partial_{\lambda, \nu}^\alpha}{C_\gamma} D^\alpha \vartheta D^\beta \mathcal{L}_{\xi^\gamma}(x, u, \ldots, \nabla^k u) \right\} = \\
\quad = \text{div}(v) \mathcal{L}(x, u, \ldots, \nabla^k u) + v \cdot \nabla_x \mathcal{L}(x, u, \ldots, \nabla^k u) - \sum_{|\alpha| = 0}^k \partial_{\lambda, \nu}^\alpha u \cdot \mathcal{L}_{\xi^\alpha}(x, u, \ldots, \nabla^k u)
\]
for a.e. \( x \in \Omega \) and for each \( v \in C^k(\Omega, \mathbb{R}^n) \).
Chapter 6. Non-existence results

Proof. The identity follows by direct computation. See section 5 of [91]. □

We now come to the main non-existence result for first order scalar-valued extremals.

**Theorem 6.2.2.** Assume that \( \Omega \) is star-shaped with respect to 0. Suppose also that

\[
\xi \cdot \nabla_\xi \mathcal{L}(x, 0, \xi) - \mathcal{L}(x, 0, \xi) \geq 0
\]

for a.e. \( x \in \Omega \) and all \( \xi \in \mathbb{R}^n \) and that there exists \( \lambda \in \mathbb{R} \) such that

\[
n \mathcal{L}(x, s, \xi) + x \cdot \nabla_x \mathcal{L}(x, s, \xi) - \lambda s D_s \mathcal{L}(x, s, \xi) - (\lambda + 1) \xi \cdot \nabla_\xi \mathcal{L}(x, s, \xi) \geq 0
\]

for a.e. \( x \in \Omega \) and each \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \). Let \( s = 0 \) or \( \xi = 0 \) whenever equality holds in (6.22). Then the elliptic boundary value problem

\[
-\text{div} \left( \nabla_\xi \mathcal{L}(x, u, \nabla u) \right) + D_s \mathcal{L}(x, u, \nabla u) = 0 \quad \text{in} \quad \Omega
\]

has no weak solution \( u \in C^1(\overline{\Omega}) \).

**Proof.** Let \( u \in C^1(\overline{\Omega}) \) be a weak solution of (6.20). By applying the divergence Theorem to identity of Proposition 6.2.1 choosing \( v(x) = x \) and \( k = 1 \), since \( u = 0 \) on \( \partial \Omega \), we get

\[
\int_{\partial \Omega} \left[ \mathcal{L}(x, 0, \nabla u) - \nabla_\xi \mathcal{L}(x, 0, \nabla u) \cdot \nabla u \right] (x \cdot \nu) d\mathcal{H}^{n-1} =
\]

\[
\int_{\Omega} \left\{ n \mathcal{L}(x, u, \nabla u) + x \cdot \nabla_x \mathcal{L}(x, u, \nabla u) - \lambda u D_s \mathcal{L}(x, u, \nabla u) - (\lambda + 1) \nabla u \cdot \nabla_\xi \mathcal{L}(x, u, \nabla u) \right\} dx.
\]

Taking into account that on \( \partial \Omega \) it is \( (x \cdot \nu) > 0 \), conditions (6.25) and (6.26) yield a contradiction. □

**Corollary 6.2.3.** Assume that there exists \( \lambda \in \mathbb{R} \) such that

\[
\sum_{i,j=1}^n \left( (n - 2\lambda - 2)a_{ij}(x, s) + x \cdot \nabla_x a_{ij}(x, s) - \lambda s D_s a_{ij}(x, s) \right) \xi_i \xi_j \geq 0
\]

for a.e. \( x \in \Omega \) and each \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \) and

\[
\lambda s g(x, s) - n G(x, s) - x \cdot \nabla_x G(x, s) > 0
\]

and for a.e. \( x \in \Omega \) and each \( s \in \mathbb{R} \backslash \{0\} \). Then the quasilinear problem

\[
- \sum_{i,j=1}^n D_j(a_{ij}(x, u) D_i u) + \frac{1}{2} \sum_{i,j=1}^n D_s a_{ij}(x, u) D_i u D_j u = g(x, u) \quad \text{in} \quad \Omega
\]

has no weak solution \( u \in C^1(\overline{\Omega}) \).
6.2. Applications to non-existence results

Proof. It comes straightforward from the previous result taking

\[ \mathcal{L}(x, s, \xi) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x, s) \xi_i \xi_j \]

for a.e. \( x \in \Omega \) and each \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\).

We now come to the main non-existence result for first order vector-valued extremals.

**Theorem 6.2.4.** Assume that \( \Omega \) is star-shaped with respect to 0. Suppose also that

\[
\xi \cdot \nabla \xi \mathcal{L}(x, 0, \xi) - \mathcal{L}(x, 0, \xi) \geq 0,
\]

for a.e. \( x \in \Omega \) and each \( \xi \in \mathbb{R}^{nN} \) and that there exists \( \lambda \in \mathbb{R} \) such that

\[
n \mathcal{L}(x, s, \xi) + x \cdot \nabla_x \mathcal{L}(x, s, \xi) - \lambda u \cdot \nabla_s \mathcal{L}(x, s, \xi) - (\lambda + 1) \xi \cdot \nabla \xi \mathcal{L}(x, s, \xi) \geq 0
\]

for a.e. \( x \in \Omega \) and each \((s, \xi) \in \mathbb{R}^n \times \mathbb{R}^{nN} \). Assume further that equality holds only when either \( s = 0 \) or \( \xi = 0 \). Then the nonlinear elliptic system

\[
\text{div} (\nabla \xi \mathcal{L}(x, u, \nabla u)) + \nabla_x \mathcal{L}(x, u, \nabla u) = 0
\]

has no weak solution \( u \in C^2(\Omega, \mathbb{R}^N) \cap C^1(\overline{\Omega}, \mathbb{R}^N) \).

Proof. Arguing as in the scalar case we obtain the following variational identity

\[
D_i \left\{ v_i \mathcal{L}(x, u, \nabla u) - \left( v_j D_j u^k + \lambda u^k \right) D_{\xi_i} \mathcal{L}(x, u, \nabla u) \right\} =
\]

\[
= D_i v_i \mathcal{L}(x, u, \nabla u) + v_i D_i \mathcal{L}(x, u, \nabla u) - \left( D_j u^k D_i v_j + u^k D_i \lambda \right) D_{\xi_i} \mathcal{L}(x, u, \nabla u) +
\]

\[
- \lambda \left( D_i u^k D_{\xi_i} \mathcal{L}(x, u, \nabla u) + u^k D_{\xi_i} \mathcal{L}(x, u, \nabla u) \right)
\]

where \( i, j \) are understood to be summed from 1 to \( n \) and \( k \) from 1 to \( N \). Therefore, it suffices to argue as in Theorem 6.2.2.

**Corollary 6.2.5.** Assume that there exists \( \lambda \in \mathbb{R} \) such that

\[
\sum_{i,j=1}^{n} \sum_{h,k=1}^{N} \left( (n - 2\lambda - 2) a_{ij}^{hk}(x, s) + x \cdot \nabla_x a_{ij}^{hk}(x, s) - \lambda s \cdot D_s a_{ij}^{hk}(x, s) \right) \xi_i \xi_j \geq 0
\]

for a.e. \( x \in \Omega \) and each \((s, \xi) \in \mathbb{R}^n \times \mathbb{R}^{nN} \) and

\[
\lambda s \cdot g(x, s) - nG(x, s) - x \cdot \nabla_x G(x, s) > 0.
\]

and for a.e. \( x \in \Omega \) and each \( s \in \mathbb{R}^N \setminus \{0\} \). Then the quasilinear system \((\ell = 1, \ldots, N)\)

\[
- \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_j \left( a_{ij}^{h\ell}(x, u) D_i u_h \right) + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} D_{s_h} a_{ij}^{h\ell}(x, u) D_i u_h D_j u_k = g_\ell(x, u) \quad \text{in} \quad \Omega
\]

has no weak solution \( u \in C^2(\Omega, \mathbb{R}^N) \cap C^1(\overline{\Omega}, \mathbb{R}^N) \).
Proof. It follows by Theorem (6.2.4) choosing
\[ \mathcal{L}(x, s, \xi) = \frac{1}{2} \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x, s) \xi_i^{h} \xi_j^{k} - G(x, s) \]
for a.e. \( x \in \Omega \) and each \((s, \xi) \in \mathbb{R}^n \times \mathbb{R}^{nN} \).

Theorem 6.2.6. Let \( \Omega \) be star-shaped with respect to the origin and
\[ \nabla_x \mathcal{L}(x, s, \xi) \cdot x - \frac{n}{p^*} D_s \mathcal{L}(x, s, \xi) s + \left\{ \frac{n}{p^*} - \frac{n}{q} \right\} \lambda |s|^q \geq 0, \tag{6.29} \]
for a.e. \( x \in \Omega \) and all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n \). Then \((\mathcal{P}_{0,\lambda})\) has no nontrivial solution \( u \in C^1(\overline{\Omega}) \).

Proof. If we define \( \mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) by setting
\[ \forall (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n : \mathcal{F}(x, s, \xi) = \mathcal{L}(x, s, \xi) - \frac{\lambda}{q} |s|^q - \frac{1}{p^*} |s|^{p^*}, \]
the first assertion follows, after some computations, by the inequality
\[ n \mathcal{F} + \nabla_x \mathcal{F} \cdot x - aD_s \mathcal{F} s - (a + 1) \nabla_\xi \mathcal{F} : \xi \geq 0 \]
where we have chosen \( a = \frac{n-p}{p^*} \) (see [91, Theorem 1]). \( \square \)

Corollary 6.2.7. Let \( \Omega \) be star-shaped with respect to the origin, \( \lambda \leq 0 \) and
\[ p^* \nabla_x \mathcal{L}(x, s, \xi) \cdot x - nD_s \mathcal{L}(x, s, \xi) s \geq 0, \tag{6.30} \]
for a.e. \( x \in \Omega \) and all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n \). Then \((\mathcal{P}_{0,\lambda})\) admits no nontrivial solution \( u \in C^1(\overline{\Omega}) \).

Proof. Since \( q < p^* \) and \( \lambda \leq 0 \), condition (6.30) implies condition (6.29). \( \square \)

Remark 6.2.8. Assume that \( \lambda \leq 0 \) and \( \mathcal{L} \) does not depend on \( x \). Then, by the previous result, the non-existence condition becomes \( D_s \mathcal{L}(s, \xi) s \leq 0 \). Note that this is precisely the contrary of our assumption (5.41). Then, from this point of view (5.41) seems to be natural.
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