Axiomatizations for Probabilistic Bisimulation*

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Abstract. We study complete axiomatizations for different notions of probabilistic bisimulation on a recursion free process algebra with probability and nondeterminism under alternating and non-alternating semantics. The axioms that do not involve probability coincide with the original axioms of Milner. The axioms that involve probability differ depending on the bisimulation under examination and on the semantics that is used, thus revealing the implications of the different choices.

1 Introduction

Probabilistic process algebras have been studied extensively in the literature [1, 3, 4, 6, 8, 13, 16], and classical concepts from concurrency theory have been extended to the probabilistic case. Probabilistic models of concurrent systems are classified in [5] into *reactive*, *generative*, and *stratified*. Both in reactive and generative systems the transitions that leave from a state are equipped with probabilities: in generative systems the sum of the probabilities of the transitions that leave from a state is required to be 1, while in reactive systems the sums of the probabilities of the transitions that leave from a state and are labeled by the same action are required to be 1. The stratified model imposes some extra structure which is not relevant for the purpose of this paper.

Motivated by the fact that neither reactive nor generative nor stratified systems model real nondeterminism in the process algebraic sense, and motivated as well by the desire to separate clearly probability from nondeterminism, in [11] a model of *probabilistic automata* is introduced and studied. Probabilistic automata, and more precisely the simple probabilistic automata of [11], are like ordinary automata (labeled transition systems) except that a transition leads to a probability distribution over states rather than to a single state. Thus, the choice between different transitions is a nondeterministic choice, while the choice of a state within a transition is a probabilistic choice. A similar model was proposed in [6] based on the Concurrent Markov Chains of [15]. In such model, also known as the *alternating model*, there is a clear distinction between *nondeterministic states*, that enable only transitions leading to a unique state, and *probabilistic states*, that enable a unique transition leading to a distribution over states. There is a strict alternation between nondeterministic and probabilistic states. Both the alternating model and the model of [11], which in contraposition to the alternating model is also known as the *non-alternating model*, are

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conservative extensions of labeled transition systems and in several contexts can be seen as the same model from the point of view of expressiveness.

Yet, the alternating and non-alternating models do have some differences that can be seen already when we study bisimulation relations. Probabilistic bisimulation was first defined in [8], then extended to the alternating model in [6] and extended to the non-alternating model in [12]. While defining probabilistic bisimulation in the non-alternating model it was shown in [12] that we obtain two different relations if we simulate a transition using deterministic and randomized schedulers, respectively. Such difference does not appear in the alternating model unless we change the definitions of probabilistic bisimulations so that probabilistic states are not taken into account.

In this paper we show the differences and similarities of the alternating and non-alternating models by analyzing the axiomatizations of the different bisimulation relations in the different frameworks. We define a process algebra without recursion and provide it with an alternating and non-alternating semantics. We then define a *strong bisimulation* relation that coincides with the relation of [6] in the alternating model and with the bisimulation of [12] in the non-alternating model. We also define the version of strong bisimulation, called *strong probabilistic bisimulation*, where a transition can be simulated by using randomized schedulers. Finally, we study the complete axiomatizations of all the relations that we introduce. Besides obtaining axioms where probability and nondeterminism are separated clearly, thus confirming the original goal behind the definitions of the models, we discover that the axiomatizations of strong bisimulation are the same in the alternating and non-alternating models. Furthermore, the axiomatizations of strong bisimulation and strong probabilistic bisimulation are the same in the alternating model, while they differ by an axiom that expresses the ability to combine transitions probabilistically in the non-alternating model.

We also study the weak bisimulations of [12], showing that the alternating and non-alternating semantics are incomparable.

Other studies of axiomatizations for probabilistic bisimulation relations appear in [2, 6, 7, 14]. Of these axiomatizations, only [6] deals with a reactive model. The axiomatization of [6] includes recursion as well.

The rest of the paper is structured as follows. Section 2 gives some preliminary definitions and notational conventions; Section 3 defines the Probabilistic Process Algebra (PPA) and its alternating and non-alternating semantics; Section 4 defines the bisimulation relations that we axiomatize; Section 5 axiomatizes the relations of Section 4, discusses the axioms, and outlines the main ideas behind the proofs of completeness; Section 6 contains some concluding remarks.

2 Preliminaries

A *discrete probability measure* over a set X is a function $\mu : 2^X \to [0,1]$ such that $\mu(X) = 1$ and for each countable family $\{X_i\}$ of pairwise disjoint elements of 2^X, $\mu(\cup_i X_i) = \sum_i \mu(X_i)$. Denote by $Disc(X)$ the set of discrete probability measures over X. Given an element x of X we denote by $\delta(x)$ the probability measure μ such that $\mu(\{x\}) = 1$, and we call it the *Dirac measure* on x. Given two measures μ_1, μ_2 and a real number $p \in [0,1]$ we define the *convex combination* $p\mu_1 + (1-p)\mu_2$ of μ_1 and μ_2 to be the probability measure μ such that, for each set Y, $\mu(Y) = p\mu_1(Y) + (1-p)\mu_2(Y)$.

A *probabilistic automaton* is a tuple $(Q, \bar{q}, \Sigma, \mathcal{D})$, where Q is a set of *states*, $\bar{q} \in Q$ is a *start state*, Σ is a set of *actions*, and $\mathcal{D} \subseteq Q \times \Sigma \times Disc(Q)$ is a *transition relation*. An ordinary automaton can be seen as a probabilistic automaton where each transition leads to a Dirac measure. Probabilistic automata are used as the basis to give an operational semantics to our probabilistic process algebra.

3 Probabilistic Process Algebra

We denote by $\mathcal{A}$ the set of observable *actions* or *labels*, and let $Act = \mathcal{L} \cup \{\tau\}$ be the full set of actions. We call τ the *silent action* and we let α range over Act.

Let *NProc* denote the set of *nondeterministic processes*, ranged over by E, and *PProc* denote the set of *probabilistic processes*, ranged over by P. Finally, let $Proc \triangleq NProc \cup PProc$ denote the set of *processes*, ranged over by Q. The syntax for our Probabilistic Process Algebra is given by the following rules:

$$E ::= \mathsf{0} \mid E + E \mid \alpha.P$$
$$P ::= \Delta(E) | P \oplus_p P$$

The expression $\mathsf{0}$ is the inactive process having no transitions. The $+$ operator is the classical nondeterministic sum as defined in [9]. Process $\alpha.P$ performs action α and then offers a probabilistic choice described by the probabilistic process P. A probabilistic process is either a Dirac distribution over a single nondeterministic process, described by $\Delta(E)$, or a combination of the distributions associated with two probabilistic processes, described by the $\oplus_p$ operator.

For notational convenience we can represent sums of nondeterministic processes by $\sum_{i \in I} E_i$ and sums of probabilistic processes by $\boldsymbol{\sum}_{i \in I}[p_i]E_i$. Such representations are justified by the fact that in this paper both the operators $+$ and $\oplus_p$ turn out to be associative and commutative. We let μ range over distributions over nondeterministic processes and sometimes we represent a distribution over nondeterministic processes by $\{[p_i]E_i\}_{i \in I}$.

Note that PPA is characterized by a strict alternation between probabilistic and nondeterministic processes as in [6]. The alternation is kept in the alternating semantics of the calculus and is removed in the non-alternating semantics.

Table 1 contains the operational semantics of PPA, where $E \xrightarrow{\alpha} \mu$ describes a transition labeled by α that leaves from E and leads to a probability distribution μ, while $P \longmapsto \mu$ states that the probability distribution associated with P is μ. The rules of Table 1 describe the transitions of a probabilistic automaton; thus, the target of a transition of Table 1 is a probability distribution over expressions rather than a single expression.

Table 1 is subdivided into three sections. The first section defines the probability distributions associated with a probabilistic process. Specifically, process $\Delta(E)$ is associated with a Dirac distribution over the single process E (rule **idle**),

Probabilistic rules

$$\textbf{idle}\quad \frac{-}{\Delta(E) \longmapsto \delta(E)} \qquad \textbf{pchoice}\quad \frac{P_1 \longmapsto \mu_1 \qquad P_2 \longmapsto \mu_2}{P_1 \oplus_p P_2 \longmapsto p\mu_1 + (1-p)\mu_2}$$

Common nondeterministic rules

$$\textbf{lchoice}\quad \frac{E_1 \xrightarrow{\alpha} \mu}{E_1 + E_2 \xrightarrow{\alpha} \mu} \qquad \textbf{rchoice}\quad \frac{E_2 \xrightarrow{\alpha} \mu}{E_1 + E_2 \xrightarrow{\alpha} \mu}$$

$$\textbf{P}-\textbf{idle}\quad \frac{P \longmapsto \mu}{P \xrightarrow{\tau} \mu}$$

Rule for non alternating model | **Rule for alternating model**

$$\textbf{NA}-\textbf{prefix}\quad \frac{P \longmapsto \mu}{\alpha.P \xrightarrow{\alpha} \mu} \qquad \textbf{A}-\textbf{prefix}\quad \frac{-}{\alpha.P \xrightarrow{\alpha} \delta(P)}$$

Table 1. Operational semantics of PPA

while the probability distribution associated with the probabilistic combination $P_1 \oplus_p P_2$ is obtained by convex combination weighted by p of the distributions associated with P_1 and P_2, respectively (rule **pchoice**). The second section of Table 1 describes the operators whose semantics does not change in the alternating and non-alternating interpretations. Specifically, the semantics of the $+$ operator is the same as in CCS (rules **lchoice** and **rchoice**). Rule **P-idle** describes the unique transition that is enabled from a probabilistic process, which moves silently to the distribution associated with the process. This rule is essential in the alternating semantics, where probabilistic processes can be reached; however, the same rule is convenient also in the non-alternating semantics to obtain an axiomatization of probabilistic bisimulation that reveals better the relationship between the two semantics. The third section of Table 1 contains the rules for action-prefixing, which constitute the key difference between the alternating and non-alternating semantics. In the non-alternating semantics process $\alpha.P$ moves with action α to the distribution identified by P (rule **NA-prefix**), while in the alternating semantics process $\alpha.P$ moves with action α to process P (rule **A-prefix**) from which a silent move leads to the distribution identified by P (cf. rule **P-idle**).

Remark 1. There is a folklore idea of how an alternating system can be translated into a non-alternating system and vice versa. Specifically, to move from an alternating system to a non-alternating system it is sufficient to remove all the probabilistic states and collapse the transitions that go through a probabilistic state, while to move from a non-alternating system to an alternating system it is sufficient to split each transition into two transitions, the first of which leads to a probabilistic state. The operational semantics of Table 1 respects the folklore transformation: for each process E the transformation of its alternating semantics coincides with its non-alternating semantics and vice versa.

4 Bisimulation

In this section we define bisimulation relations in the strong and weak version based on deterministic and randomized schedulers. In the non-alternating model our definition of strong and weak (probabilistic) bisimulation coincide with those of [12]; in the alternating model strong bisimulation coincides with the strong bisimulation of [6], while weak probabilistic bisimulation coincides with the weak bisimulation of [10].

4.1 Lifting Equivalence Relations

An equivalence relation over *Proc* can be lifted to a relation over distributions over *Proc* by stating that two distributions are equivalent if they assign the same probability to the same equivalence classes [8].

Formally, let $\mathcal{R}$ be an equivalence relation over *Proc*. Two probability distributions μ_1 and μ_2 are $\mathcal{R}$-equivalent, written $\mu \ \mathcal{R}_p \ \mu'$, iff for every equivalence class $\mathcal{E} \in Proc/\ \mathcal{R}$ we have $\mu(\mathcal{E}) = \mu'(\mathcal{E})$.

4.2 Strong Bisimulation

An equivalence relation $\mathcal{R} \subseteq Proc \times Proc$ is a *strong bisimulation* iff, for all $Q_1, Q_2 \in Proc$ such that $Q_1 \ \mathcal{R} \ Q_2$, and for all $\alpha \in Act$,

- if $Q_1 \xrightarrow{\alpha} \mu_1$, then there exists μ_2 such that $Q_2 \xrightarrow{\alpha} \mu_2$ and $\mu_1 \ \mathcal{R}_p \ \mu_2$;
- if $Q_2 \xrightarrow{\alpha} \mu_2$, then there exists μ_1 such that $Q_1 \xrightarrow{\alpha} \mu_1$ and $\mu_1 \ \mathcal{R}_p \ \mu_2$.

We write $Q_1 \sim Q_2$ whenever there is a strong bisimulation that relates Q_1, Q_2.

Proposition 1. *Strong bisimulation is a congruence in PPA.*

In a strong bisimulation a transition of a process must be simulated by a single transition of the other process chosen deterministically among the transitions that are enabled. It was observed in [12] that deterministic schedulers may not be enough in a randomized setting.

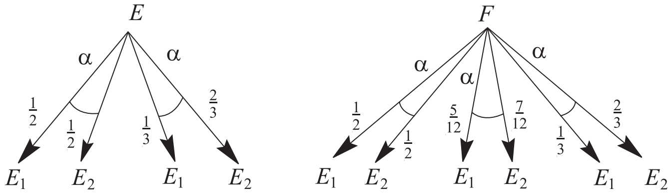

Fig. 1. Two processes not strongly bisimilar

Example 1. Consider $E \triangleq \alpha.(\Delta(E_1) \oplus_{1/2} \Delta(E_2)) + \alpha.(\Delta(E_1) \oplus_{1/3} \Delta(E_2))$ and $F \triangleq \alpha.(\Delta(E_1) \oplus_{1/2} \Delta(E_2)) + \alpha.(\Delta(E_1) \oplus_{5/12} \Delta(E_2)) + \alpha.(\Delta(E_1) \oplus_{1/3} \Delta(E_2))$ whose non-alternating semantics is represented in Figure 1. Each bundle of edges corresponds to a transition. The difference between E and F is that F enables an additional transition which is obtained by combining probabilistically the two transitions of E. There is no strong bisimulation between E and F if E_1 and E_2 are not bisimilar; however, E and F would be bisimilar if we permit the use of randomized schedulers to simulate the extra transition of F.

Example 1 suggests a new bisimulation relation where it is possible to combine several transitions labeled by the same action in a unique transition. We say that there is a *combined transition* labeled by action α from a process E to a distribution μ, denoted by $E \xrightarrow{\alpha}_C \mu$, iff there exists a collection $\{\mu_i, p_i\}_{i \in I}$ of distributions and probabilities such that $\sum p_i = 1$, $\mu = \sum p_i \mu_i$, and $\forall i : E \xrightarrow{\alpha} \mu_i$.

An equivalence relation $\mathcal{R} \subseteq Proc \times Proc$ is a *strong probabilistic bisimulation* iff, for all $Q_1, Q_2 \in Proc$ such that $Q_1 \mathcal{R} Q_2$, and for all $\alpha \in Act$,

- if $Q_1 \xrightarrow{\alpha} \mu_1$, then there exists μ_2 such that $Q_2 \xrightarrow{\alpha}_C \mu_2$ and $\mu_1 \mathcal{R}_p \mu_2$;
- if $Q_2 \xrightarrow{\alpha} \mu_2$, then there exists μ_1 such that $Q_1 \xrightarrow{\alpha}_C \mu_1$ and $\mu_1 \mathcal{R}_p \mu_2$.

We write $Q_1 \sim_C Q_2$ whenever there is a strong probabilistic bisimulation that relates Q_1 and Q_2.

Proposition 2. *Strong probabilistic bisimulation is a congruence in PPA.*

It is easy to observe that strong bisimulation is just a particular case of strong probabilistic bisimulation. An important result is that in the alternating semantics strong bisimulation coincides with strong probabilistic bisimulation (cf. Proposition 3). Thus, randomized schedulers do not add any extra power to the ability of simulating a transition. Roughly speaking, in the alternating model each probability distribution is declared explicitly through a probabilistic state before being drawn. Strong bisimulation must preserve the declarations as well, and on the other hand there is no way to declare the combination of two transitions.

Proposition 3. *Under the alternating semantics a strong probabilistic bisimulation is also a strong bisimulation.*

$$\frac{E \xrightarrow{\alpha} \mu}{E \overset{\alpha}{\Longrightarrow} \mu} \qquad \frac{E \xrightarrow{\tau} \mu}{E \Longrightarrow \mu} \qquad \frac{-}{E \Longrightarrow \delta(E)}$$

$$\frac{E \overset{\alpha}{\Longrightarrow} \mu \qquad \forall_{E_i \in \mu} E_i \Longrightarrow \mu_i}{E \overset{\alpha}{\Longrightarrow} \sum_{E_i \in \mu} \mu(E_i)\mu_i} \qquad \frac{E \Longrightarrow \mu \qquad \forall_{E_i \in \mu} E_i \overset{\alpha}{\Longrightarrow} \mu_i}{E \overset{\alpha}{\Longrightarrow} \sum_{E_i \in \mu} \mu(E_i)\mu_i}$$

Table 2. Weak transitions

Proof sketch. Let $\sim_C$ be a strong probabilistic bisimulation and suppose $Q_1 \sim_C Q_2$. If Q_1 and Q_2 are probabilistic processes, then they enable only one transition, the silent transition that selects probabilistically one process. Thus, there is nothing to combine. If Q_1 and Q_2 are nondeterministic processes and $Q_1 \xrightarrow{\alpha} \mu$, then μ is a Dirac distribution over some probabilistic process P. The combined transition $Q_2 \xrightarrow{\alpha}_C \mu'$ that simulates $Q_1 \xrightarrow{\alpha} \mu$ leads to a distribution that assigns probability 1 to the equivalence class of P. Thus, any transition from Q_2 that contributes to $Q_2 \xrightarrow{\alpha}_C \mu'$ leads to distribution that assigns probability 1 to the equivalence class of P. This shows that $Q_1 \sim Q_2$.

4.3 Weak Bisimulation

Weak bisimulation is the same as strong bisimulation except that we replace transitions by weak transitions. That is, we are not interested in observing the silent behavior of a system. A weak transition, whose formal definition is given in Table 2, is a probabilistic extension of the weak transitions of [9]. We schedule several transitions as long as they always lead to the occurrence of a single external action α, possibly interleaved by silent actions. For notational convenience, given a sequence s of actions in *Act*, we denote by $\widehat{s}$ the sequence obtained from s by removing all τ's.

An equivalence relation $\mathcal{R} \subseteq Proc \times Proc$ is a *weak bisimulation* iff, for all $Q_1, Q_2 \in Proc$ such that $Q_1 \; \mathcal{R} \; Q_2$, and for all $\alpha \in Act$,

- if $Q_1 \xrightarrow{\alpha} \mu_1$ then there exists μ_2 such that $Q_2 \overset{\widehat{\alpha}}{\Longrightarrow} \mu_2$ and $\mu_1 \; \mathcal{R}_p \; \mu_2$;
- if $Q_2 \xrightarrow{\alpha} \mu_2$ then there exists μ_1 such that $Q_1 \overset{\widehat{\alpha}}{\Longrightarrow} \mu_1$ and $\mu_1 \; \mathcal{R}_p \; \mu_2$.

We write $Q_1 \approx Q_2$ whenever there is a weak bisimulation that relates Q_1 and Q_2.

We can define a *weak combined transition relation* ($\Longrightarrow_C$), as we have done in the strong case, by combining simple weak transitions. Thus, it is possible to define *weak probabilistic bisimulation* by replacing weak transitions by weak combined transitions in the definition above.

4.4 Observation Congruence

As in ordinary CCS [9], weak bisimulation is not preserved by the nondeterministic choice operator $+$. The classical example is given by the pair of processes $a.0 \approx \tau.a.0$, which are equivalent both according to weak bisimulation and weak probabilistic bisimulation, where $a.0 + b.0 \not\approx \tau.a.0 + b.0$. Following the classical approach of [9], we define *observation congruence* and *probabilistic observation congruence*.

Two processes Q_1, Q_2 are *congruent*, written $Q_1 = Q_2$, if Q_1 and Q_2 are both nondeterministic or both probabilistic, and for all $\alpha \in Act$,

- if $Q_1 \xrightarrow{\alpha} \mu_1$ then there exists μ_2 such that $Q_2 \stackrel{\alpha}{\Longrightarrow} \mu_2$ and $\mu_1 \approx \mu_2$
- if $Q_2 \xrightarrow{\alpha} \mu_2$ then there exists μ_1 such that $Q_1 \stackrel{\alpha}{\Longrightarrow} \mu_1$ and $\mu_1 \approx \mu_2$

Two processes Q_1, Q_2 are *probabilistically congruent*, written $Q_1 =_C Q_2$, if Q_1 and Q_2 are both nondeterministic or both probabilistic, and for all $\alpha \in Act$,

- if $Q_1 \xrightarrow{\alpha} \mu_1$ then there exists μ_2 such that $Q_2 \stackrel{\alpha}{\Longrightarrow} \mu_2$ and $\mu_1 \approx_C \mu_2$
- if $Q_2 \xrightarrow{\alpha} \mu_2$ then there exists μ_1 such that $Q_1 \stackrel{\alpha}{\Longrightarrow} \mu_1$ and $\mu_1 \approx_C \mu_2$

The only difference between congruence and weak bisimulation is that in the former there is $\stackrel{\alpha}{\Longrightarrow}$ instead of $\stackrel{\widehat{\alpha}}{\Longrightarrow}$. This implies that every τ-transition of Q_1 is related with *at least* one τ-transition of Q_2, and vice versa. Observe that this strong relationship is requested only for the first transitions of both Q_1 and Q_2: in fact, it is sufficient that $\mu_1 \approx_p \mu_2$, not $\mu_1 =_p \mu_2$.

Proposition 4. *The relations $=$ and $=_C$ are congruences in PPA.*

5 Axiomatizations

5.1 Discussion of the Axioms

The axioms that characterize completely the bisimulation relations of this paper are listed in Table 3. The left side of Table 3 contains the axioms for the non-alternating semantics of PPA, while the right part contains the axioms for the alternating semantics of PPA. Table 3 is also subdivided into four horizontal sections. The first and third sections axiomatize strong bisimulation. By adding the second section we obtain complete axiomatizations for observation congruence. Finally, by adding the fourth section we obtain complete axiomatizations for the probabilistic versions of our bisimulations, where axiom **CW** holds only for the weak relations. Thus, sections 1, 3 and 4 provide complete axiomatizations for the strong probabilistic bisimulations.

Observe that there is no **C** axiom in the right column of Table 3, which confirms that strong bisimulation is the same under randomized and non-randomized schedulers in the alternating semantics. Furthermore, there is no **CW** axiom in the left column of Table 3, which shows that randomization adds some restricted power to the ability of simulating a weak transition in the alternating model.

Axiom **CW** does not hold in the non-alternating semantics since the term $\tau.P$ reached after the α-labeled transition of $\alpha.(P + \Delta(\tau.P))$ cannot be simulated in general by the distribution identified by P in $\alpha.P$. See also the discussion about axiom **A8**.

Observe that the first and third sections of Table 3 contain the same axioms in the two columns. This confirms that under strong bisimulation with deterministic schedulers the alternating and non-alternating models are indeed the same. We can observe a difference between alternating and non-alternating semantics in the second section of Table 3. Specifically, axioms **A6-7** of the right column are more restrictive than the axioms of the left column (P_i replaced by P). On the other hand, axiom **A8** holds only in the alternating semantics, thus showing that weak bisimulations are incomparable. Axiom **A8** expresses the informal idea that in the alternating model each distribution must be declared before being drawn. Thus, adding further declarations does not matter. The left version of axiom **A5** can be replaced by its right version. We have kept both versions to illustrate better the analogies with the τ-laws of Milner.

Another important observation is that the axiomatizations of Table 3 keep most of the structure of the axiomatizations for ordinary CCS [9]. The axioms of the first section are exactly the axioms for strong bisimulation on CCS, and the axioms of the third sections add the ingredients that are need for the new probabilistic choice operator. The τ-laws of the second section have the same structure of the τ-laws of Milner, except that within a prefix we have the probabilistic choice operator. If we consider processes without the probabilistic choice operator, then our τ-laws coincide with the τ-laws of Milner.

5.2 Proof Sketches

The proofs of the completeness results are similar to the corresponding proofs for CCS [9]: a process is reduced to a normal form, possibly saturated, and then processes are compared almost syntactically piece by piece. In this section we give an overview of the normal forms that are needed in the proofs.

Definition 1. *A nondeterministic process E is in* normal form *(NF) if*

$$E \equiv \sum_{i \in I} \alpha_i . \sum_{j \in J_i} [p_j^i] E_j^i$$

where the processes E_j^i are in normal form as well.

Getting a process in normal form is almost immediate since it is sufficient to remove all exceeding 0's by using axiom **A4** and the congruence rules.

Definition 2. *A nondeterministic process E is in* strict normal form *(SNF) if*

$$E \equiv \sum_{i \in I} \alpha_i . \sum_{j \in J_i} [p_j^i] E_j^i$$

where $\forall_i \forall_{j,j' \in J_i}$ if $\mathcal{S} \vdash E_j^i = E_{j'}^i$, then $j = j'$. With $\mathcal{S}$ we denote the axioms of the first and third sections of Table 3.

	Non alternating semantics	Alternating semantics
A1	$E + F = F + E$	$E + F = F + E$
A2	$E + (F + G) = (E + F) + G$	$E + (F + G) = (E + F) + G$
A3	$E + E = E$	$E + E = E$
A4	$E + 0 = E$	$E + 0 = E$
A5	$\alpha.(\Delta(\tau.\Delta(E)) \oplus_p P) = \alpha.(\Delta(E) \oplus_p P)$	$\Delta(\tau.\Delta(E)) = \Delta(E)$
A6	$\tau.\sum_{i \in I} [p_i](E_i + \alpha.P_i) + \alpha.\sum_{i \in I} [p_i]P_i = \tau.\sum_{i \in I} [p_i](E_i + \alpha.P_i)$	$\tau.\sum_{i \in I} [p_i](E_i + \alpha.P) + \alpha.P = \tau.\sum_{i \in I} [p_i](E_i + \alpha.P)$
A7	$\alpha.\sum_{i \in I} [p_i](E_i + \tau.P_i) + \alpha.\sum_{i \in I} [p_i]P_i = \alpha.\sum_{i \in I} [p_i](E_i + \tau.P_i)$	$\alpha.\sum_{i \in I} [p_i](E_i + \tau.P) + \alpha.P = \alpha.\sum_{i \in I} [p_i](E_i + \tau.P)$
A8	-	$\alpha.P = \alpha.\Delta(\tau.P)$
P1	$P \oplus_p Q = Q \oplus_{(1-p)} P$	$P \oplus_p Q = Q \oplus_{(1-p)} P$
P2	$P \oplus_{p_1} (Q \oplus_{\frac{p_2}{1-p_1}} R) = (P \oplus_{\frac{p_1}{p_1+p_2}} Q) \oplus_{(p_1+p_2)} R$	$P \oplus_{p_1} (Q \oplus_{\frac{p_2}{1-p_1}} R) = (P \oplus_{\frac{p_1}{p_1+p_2}} Q) \oplus_{(p_1+p_2)} R$
P3	$P \oplus_p P = P$	$P \oplus_p P = P$
C	$\alpha.P_1 + \alpha.P_2 = \alpha.P_1 + \alpha.P_2 + \alpha.(P_1 \oplus_p P_2)$	-
CW	-	$\alpha.(P \oplus \Delta(\tau.P)) = \alpha.P$

Table 3. Axioms for strong and weak bisimulations

To get a process in strict normal form we first convert the process to normal form. Then, whenever we find two elements E^i_j and $E^i_{j'}$ that are provably equivalent, we use axiom **P3** to collapse them. Of course we need also axioms **P1** and **P2** to get the two terms next to each other.

Processes in strict normal form are sufficient for the proof of completeness for strong bisimulation that works prefix by prefix as in [9]. To handle strong probabilistic bisimulation we use axiom **C** to build the missing summands that originate from convex combinations of other summands. Thus, we reduce strong probabilistic bisimulation to strong bisimulation.

To deal with weak bisimulation we need to saturate a process as in [9]. For this purpose we define complete normal forms.

Definition 3. *A nondeterministic process E is in* complete normal form *(CNF) if*

- $E \equiv \sum_{i \in I} \alpha_i . \sum_{j \in J_i} [p^i_j] E^i_j$
- *the processes E^i_j are in CNF*
- *if $E \stackrel{\alpha}{\Longrightarrow} \mu$, then $E \stackrel{\alpha}{\longrightarrow} \mu$.*

The saturation process to get an expression in complete normal form consists of using axiom **A6** to move out of a τ-prefix each transition labeled by some external action. The final step is to get a strict complete normal form in the same way as we do for strong bisimulation. When axiomatizing weak probabilistic bisimulation, once again we use axiom **C** to create the missing summands.

The normal form for weak bisimulation in the alternating semantics differs from the normal form in the non-alternating semantics in that axiom **A6** allows us to saturate only those transitions that lead to Dirac distributions.

Definition 4. *A nondeterministic process E is in* alternating complete normal form *(ACNF) if*

- $E \equiv \sum_{i \in I} \alpha_i . \sum_{j \in J_i} [p^i_j] E^i_j$
- *the processes E^i_j are in ACNF*
- *if $E \stackrel{\alpha}{\Longrightarrow} \delta(P)$, then $E \stackrel{\alpha}{\longrightarrow} \delta(P)$.*

6 Concluding Remarks

We have studied axiomatizations of bisimulation relations for a recursion free fragment of a probabilistic process algebra that includes probabilistic and nondeterministic choices. Our analysis included strong and weak bisimulation, deterministic and randomized schedulers, alternating and non-alternating semantics.

The axioms have a structure consistent with the original axioms of Milner and separate clearly the concerns of nondeterminism and probability. The axiomatizations that we have found also highlight the main differences and similarities of the alternating and non-alternating models of concurrent probabilistic systems.

We are currently planning to extend our axiomatizations to a probabilistic process algebra with recursion and parallel composition. We do not expect any special surprises with parallel composition since a probabilistic generalization of the expansion law of Milner is easy to derive.

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