Those who cannot remember the past are doomed to repeat it.

— George Santayana, *The Life of Reason, Book I: Introduction and Reason in Common Sense* (1905)

The 1950s were not good years for mathematical research. We had a very interesting gentleman in Washington named Wilson. He was secretary of Defense, and he actually had a pathological fear and hatred of the word 'research'*. I'm not using the term lightly; I'm using it precisely. His face would suffuse, he would turn red, and he would get violent if people used the term* 'research' *in his presence. You can imagine how he felt, then, about the term* 'mathematical'*. The RAND Corporation was employed by the Air Force, and the Air Force had Wilson as its boss, essentially. Hence, I felt I had to do something to shield Wilson and the Air Force from the fact that I was really doing mathematics inside the RAND Corporation. What title, what name, could I choose?*

— Richard Bellman, on the origin of his term 'dynamic programming' (1984)

If we all listened to the professor, we may be all looking for professor jobs.

— Pittsburgh Steelers' head coach Bill Cowher, responding to David Romer's dynamic-programming analysis of football strategy (2003)

5 Dynamic Programming

5.1 Fibonacci Numbers

5.1.1 Recursive Definitions Are Recursive Algorithms

The Fibonacci numbers F_n, named after Leonardo Fibonacci Pisano[1], the mathematician who popularized 'algorism' in Europe in the 13th century, are defined as follows: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. The recursive definition of Fibonacci numbers immediately gives us a recursive algorithm for computing them:

```
RecFibo(n):
  if (n < 2)
      return n
  else
      return RecFibo(n − 1) + RecFibo(n − 2)
```

How long does this algorithm take? Except for the recursive calls, the entire algorithm requires only a constant number of steps: one comparison and possibly one addition. If $T(n)$ represents the number of recursive calls to RecFibo, we have the recurrence

$$T(0) = 1, \quad T(1) = 1, \quad T(n) = T(n-1) + T(n-2) + 1.$$

This looks an awful lot like the recurrence for Fibonacci numbers! The annihilator method gives us an asymptotic bound of $\Theta(\phi^n)$, where $\phi = (\sqrt{5}+1)/2 \approx 1.61803398875$, the so-called *golden ratio*, is the largest root of the polynomial $r^2 - r - 1$. But it's fairly easy to prove (hint, hint) the exact solution $\boldsymbol{T(n) = 2F_{n+1} - 1}$. In other words, computing F_n using this algorithm takes more than twice as many steps as just counting to F_n!

Another way to see this is that the RecFibo is building a big binary tree of additions, with nothing but zeros and ones at the leaves. Since the eventual output is F_n, our algorithm must call RecRibo(1) (which returns 1) exactly F_n times. A quick inductive argument implies that RecFibo(0) is called exactly F_{n-1} times. Thus, the recursion tree has $F_n + F_{n-1} = F_{n+1}$ leaves, and therefore, because it's a full binary tree, it must have $2F_{n+1} - 1$ nodes.

[1] literally, "Leonardo, son of Bonacci, of Pisa"

5.1.2 Memo(r)ization: Remember Everything

The obvious reason for the recursive algorithm's lack of speed is that it computes the same Fibonacci numbers over and over and over. A single call to RECURSIVEFIBO(n) results in one recursive call to RECURSIVEFIBO($n-1$), two recursive calls to RECURSIVEFIBO($n-2$), three recursive calls to RECURSIVEFIBO($n-3$), five recursive calls to RECURSIVEFIBO($n-4$), and in general, F_{k-1} recursive calls to RECURSIVEFIBO($n-k$), for any $0 \le k < n$. For each call, we're recomputing some Fibonacci number from scratch.

We can speed up our recursive algorithm considerably just by writing down the results of our recursive calls and looking them up again if we need them later. This process was dubbed *memoization* by Richard Michie in the late 1960s.[2]

```
MEMFIBO(n):
  if (n < 2)
      return n
  else
      if F[n] is undefined
          F[n] ← MEMFIBO(n − 1) + MEMFIBO(n − 2)
      return F[n]
```

Memoization clearly decreases the running time of the algorithm, but by how much? If we actually trace through the recursive calls made by MEMFIBO, we find that the array $F[\,]$ is filled from the bottom up: first $F[2]$, then $F[3]$, and so on, up to $F[n]$. This pattern can be verified by induction: Each entry $F[i]$ is filled only after its predecessor $F[i-1]$. If we ignore the time spent in recursive calls, it requires only constant time to evaluate the recurrence for each Fibonacci number F_i. But by design, the recurrence for F_i is evaluated only once! We conclude that MEMFIBO performs only $O(n)$ additions, an *exponential* improvement over the naïve recursive algorithm!

5.1.3 Dynamic Programming: Fill Deliberately

But once we see how the array $F[\,]$ is filled, we can replace the recursion with a simple loop that intentionally fills the array in order, instead of relying on the complicated recursion to do it for us 'accidentally'.

```
ITERFIBO(n):
  F[0] ← 0
  F[1] ← 1
  for i ← 2 to n
      F[i] ← F[i − 1] + F[i − 2]
  return F[n]
```

Now the time analysis is immediate: ITERFIBO clearly uses ***O(n) additions*** and stores $O(n)$ *integers*.

This gives us our first explicit *dynamic programming* algorithm. The dynamic programming paradigm was developed by Richard Bellman in the mid-1950s, while working at the RAND Corporation. Bellman deliberately chose the name 'dynamic programming' to hide the mathematical character of his work from his military bosses, who were actively hostile toward anything resembling mathematical research. Here, the word 'programming' does not refer to writing code, but rather to the older sense of *planning* or *scheduling*, typically by filling in a table. For example, sports programs and theater programs are schedules of important events (with ads); television programming involves filling each available time slot with a show (and ads); degree programs are schedules of classes to be taken (with ads). The Air

[2] "My name is Elmer J. Fudd, millionaire. I own a mansion and a yacht."

Force funded Bellman and others to develop methods for constructing training and logistics schedules, or as they called them, 'programs'. The word 'dynamic' is meant to suggest that the table is filled in over time, rather than all at once (as in 'linear programming', which we will see later in the semester).[3]

5.1.4 Don't Remember Everything After All

In many dynamic programming algorithms, it is not necessary to retain *all* intermediate results through the entire computation. For example, we can significantly reduce the space requirements of our algorithm IterFibo by maintaining only the two newest elements of the array:

```
IterFibo2(n):
  prev ← 1
  curr ← 0
  for i ← 1 to n
      next ← curr + prev
      prev ← curr
      curr ← next
  return curr
```

(This algorithm uses the non-standard but perfectly consistent base case $F_{-1} = 1$ so that IterFibo2(0) returns the correct value 0.)

5.1.5 Faster! Faster!

Even this algorithm can be improved further, using the following wonderful fact:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x+y \end{bmatrix}$$

In other words, multiplying a two-dimensional vector by the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ does exactly the same thing as one iteration of the inner loop of IterFibo2. This might lead us to believe that multiplying by the matrix n times is the same as iterating the loop n times:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix}.$$

A quick inductive argument proves this fact. So if we want the nth Fibonacci number, we just have to compute the nth power of the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. If we use repeated squaring, computing the nth power of something requires only $O(\log n)$ multiplications. In this case, that means $O(\log n)$ 2×2 matrix multiplications, each of which reduces to a constant number of integer multiplications and additions. Thus, we can compute F_n in only ***O*(log *n*) *integer arithmetic operations***.

This is an exponential speedup over the standard iterative algorithm, which was already an exponential speedup over our original recursive algorithm. Right?

5.1.6 Whoa! Not so fast!

Well, not exactly. Fibonacci numbers grow exponentially fast. The nth Fibonacci number is approximately $n \log_{10} \phi \approx n/5$ decimal digits long, or $n \log_2 \phi \approx 2n/3$ bits. So we can't possibly compute F_n in logarithmic time — we need $\Omega(n)$ time just to write down the answer!

[3] "I thought dynamic programming was a good name. It was something not even a Congressman could object to. So I used it as an umbrella for my activities."

The way out of this apparent paradox is to observe that ***we can't perform arbitrary-precision arithmetic in constant time***. Multiplying two n-digit numbers using fast Fourier transforms (described in a different lecture note) requires $O(n \log n \log\log n)$ time. Thus, the matrix-based algorithm's actual running time obeys the recurrence $T(n) = T(\lfloor n/2 \rfloor) + O(n \log n \log\log n)$, which solves to $T(n) = \boldsymbol{O(n \log n \log\log n)}$ using recursion trees.

Is this slower than our "linear-time" iterative algorithm? No! Addition isn't free, either. Adding two n-digit numbers takes $O(n)$ time, so the running time of the iterative algorithm is $\boldsymbol{O(n^2)}$. (Do you see why?) Our matrix algorithm really is faster than our iterative algorithm, but not exponentially faster.

In the original recursive algorithm, the extra cost of arbitrary-precision arithmetic is overwhelmed by the huge number of recursive calls. The correct recurrence is $T(n) = T(n-1) + T(n-2) + O(n)$, for which the annihilator method still implies the solution $T(n) = O(\phi^n)$.

5.2 Longest Increasing Subsequence

In a previous lecture, we developed a recursive algorithm to find the length of the longest increasing subsequence of a given sequence of numbers. Given an array $A[1..n]$, the length of the longest increasing subsequences is computed by the function call LISBIGGER$(-\infty, A[1..n])$, where LISBIGGER is the following recursive algorithm:

```
LISBIGGER(prev, A[1..n]):
  if n = 0
      return 0
  else
      max ← LISBIGGER(prev, A[2..n])
      if A[1] > prev
          L ← 1 + LISBIGGER(A[1], A[2..n])
          if L > max
              max ← L
      return max
```

We can simplify our notation slightly with two simple observations. First, the input variable *prev* is always either $-\infty$ or an element of the input array. Second, the second argument of LISBIGGER is always a *suffix* of the original input array. If we add a new sentinel value $A[0] = -\infty$ to the input array, we can identify any recursive subproblem with two array indices.

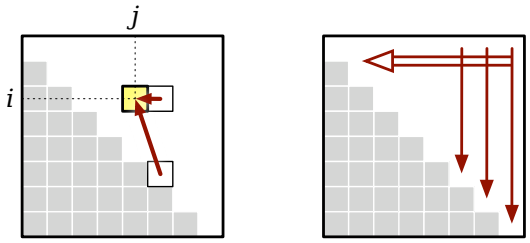
Thus, we can rewrite the recursive algorithm as follows. Add the sentinel value $A[0] = -\infty$. Let $LIS(i, j)$ denote the length of the longest increasing subsequence of $A[j..n]$ with all elements larger than $A[i]$. Our goal is to compute $LIS(0, 1)$. For all $i < j$, we have

$$LIS(i,j) = \begin{cases} 0 & \text{if } j > n \\ LIS(i, j+1) & \text{if } A[i] \geq A[j] \\ \max\{LIS(i, j+1),\ 1 + LIS(j, j+1)\} & \text{otherwise} \end{cases}$$

Because each recursive subproblem can be identified by two indices i and j, we can store the intermediate values in a two-dimensional array $\boldsymbol{LIS[0..n, 1..n]}$. Since there are $O(n^2)$ entries in the table, our memoized algorithm uses $\boldsymbol{O(n^2)}$ ***space***.[4] Each entry in the table can be computed in $O(1)$ time once we know its predecessors, so our memoized algorithm runs in $\boldsymbol{O(n^2)}$ ***time***.

[4] In fact, we only need half of this array, because we always have $i < j$. But even if we cared about constant factors in this class (we don't), this would be the wrong time to worry about them. The first order of business is to find an algorithm that actually *works*; once we have that, then we can think about optimizing it.

It's not immediately clear what order the recursive algorithm fills the rest of the table; all we can tell from the recurrence is that each entry $LIS[i, j]$ is filled in *after* the entries $LIS[i, j+1]$ and $LIS[j, j+1]$ in the next columns. But just this partial information is enough to give us an explicit evaluation order. If we fill in our table one column at a time, from right to left, then whenever we reach an entry in the table, the entries it depends on are already available.



Dependencies in the memoization table for longest increasing subsequence, and a legal evaluation order

Finally, putting everything together, we obtain the following dynamic programming algorithm:

```
LIS(A[1..n]):
  A[0] ← −∞                          ⟨⟨Add a sentinel⟩⟩
  for i ← 0 to n                     ⟨⟨Base cases⟩⟩
      LIS[i, n+1] ← 0

  for j ← n downto 1
      for i ← 0 to j − 1
          if A[i] ≥ A[j]
              LIS[i, j] ← LIS[i, j+1]
          else
              LIS[i, j] ← max{LIS[i, j+1], 1 + LIS[j, j+1]}

  return LIS[0, 1]
```

As expected, the algorithm clearly uses $\boldsymbol{O(n^2)}$ ***time and space***. However, we can reduce the space to $O(n)$ by only maintaining the two most recent columns of the table, $LIS[\cdot, j]$ and $LIS[\cdot, j+1]$.[5]

This is not the only recursive strategy we could use for computing longest increasing subsequences efficiently. Here is another recurrence that gives us the $O(n)$ space bound for free. Let $LIS'(i)$ denote the length of the longest increasing subsequence of $A[i..n]$ that starts with $A[i]$. Our goal is to compute $LIS'(0) - 1$; we subtract 1 to ignore the sentinel value $-\infty$. To define $LIS'(i)$ recursively, we only need to specify the *second* element in subsequence; the Recursion Fairy will do the rest.

$$LIS'(i) = 1 + \max\left\{LIS'(j) \mid j > i \text{ and } A[j] > A[i]\right\}$$

Here, I'm assuming that $\max\varnothing = 0$, so that the base case is $L'(n) = 1$ falls out of the recurrence automatically. Memoizing this recurrence requires only $\boldsymbol{O(n)}$ ***space***, and the resulting algorithm runs in $\boldsymbol{O(n^2)}$ ***time***. To transform this memoized recurrence into a dynamic programming algorithm, we only need to guarantee that $LIS'(j)$ is computed before $LIS'(i)$ whenever $i < j$.

[5] See, I told you not to worry about constant factors yet!

```
LIS2(A[1..n]):
  A[0] = -∞                          ⟨⟨Add a sentinel⟩⟩

  for i ← n downto 0
      LIS'[i] ← 1
      for j ← i + 1 to n
          if A[j] > A[i] and 1 + LIS'[j] > LIS'[i]
              LIS'[i] ← 1 + LIS'[j]

  return LIS'[0] - 1                 ⟨⟨Don't count the sentinel⟩⟩
```

5.3 The Pattern: Smart Recursion

In a nutshell, dynamic programming is *recursion without repetition*. Dynamic programming algorithms store the solutions of intermediate subproblems, often *but not always* in some kind of array or table. Many algorithms students make the mistake of focusing on the table (because tables are easy and familiar) instead of the *much* more important (and difficult) task of finding a correct recurrence. As long as we memoize the correct recurrence, an explicit table isn't really necessary, but if the recursion is incorrect, nothing works.

Dynamic programming is *not* about filling in tables.
It's about smart recursion!

Dynamic programming algorithms are almost always developed in two distinct stages.

1. **Formulate the problem recursively.** Write down a recursive formula or algorithm for the whole problem in terms of the answers to smaller subproblems. This is the hard part.

2. **Build solutions to your recurrence from the bottom up.** Write an algorithm that starts with the base cases of your recurrence and works its way up to the final solution, by considering intermediate subproblems in the correct order. This stage can be broken down into several smaller, relatively mechanical steps:

   (a) **Identify the subproblems.** What are all the different ways can your recursive algorithm call itself, starting with some initial input? For example, the argument to RecFibo is always an integer between 0 and n.

   (b) **Analyze space and running time.** The number of possible distinct subproblems determines the space complexity of your memoized algorithm. To compute the time complexity, add up the running times of all possible subproblems, *ignoring the recursive calls*. For example, if we already know F_{i-1} and F_{i-2}, we can compute F_i in $O(1)$ time, so computing the first n Fibonacci numbers takes $O(n)$ time.

   (c) **Choose a data structure to memoize intermediate results.** For most problems, each recursive subproblem can be identified by a few integers, so you can use a multidimensional array. For some problems, however, a more complicated data structure is required.

   (d) **Identify dependencies between subproblems.** Except for the base cases, every recursive subproblem depends on other subproblems—which ones? Draw a picture of your data structure, pick a generic element, and draw arrows from each of the other elements it depends on. Then formalize your picture.

   (e) **Find a good evaluation order.** Order the subproblems so that each subproblem comes *after* the subproblems it depends on. Typically, this means you should consider the base cases

first, then the subproblems that depends only on base cases, and so on. More formally, the dependencies you identified in the previous step define a partial order over the subproblems; in this step, you need to find a linear extension of that partial order. ***Be careful!***

(f) **Write down the algorithm.** You know what order to consider the subproblems, and you know how to solve each subproblem. So do that! If your data structure is an array, this usually means writing a few nested for-loops around your original recurrence. ***You don't need to do this on homework or exams.***

Of course, you have to prove that each of these steps is correct. If your recurrence is wrong, or if you try to build up answers in the wrong order, your algorithm won't work!

5.4 Warning: Greed is Stupid

If we're very very very very lucky, we can bypass all the recurrences and tables and so forth, and solve the problem using a *greedy* algorithm. The general greedy strategy is look for the best first step, take it, and then continue. While this approach seems very natural, it almost never works; optimization problems that can be solved correctly by a greedy algorithm are *very* rare. Nevertheless, for many problems that should be solved by dynamic programming, many students' first intuition is to apply a greedy strategy.

For example, a greedy algorithm for the edit distance problem might look for the longest common substring of the two strings, match up those substrings (since those substitutions don't cost anything), and then recursively look for the edit distances between the left halves and right halves of the strings. If there is no common substring—that is, if the two strings have no characters in common—the edit distance is clearly the length of the larger string. If this sounds like a stupid hack to you, pat yourself on the back. It isn't even *close* to the correct solution.

Everyone should tattoo the following sentence on the back of their hands, right under all the rules about logarithms and big-Oh notation:

Greedy algorithms never work!
Use dynamic programming instead!

What, never?
No, never!
What, *never*?
Well. . . hardly ever.[6]

A different lecture note describes the effort required to prove that greedy algorithms are correct, in the rare instances when they are. **You will not receive *any* credit for *any* greedy algorithm for *any* problem in this class without a *formal* proof of correctness.** We'll push through the formal proofs for several greedy algorithms later in the semester.

5.5 Edit Distance

The *edit distance* between two words—sometimes also called the *Levenshtein distance*—is the minimum number of letter insertions, letter deletions, and letter substitutions required to transform one word into another. For example, the edit distance between `FOOD` and `MONEY` is at most four:

[6] Greedy methods hardly ever work! So give three cheers, and one cheer more, for the hardy Captain of the *Pinafore*! Then give three cheers, and one cheer more, for the Captain of the *Pinafore*!

$$\underline{\texttt{F}}\texttt{OOD} \rightarrow \texttt{MO}\underline{\texttt{O}}\texttt{D} \rightarrow \texttt{MON}\underset{\wedge}{\underline{\texttt{D}}} \rightarrow \texttt{MONE}\underline{\texttt{D}} \rightarrow \texttt{MONEY}$$

A better way to display this editing process is to place the words one above the other, with a gap in the first word for every insertion, and a gap in the second word for every deletion. Columns with two *different* characters correspond to substitutions. Thus, the number of editing steps is just the number of columns that don't contain the same character twice.

```
F  O  O     D
M  O  N  E  Y
```

It's fairly obvious that you can't get from FOOD to MONEY in three steps, so their edit distance is exactly four. Unfortunately, this is not so easy in general. Here's a longer example, showing that the distance between ALGORITHM and ALTRUISTIC is at most six. Is this optimal?

```
A  L  G  O  R     I     T  H  M
A  L     T  R  U  I  S  T  I  C
```

To develop a dynamic programming algorithm to compute the edit distance between two strings, we first need to develop a recursive definition. Our gap representation for edit sequences has a crucial "optimal substructure" property. Suppose we have the gap representation for the shortest edit sequence for two strings. **If we remove the last column, the remaining columns must represent the shortest edit sequence for the remaining substrings.** We can easily prove this by contradiction. If the substrings had a shorter edit sequence, we could just glue the last column back on and get a shorter edit sequence for the original strings. Once we figure out what should go in the last column, the Recursion Fairy will magically give us the rest of the optimal gap representation.

So let's recursively define the edit distance between two strings $A[1..m]$ and $B[1..n]$, which we denote by $Edit(A[1..m], B[1..n])$. If neither string is empty, there are three possibilities for the last column in the shortest edit sequence:

- **Insertion:** The last entry in the bottom row is empty. In this case, the edit distance is equal to $Edit(A[1..m-1], B[1..n]) + 1$. The $+1$ is the cost of the final insertion, and the recursive expression gives the minimum cost for the other columns.
- **Deletion:** The last entry in the top row is empty. In this case, the edit distance is equal to $Edit(A[1..m], B[1..n-1]) + 1$. The $+1$ is the cost of the final deletion, and the recursive expression gives the minimum cost for the other columns.
- **Substitution:** Both rows have characters in the last column. If the characters are the same, the substitution is free, so the edit distance is equal to $Edit(A[1..m-1], B[1..n-1])$. If the characters are different, then the edit distance is equal to $Edit(A[1..m-1], B[1..n-1]) + 1$.

The edit distance between A and B is the smallest of these three possibilities:[7]

$$Edit(A[1..m], B[1..n]) = \min \left\{ \begin{array}{l} Edit(A[1..m-1], B[1..n]) + 1 \\ Edit(A[1..m], B[1..n-1]) + 1 \\ Edit(A[1..m-1], B[1..n-1]) + \big[A[m] \neq B[n]\big] \end{array} \right\}$$

[7] Once again, I'm using Iverson's bracket notation $[P]$ to denote the *indicator variable* for the logical proposition P, which has value 1 if P is true and 0 if P is false.

This recurrence has two easy base cases. The only way to convert the empty string into a string of n characters is by performing n insertions. Similarly, the only way to convert a string of m characters into the empty string is with m deletions, Thus, if ε denotes the empty string, we have

$$Edit(A[1..m], \varepsilon) = m, \qquad Edit(\varepsilon, B[1..n]) = n.$$

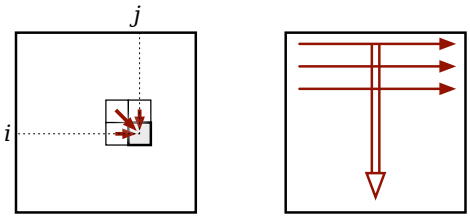
Both of these expressions imply the trivial base case $Edit(\varepsilon, \varepsilon) = 0$.

Now notice that the arguments to our recursive subproblems are always *prefixes* of the original strings A and B. Thus, we can simplify our notation considerably by using the lengths of the prefixes, instead of the prefixes themselves, as the arguments to our recursive function. So let's write $Edit(i, j)$ as shorthand for $Edit(A[1..i], B[1..j])$. This function satisfies the following recurrence:

$$Edit(i,j) = \begin{cases} i & \text{if } j = 0 \\ j & \text{if } i = 0 \\ \min \left\{ \begin{array}{l} Edit(i-1, j) + 1, \\ Edit(i, j-1) + 1, \\ Edit(i-1, j-1) + \big[A[i] \neq B[j]\big] \end{array} \right. & \text{otherwise} \end{cases}$$

The edit distance between the original strings A and B is just $Edit(m, n)$. This recurrence translates directly into a recursive algorithm; the precise running time is not obvious, but it's clearly exponential in m and n. ***Fortunately, we don't care about the precise running time of the recursive algorithm.*** The recursive running time wouldn't tell us anything about our eventual dynamic programming algorithm, so we're just not going to bother computing it.[8]

Because each recursive subproblem can be identified by two indices i and j, we can memoize intermediate values in a two-dimensional array $\boldsymbol{Edit[0..m, 0..n]}$. Note that the index ranges start at zero to accommodate the base cases. Since there are $\Theta(mn)$ entries in the table, our memoized algorithm uses $\boldsymbol{\Theta(mn)}$ ***space***. Since each entry in the table can be computed in $\Theta(1)$ time once we know its predecessors, our memoized algorithm runs in $\boldsymbol{\Theta(mn)}$ ***time***.



Dependencies in the memoization table for edit distance, and a legal evaluation order

[8] In case you're curious, the running time of the unmemoized recursive algorithm obeys the following recurrence:

$$T(m,n) = \begin{cases} O(1) & \text{if } n = 0 \text{ or } m = 0, \\ T(m, n-1) + T(m-1, n) + T(n-1, m-1) + O(1) & \text{otherwise.} \end{cases}$$

I don't know of a general closed-form solution for this mess, but we can derive an upper bound by defining a new function

$$T'(N) = \max_{n+m=N} T(n,m) = \begin{cases} O(1) & \text{if } N = 0, \\ 2T(N-1) + T(N-2) + O(1) & \text{otherwise.} \end{cases}$$

The annihilator method implies that $T'(N) = O((1+\sqrt{2})^N)$. Thus, the running time of our recursive edit-distance algorithm is *at most* $T'(n+m) = O((1+\sqrt{2})^{n+m})$.

Each entry $Edit[i, j]$ depends only on its three neighboring entries $Edit[i-1, j]+1$, $Edit[i, j-1]+1$, and $Edit[i-1, j-1]$. If we fill in our table in the standard row-major order—row by row from top down, each row from left to right—then whenever we reach an entry in the table, the entries it depends on are already available. Putting everything together, we obtain the following dynamic programming algorithm:

```
EDITDISTANCE(A[1..m], B[1..n]):
  for j ← 1 to n
      Edit[0, j] ← j

  for i ← 1 to m
      Edit[i, 0] ← i
      for j ← 1 to n
          if A[i] = B[j]
              Edit[i, j] ← min{Edit[i − 1, j] + 1, Edit[i, j − 1] + 1, Edit[i − 1, j − 1]}
          else
              Edit[i, j] ← min{Edit[i − 1, j] + 1, Edit[i, j − 1] + 1, Edit[i − 1, j − 1] + 1}

  return Edit[m, n]
```

Here's the resulting table for ALGORITHM → ALTRUISTIC. Bold numbers indicate places where characters in the two strings are equal. The arrows represent the predecessor(s) that actually define each entry. Each direction of arrow corresponds to a different edit operation: horizontal=deletion, vertical=insertion, and diagonal=substitution. Bold diagonal arrows indicate "free" substitutions of a letter for itself. Any path of arrows from the top left corner to the bottom right corner of this table represents an optimal edit sequence between the two strings. (There can be many such paths.) Moreover, since we can compute these arrows in a post-processing phase from the values stored in the table, we can reconstruct the actual optimal editing sequence in $O(n+m)$ additional time.

		A	L	G	O	R	I	T	H	M
	0	→1	→2	→3	→4	→5	→6	→7	→8	→9
	↓	↘								
A	1	**0**	→1	→2	→3	→4	→5	→6	→7	→8
	↓	↓	↘							
L	2	1	**0**	→1	→2	→3	→4	→5	→6	→7
	↓	↓	↓	↘	↘	↘	↘	↘		
T	3	2	1	1	→2	→3	→4	**→4**	→5	→6
	↓	↓	↓	↓	↘	↘		↘	↘	
R	4	3	2	2	2	**2**	→3	→4	→5	→6
	↓	↓	↓	↘	↓	↘	↘	↘	↘	↘
U	5	4	3	3	3	3	3	→4	→5	→6
	↓	↓	↓	↘	↓	↘	↘	↘	↘	↘
I	6	5	4	4	4	4	**3**	→4	→5	→6
	↓	↓	↓	↘	↓	↘	↓	↘	↘	↘
S	7	6	5	5	5	5	4	4	5	6
	↓	↓	↓	↘	↓	↘	↓	↘	↘	↘
T	8	7	6	6	6	6	5	**4**	→5	→6
	↓	↓	↓	↘	↓	↘	↘	↓	↘	↘
I	9	8	7	7	7	7	**6**	5	5	→6
	↓	↓	↓	↘	↓	↘	↓	↓	↘	↘
C	10	9	8	8	8	8	7	6	6	6

The edit distance between ALGORITHM and ALTRUISTIC is indeed six. There are three paths through this table from the top left to the bottom right, so there are three optimal edit sequences:

```
 A  L  G  O  R  I     T  H  M
 A  L  T  R  U  I  S  T  I  C

A  L  G  O  R     I     T  H  M
A  L     T  R  U  I  S  T  I  C

A  L  G  O  R     I     T  H  M
A  L  T     R  U  I  S  T  I  C
```

5.6 Optimal Binary Search Trees

In an earlier lecture, we developed a recursive algorithm for the optimal binary search tree problem. We are given a sorted array $A[1..n]$ of search keys and an array $f[1..n]$ of frequency counts, where $f[i]$ is the number of searches to $A[i]$. Our task is to construct a binary search tree for that set such that the total cost of all the searches is as small as possible. We developed the following recurrence for this problem:

$$OptCost(f[1..n]) = \min_{1 \le r \le n} \left\{ OptCost(f[1..r-1]) \; + \; \sum_{i=1}^{n} f[i] \; + \; OptCost(f[r+1..n]) \right\}$$

To put this recurrence in more standard form, fix the frequency array f, and let $OptCost(i, j)$ denote the total search time in the optimal search tree for the subarray $A[i..j]$. To simplify notation a bit, let $F(i, j)$ denote the total frequency count for all the keys in the interval $A[i..j]$:

$$F(i, j) := \sum_{k=i}^{j} f[k]$$

We can now write

$$\boxed{OptCost(i, j) = \begin{cases} 0 & \text{if } j < i \\ F(i, j) + \min\limits_{i \le r \le j} \big(OptCost(i, r-1) + OptCost(r+1, j) \big) & \text{otherwise} \end{cases}}$$

The base case might look a little weird, but all it means is that the total cost for searching an empty set of keys is zero.

The algorithm will be somewhat simpler and more efficient if we precompute all possible values of $F(i, j)$ and store them in an array. Computing each value $F(i, j)$ using a separate for-loop would $O(n^3)$ time. A better approach is to turn the recurrence

$$F(i, j) = \begin{cases} f[i] & \text{if } i = j \\ F(i, j-1) + f[j] & \text{otherwise} \end{cases}$$

into the following $O(n^2)$-time dynamic programming algorithm:

```
INITF(f[1..n]):
  for i ← 1 to n
      F[i, i-1] ← 0
      for j ← i to n
          F[i, j] ← F[i, j-1] + f[i]
```

This will be used as an initialization subroutine in our final algorithm.

So now let's compute the optimal search tree cost $OptCost(1,n)$ from the bottom up. We can store all intermediate results in a table $OptCost[1..n, 0..n]$. Only the entries $OptCost[i,j]$ with $j \geq i-1$ will actually be used. The base case of the recurrence tells us that any entry of the form $OptCost[i,i-1]$ can immediately be set to 0. For any other entry $OptCost[i,j]$, we can use the following algorithm fragment, which comes directly from the recurrence:

```
COMPUTEOPTCOST(i, j):
  OptCost[i, j] ← ∞
  for r ← i to j
      tmp ← OptCost[i, r − 1] + OptCost[r + 1, j]
      if OptCost[i, j] > tmp
          OptCost[i, j] ← tmp
  OptCost[i, j] ← OptCost[i, j] + F[i, j]
```

The only question left is what order to fill in the table.

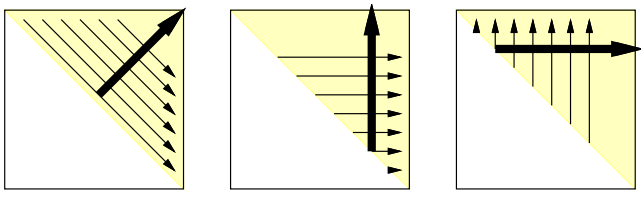
Each entry $OptCost[i,j]$ depends on all entries $OptCost[i,r-1]$ and $OptCost[r+1,j]$ with $i \leq k \leq j$. In other words, every entry in the table depends on all the entries directly to the left or directly below. In order to fill the table efficiently, we must choose an order that computes all those entries before $OptCost[i,j]$. There are at least three different orders that satisfy this constraint. The one that occurs to most people first is to scan through the table one diagonal at a time, starting with the trivial base cases $OptCost[i,i-1]$. The complete algorithm looks like this:

```
OPTIMALSEARCHTREE(f[1..n]):
  INITF(f[1..n])
  for i ← 1 to n
      OptCost[i, i − 1] ← 0
  for d ← 0 to n − 1
      for i ← 1 to n − d
          COMPUTEOPTCOST(i, i + d)
  return OptCost[1, n]
```

We could also traverse the array row by row from the bottom up, traversing each row from left to right, or column by column from left to right, traversing each columns from the bottom up. These two orders give us the following algorithms:

```
OPTIMALSEARCHTREE2(f[1..n]):
  INITF(f[1..n])
  for i ← n downto 1
      OptCost[i, i − 1] ← 0
      for j ← i to n
          COMPUTEOPTCOST(i, j)
  return OptCost[1, n]
```

```
OPTIMALSEARCHTREE3(f[1..n]):
  INITF(f[1..n])
  for j ← 0 to n
      OptCost[j + 1, j] ← 0
      for i ← j downto 1
          COMPUTEOPTCOST(i, j)
  return OptCost[1, n]
```



Three different evaluation orders for the table $OptCost[i,j]$.

No matter which of these orders we actually use, the resulting algorithm runs in $\Theta(n^3)$ ***time*** and uses $\Theta(n^2)$ ***space***.

We could have predicted these space and time bounds directly from the original recurrence.

$$OptCost(i,j) = \begin{cases} 0 & \text{if } j = i - i \\ F(i,j) + \min\limits_{i \le r \le j} \big(OptCost(i, r-1) + OptCost(r+1, j) \big) & \text{otherwise} \end{cases}$$

First, the function has two arguments, each of which can take on any value between 1 and n, so we probably need a table of size $O(n^2)$. Next, there are *three* variables in the recurrence (i, j, and r), each of which can take any value between 1 and n, so it should take us $O(n^3)$ time to fill the table.

5.7 Dynamic Programming on Trees

So far, all of our dynamic programming example use a multidimensional array to store the results of recursive subproblems. However, as the next example shows, this is not always the most appropriate date structure to use.

A ***independent set*** in a graph is a subset of the vertices that have no edges between them. Finding the largest independent set in an arbitrary graph is extremely hard; in fact, this is one of the canonical NP-hard problems described in another lecture note. But from some special cases of graphs, we can find the largest independent set efficiently. In particular, when the input graph is a tree (a connected and acyclic graph) with n vertices, we can compute the largest independent set in $O(n)$ time.

In the recursion notes, we saw a recursive algorithm for computing the size of the largest independent set in an arbitrary graph:

```
MaximumIndSetSize(G):
  if G = ∅
      return 0

  v ← any node in G
  withv ← 1 + MaximumIndSetSize(G \ N(v))
  withoutv ← MaximumIndSetSize(G \ {v})
  return max{withv, withoutv}.
```

Here, $N(v)$ denotes the *neighborhood* of v: the set containing v and all of its neighbors. As we observed in the other lecture notes, this algorithm has a worst-case running time of $O(2^n \text{ poly}(n))$, where n is the number of vertices in the input graph.

Now suppose we require that the input graph is a tree; we will call this tree T instead of G from now on. We need to make a slight change to the algorithm to make it truly recursive. The subgraphs $T \setminus \{v\}$ and $T \setminus N(v)$ are forests, which may have more than one component. But the largest independent set in a disconnected graph is just the union of the largest independent sets in its components, so we can separately consider each tree in these forests. Fortunately, this has the added benefit of making the recursive algorithm more efficient, especially if we can choose the node v such that the trees are all significantly smaller than T. Here is the modified algorithm:

```
MaximumIndSetSize(T):
  if T = ∅
      return 0
  v ← any node in T
  withv ← 1
  for each tree T' in T \ N(v)
      withv ← withv + MaximumIndSetSize(T')
  withoutv ← 0
  for each tree T' in T \ {v}
      withoutv ← withoutv + MaximumIndSetSize(T')
  return max{withv, withoutv}.
```

Now let's try to memoize this algorithm. Each recursive subproblem considers a subtree (that is, a connected subgraph) of the original tree T. Unfortunately, a single tree T can have exponentially many subtrees, so we seem to be doomed from the start!

Fortunately, there's a degree of freedom that we have not yet exploited: *We get to choose the vertex v.* We need a recipe—an algorithm!—for choosing v in each subproblem that limits the number of different subproblems the algorithm considers. To make this work, we impose some additional structure on the original input tree. Specifically, we declare one of the vertices of T to be the *root*, and we orient all the edges of T away from that root. Then we let v be the root of the input tree; this choice guarantees that each recursive subproblem considers a *rooted* subtree of T. Each vertex in T is the root of exactly one subtree, so now the number of distinct subproblems is exactly n. We can further simplify the algorithm by only passing a single node instead of the entire subtree:

```
MaximumIndSetSize(v):
  withv ← 1
  for each grandchild x of v
      withv ← withv + MaximumIndSetSize(x)
  withoutv ← 0
  for each child w of v
      withoutv ← withoutv + MaximumIndSetSize(w)
  return max{withv, withoutv}.
```

What data structure should we use to store intermediate results? The most natural choice is the tree itself! Specifically, for each node v, we store the result of MaximumIndSetSize(v) in a new field $v.MIS$. (We *could* use an array, but then we'd have to add a new field to each node anyway, pointing to the corresponding array entry. Why bother?)

What's the running time of the algorithm? The non-recursive time associated with each node v is proportional to the number of children and grandchildren of v; this number can be very different from one vertex to the next. But we can turn the analysis around: Each vertex contributes a constant amount of time to its parent and its grandparent! Since each vertex has at most one parent and at most one grandparent, the total running time is $O(n)$.

What's a good order to consider the subproblems? The subproblem associated with any node v depends on the subproblems associated with the children and grandchildren of v. So we can visit the nodes in any order, provided that all children are visited before their parent. In particular, we can use a straightforward post-order traversal.

Here is the resulting dynamic programming algorithm. Yes, it's still recursive. I've swapped the evaluation of the with-v and without-v cases; we need to visit the kids first anyway, so why not consider the subproblem that depends directly on the kids first?

```
MaximumIndSetSize(v):
  withoutv ← 0
  for each child w of v
      withoutv ← withoutv + MaximumIndSetSize(w)
  withv ← 1
  for each grandchild x of v
      withv ← withv + x.MIS
  v.MIS ← max{withv, withoutv}
  return v.MIS
```

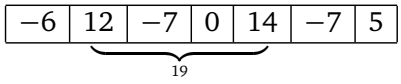
Another option is to store *two* values for each rooted subtree: the size of the largest independent set *that includes the root*, and the size of the largest independent set *that excludes the root*. This gives us an even simpler algorithm, with the same $O(n)$ running time.

```
MaximumIndSetSize(v):
  v.MISno ← 0
  v.MISyes ← 1
  for each child w of v
      v.MISno ← v.MISno + MaximumIndSetSize(w)
      v.MISyes ← v.MISyes + w.MISno
  return max{v.MISyes, v.MISno}
```

Exercises

1. Suppose you are given an array $A[1..n]$ of numbers, which may be positive, negative, or zero.

   (a) Describe and analyze an algorithm that finds the largest sum of of elements in a contiguous subarray $A[i..j]$. For example, if the array contains the numbers $(-6, 12, -7, 0, 14, -7, 5)$, then the largest sum of any contiguous subarray is $19 = 12 - 7 + 0 + 14$.

   (b) Describe and analyze an algorithm that finds the largest *product* of of elements in a contiguous subarray $A[i..j]$. You may *not* assume that all the numbers in the input array are integers.

2. This series of exercises asks you to develop efficient algorithms to find optimal *subsequences* of various kinds. A subsequence is anything obtained from a sequence by extracting a subset of elements, but keeping them in the same order; the elements of the subsequence need not be contiguous in the original sequence. For example, the strings `C`, `DAMN`, `YAIOAI`, and `DYNAMICPROGRAMMING` are all subsequences of the sequence `DYNAMICPROGRAMMING`.

   (a) Let $A[1..m]$ and $B[1..n]$ be two arbitrary arrays. A *common subsequence* of A and B is another sequence that is a subsequence of both A and B. Describe an efficient algorithm to compute the length of the *longest* common subsequence of A and B.

   (b) Let $A[1..m]$ and $B[1..n]$ be two arbitrary arrays. A *common supersequence* of A and B is another sequence that contains both A and B as subsequences. Describe an efficient algorithm to compute the length of the *shortest* common supersequence of A and B.

(c) Call a sequence $X[1..n]$ of numbers *oscillating* if $X[i] < X[i+1]$ for all even i, and $X[i] > X[i+1]$ for all odd i. Describe an efficient algorithm to compute the length of the longest oscillating subsequence of an arbitrary array A of integers.

(d) Describe an efficient algorithm to compute the length of the shortest oscillating supersequence of an arbitrary array A of integers.

(e) Call a sequence $X[1..n]$ of numbers *accelerating* if $2 \cdot X[i] < X[i-1] + X[i+1]$ for all i. Describe an efficient algorithm to compute the length of the longest accelerating subsequence of an arbitrary array A of integers.

*(f) Recall that a sequence $X[1..n]$ of numbers is *increasing* if $X[i] < X[i+1]$ for all i. Describe an efficient algorithm to compute the length of the *longest common increasing subsequence* of two given arrays of integers. For example, $\langle 1, 4, 5, 6, 7, 9\rangle$ is the longest common increasing subsequence of the sequences $\langle 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3\rangle$ and $\langle 1, 4, 1, 4, 2, 1, 3, 5, 6, 2, 3, 7, 3, 0, 9, 5\rangle$.

3. You and your eight-year-old nephew Elmo decide to play a simple card game. At the beginning of the game, the cards are dealt face up in a long row. Each card is worth a different number of points. After all the cards are dealt, you and Elmo take turns removing either the leftmost or rightmost card from the row, until all the cards are gone. At each turn, you can decide which of the two cards to take. The winner of the game is the player that has collected the most points when the game ends.

   Having never taken an algorithms class, Elmo follows the obvious greedy strategy—when it's his turn, Elmo *always* takes the card with the higher point value. Your task is to find a strategy that will beat Elmo whenever possible. (It might seem mean to beat up on a little kid like this, but Elmo absolutely *hates* it when grown-ups let him win.)

   (a) Prove that you should not also use the greedy strategy. That is, show that there is a game that you can win, but only if you do *not* follow the same greedy strategy as Elmo.

   (b) Describe and analyze an algorithm to determine, given the initial sequence of cards, the maximum number of points that you can collect playing against Elmo.

4. It's almost time to show off your flippin' sweet dancing skills! Tomorrow is the big dance contest you've been training for your entire life, except for that summer you spent with your uncle in Alaska hunting wolverines. You've obtained an advance copy of the the list of n songs that the judges will play during the contest, in chronological order.

   You know all the songs, all the judges, and your own dancing ability extremely well. For each integer k, you know that if you dance to the kth song on the schedule, you will be awarded exactly $Score[k]$ points, but then you will be physically unable to dance for the next $Wait[k]$ songs (that is, you cannot dance to songs $k+1$ through $k + Wait[k]$). The dancer with the highest total score at the end of the night wins the contest, so you want your total score to be as high as possible.

   Describe and analyze an efficient algorithm to compute the maximum total score you can achieve. The input to your sweet algorithm is the pair of arrays $Score[1..n]$ and $Wait[1..n]$.

5. A palindrome is any string that is exactly the same as its reversal, like `I`, or `DEED`, or `RACECAR`, or `AMANAPLANACATACANALPANAMA`.

(a) Describe and analyze an algorithm to find the length of the longest *subsequence* of a given string that is also a palindrome. For example, the longest palindrome subsequence of MAHDYNAMICPROGRAMZLETMESHOWYOUTHEM is MHYMRORMYHM, so given that string as input, your algorithm should output the number 11.

(b) Any string can be decomposed into a sequence of palindromes. For example, the string BUBBASEESABANANA ('Bubba sees a banana.') can be broken into palindromes in the following ways (and many others):

```
                BUB + BASEESAB + ANANA
        B + U + BB + A + SEES + ABA + NAN + A
        B + U + BB + A + SEES + A + B + ANANA
B + U + B + B + A + S + E + E + S + A + B + A + N + A + N + A
```

Describe and analyze an efficient algorithm to find the smallest number of palindromes that make up a given input string. For example, given the input string BUBBASEESABANANA, your algorithm would return the integer 3.

6. Suppose you have a subroutine QUALITY that can compute the 'quality' of any given string $A[1..k]$ in $O(k)$ time. For example, the quality of a string might be 1 if the string is a Québecois curse word, and 0 otherwise.

   Given an arbitrary input string $T[1..n]$, we would like to break it into contiguous substrings, such that the total quality of all the substrings is as large as possible. For example, the string SAINTCIBOIREDESACRAMENTDECRISSE can be decomposed into the substrings SAINT + CIBOIRE + DE + SACRAMENT + DE + CRISSE, of which three (or possibly four) are *sacres*.

   Describe an algorithm that breaks a string into substrings of maximum total quality, using the QUALITY subroutine.

7. Consider two horizontal lines ℓ_1 and ℓ_2 in the plane. Suppose we are given (the x-coordinates of) n distinct points $a_1, a_2, \ldots, a_n$ on ℓ_1 and n distinct points $b_1, b_2, \ldots, b_n$ on ℓ_2. (The points a_i and b_j are *not* necessarily indexed in order from left to right, or in the same order.) Design an algorithm to compute the largest set S of non-intersecting line segments satisfying to the following restrictions:

   (a) Each segment in S connects some point a_i to the corresponding point b_i.

   (b) No two segments in S intersect.

8. Suppose you are given an $m \times n$ bitmap, represented by an array $M[1..m, 1..n]$ whose entries are all 0 or 1. A *solid block* is a subarray of the form $M[i..i', j..j']$ in which every bit is equal to 1. Describe and analyze an efficient algorithm to find a solid block in M with maximum area.

9. You are driving a bus along a highway, full of rowdy, hyper, thirsty students and a soda fountain machine. Each minute that a student is on your bus, that student drinks one ounce of soda. Your goal is to drop the students off quickly, so that the total amount of soda consumed by all students is as small as possible.

   You know how many students will get off of the bus at each exit. Your bus begins somewhere along the highway (probably not at either end) and move s at a constant speed of 37.4 miles per

hour. You must drive the bus along the highway; however, you may drive forward to one exit then backward to an exit in the opposite direction, switching as often as you like. (You can stop the bus, drop off students, and turn around instantaneously.)

Describe an efficient algorithm to drop the students off so that they drink as little soda as possible. Your input consists of the bus route (a list of the exits, together with the travel time between successive exits), the number of students you will drop off at each exit, and the current location of your bus (which you may assume is an exit).

10. In a previous life, you worked as a cashier in the lost Antarctican colony of Nadira, spending the better part of your day giving change to your customers. Because paper is a very rare and valuable resource in Antarctica, cashiers were required by law to use the fewest bills possible whenever they gave change. Thanks to the numerological predilections of one of its founders, the currency of Nadira, called Dream Dollars, was available in the following denominations: \$1, \$4, \$7, \$13, \$28, \$52, \$91, \$365.[9]

    (a) The greedy change algorithm repeatedly takes the largest bill that does not exceed the target amount. For example, to make \$122 using the greedy algorithm, we first take a \$91 bill, then a \$28 bill, and finally three \$1 bills. Give an example where this greedy algorithm uses more Dream Dollar bills than the minimum possible.

    (b) Describe and analyze a recursive algorithm that computes, given an integer k, the minimum number of bills needed to make k Dream Dollars. (Don't worry about making your algorithm fast; just make sure it's correct.)

    (c) Describe a dynamic programming algorithm that computes, given an integer k, the minimum number of bills needed to make k Dream Dollars. (This one needs to be fast.)

11. What excitement! The Champaign Spinners and the Urbana Dreamweavers have advanced to meet each other in the World Series of Basket-weaving! The World Champions will be decided by a best-of- $2n-1$ series of head-to-head weaving matches, and the first to win n matches will take home the coveted Golden Basket (for example, a best-of-7 series requiring four match wins, but we will keep the generalized case). We know that for any given match there is a constant probability p that Champaign will win, and a subsequent probability $q = 1 - p$ that Urbana will win.

    Let $P(i, j)$ be the probability that Champaign will win the series given that they still need i more victories, whereas Urbana needs j more victories for the championship. $P(0, j) = 1$ for any j, because Champaign needs no more victories to win. Similarly, $P(i, 0) = 0$ for any i, as Champaign cannot possibly win if Urbana already has. $P(0, 0)$ is meaningless. Champaign wins any particular match with probability p and loses with probability q, so

    $$P(i, j) = p \cdot P(i-1, j) + q \cdot P(i, j-1)$$

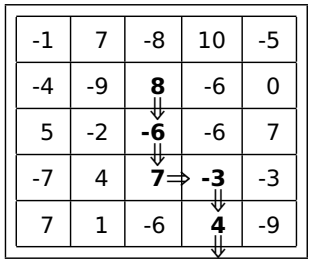
    for any $i \geq 1$ and $j \geq 1$.

    Describe and analyze an efficient algorithm that computes the probability that Champaign will win the series (that is, calculate $P(n, n)$), given the parameters n, p, and q as input.

[9] For more details on the history and culture of Nadira, including images of the various denominations of Dream Dollars, see http://www.dream-dollars.com.

12. *Vankin's Mile* is a solitaire game played on an $n \times n$ square grid. The player starts by placing a token on any square of the grid. Then on each turn, the player moves the token either one square to the right or one square down. The game ends when player moves the token off the edge of the board. Each square of the grid has a numerical value, which could be positive, negative, or zero. The player starts with a score of zero; whenever the token lands on a square, the player adds its value to his score. The object of the game is to score as many points as possible.

    For example, given the grid below, the player can score $8-6+7-3+4=10$ points by placing the initial token on the 8 in the second row, and then moving down, down, right, down, down. (This is *not* the best possible score for these values.)



    Describe and analyze an efficient algorithm to compute the maximum possible score for a game of Vankin's Mile, given the $n \times n$ array of values as input.

13. A *shuffle* of two strings X and Y is formed by interspersing the characters into a new string, keeping the characters of X and Y in the same order. For example, '`bananaananas`' is a shuffle of '`banana`' and '`ananas`' in several different ways.

$$\texttt{banana}_{\texttt{ananas}} \qquad \texttt{ban}_{\texttt{ana}}\texttt{ana}_{\texttt{nas}} \qquad \texttt{b}_{\texttt{an}}\texttt{an}_{\texttt{a}}\texttt{a}_{\texttt{na}}\texttt{na}_{\texttt{s}}$$

    The strings '`prodgyrnamammiincg`' and '`dyprongarmammicing`' are both shuffles of '`dynamic`' and '`programming`':

$$\texttt{pro}^{\texttt{d}}\texttt{g}^{\texttt{y}}\texttt{r}^{\texttt{nam}}\texttt{ammi}^{\texttt{i}}\texttt{n}^{\texttt{c}}\texttt{g} \qquad \texttt{dy}_{\texttt{pro}}\texttt{n}_{\texttt{g}}\texttt{a}_{\texttt{r}}\texttt{m}_{\texttt{amm}}\texttt{ic}_{\texttt{ing}}$$

    Given three strings $A[1..m]$, $B[1..n]$, and $C[1..m+n]$, describe and analyze an algorithm to determine whether C is a shuffle of A and B.

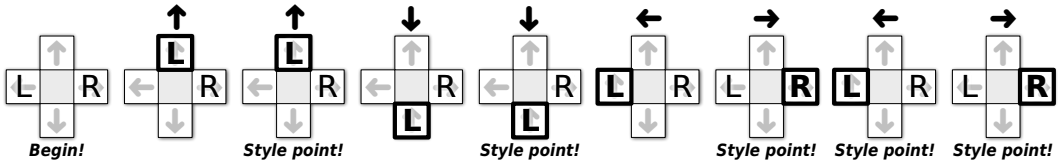
14. Describe and analyze an efficient algorithm to find the length of the longest contiguous substring that appears both forward and backward in an input string $T[1..n]$. The forward and backward substrings must not overlap. Here are several examples:

    - Given the input string `ALGORITHM`, your algorithm should return 0.
    - Given the input string `RECURSION`, your algorithm should return 1, for the substring `R`.
    - Given the input string `REDIVIDE`, your algorithm should return 3, for the substring `EDI`. (The forward and backward substrings must not overlap!)
    - Given the input string `DYNAMICPROGRAMMINGMANYTIMES`, your algorithm should return 4, for the substring `YNAM`. (In particular, it should *not* return 6, for the subsequence `YNAMIR`).

15. ***Dance Dance Revolution*** is a dance video game, first introduced in Japan by Konami in 1998. Players stand on a platform marked with four arrows, pointing forward, back, left, and right, arranged in a cross pattern. During play, the game plays a song and scrolls a sequence of n arrows (←, ↑, ↓, or →) from the bottom to the top of the screen. At the precise moment each arrow reaches the top of the screen, the player must step on the corresponding arrow on the dance platform. (The arrows are timed so that you'll step with the beat of the song.)

    You are playing a variant of this game called "Vogue Vogue Revolution", where the goal is to play perfectly but move as little as possible. When an arrow reaches the top of the screen, if one of your feet is already on the correct arrow, you are awarded one style point for maintaining your current pose. If neither foot is on the right arrow, you must move one (and *only* one) of your feet from its current location to the correct arrow on the platform. If you ever step on the wrong arrow, or fail to step on the correct arrow, or move more than one foot at a time, or move either foot when you are already standing on the correct arrow, all your style points are taken away and you lose the game.

    How should you move your feet to maximize your total number of style points? For purposes of this problem, assume you always start with you left foot on ← and you right foot on →, and that you've memorized the entire sequence of arrows. For example, if the sequence is ↑↑↓↓←→←→, you can earn 5 style points by moving you feet as shown below:



    (a) ***Prove*** that for *any* sequence of n arrows, it is possible to earn at least $n/4 - 1$ style points.

    (b) Describe an efficient algorithm to find the maximum number of style points you can earn during a given VVR routine. The input to your algorithm is an array $Arrow[1..n]$ containing the sequence of arrows.

16. Suppose we want to display a paragraph of text on a computer screen. The text consists of n words, where the ith word is p_i pixels wide. We want to break the paragraph into several lines, each exactly P pixels long. Depending on which words we put on each line, we will need to insert different amounts of white space between the words. The paragraph should be fully justified, meaning that the first word on each line starts at its leftmost pixel, and *except for the last line*, the last character on each line ends at its rightmost pixel. There must be at least one pixel of white-space between any two words on the same line.

    Define the *slop* of a paragraph layout as the sum over all lines, *except the last*, of the cube of the number of extra white-space pixels in each line (not counting the one pixel required between every adjacent pair of words). Specifically, if a line contains words i through j, then the amount of extra white space on that line is $P - j + i - \sum_{k=i}^{j} p_k$. Describe a dynamic programming algorithm to print the paragraph with minimum slop.

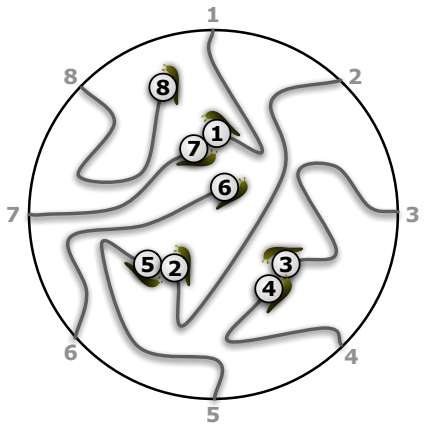
17. *[Stolen from abhi shelat.]* You have mined a large slab of marble from your quarry. For simplicity, suppose the marble slab is a rectangle measuring n inches in height and m inches in width. You want to cut the slab into smaller rectangles of various sizes—some for kitchen countertops, some

for large sculpture projects, others for memorial headstones. You have a marble saw that can make either horizontal or vertical cuts across any rectangular slab. At any given time, you can query the spot price $p_{x,y}$ of an $x \times y$ marble rectangle, for any positive integers x and y. These prices will vary with demand, so do not make any assumptions about them; in particular, larger rectangles may have much smaller spot prices. Given the spot prices, describe an algorithm to compute how to subdivide an $n \times m$ marble slab to maximize your profit.

18. Every year, as part of its annual meeting, the Antarctican Snail Lovers of Upper Glacierville hold a Round Table Mating Race. Several high-quality breeding snails are placed at the edge of a round table. The snails are numbered in order around the table from 1 to n. During the race, each snail wanders around the table, leaving a trail of slime behind it. The snails have been specially trained never to fall off the edge of the table or to cross a slime trail, even their own. If two snails meet, they are declared a breeding pair, removed from the table, and whisked away to a romantic hole in the ground to make little baby snails. Note that some snails may never find a mate, even if the race goes on forever.

    For every pair of snails, the Antarctican SLUG race organizers have posted a monetary reward, to be paid to the owners if that pair of snails meets during the Mating Race. Specifically, there is a two-dimensional array $M[1..n, 1..n]$ posted on the wall behind the Round Table, where $M[i, j] = M[j, i]$ is the reward to be paid if snails i and j meet.

    Describe and analyze an algorithm to compute the maximum total reward that the organizers could be forced to pay, given the array M as input.



The end of a typical Antarctican SLUG race. Snails 6 and 8 never find mates.
The organizers must pay $M[3,4] + M[2,5] + M[1,7]$.

19. Let P be a set of points in the plane in *convex position*. Intuitively, if a rubber band were wrapped around the points, then every point would touch the rubber band. More formally, for any point p in P, there is a line that separates p from the other points in P. Moreover, suppose the points are indexed $P[1], P[2], \ldots, P[n]$ in counterclockwise order around the 'rubber band', starting with the leftmost point $P[1]$.

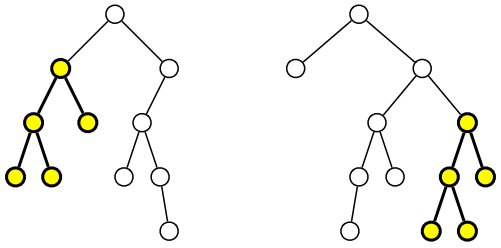
    This problem asks you to solve a special case of the traveling salesman problem, where the salesman must visit every point in P, and the cost of moving from one point $p \in P$ to another point $q \in P$ is the Euclidean distance $|pq|$.

(a) Describe a simple algorithm to compute the shortest *cyclic* tour of P.

(b) A *simple* tour is one that never crosses itself. Prove that the shortest tour of P must be simple.

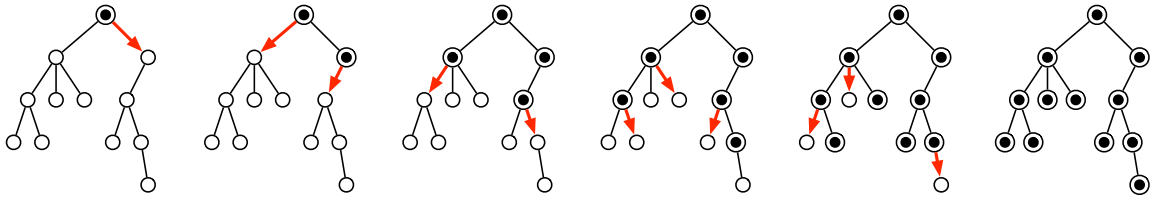
(c) Describe and analyze an efficient algorithm to compute the shortest tour of P that starts at the leftmost point $P[1]$ and ends at the rightmost point $P[r]$.

20. Recall that a *subtree* of a (rooted, ordered) binary tree T consists of a node and all its descendants. Design and analyze an efficient algorithm to compute the ***largest common subtree*** of two given binary trees T_1 and T_2; this is the largest subtree of T_1 that is isomorphic to a subtree in T_2. The contents of the nodes are irrelevant; we are only interested in matching the underlying combinatorial structure.



Two binary trees, with their largest common subtree emphasized

21. Suppose we need to distribute a message to all the nodes in a rooted tree. Initially, only the root node knows the message. In a single round, any node that knows the message can forward it to at most one of its children. Design an algorithm to compute the minimum number of rounds required for the message to be delivered to all nodes.



A message being distributed through a tree in five rounds.

22. A company is planning a party for its employees. The employees in the company are organized into a strict hierarchy, that is, a tree with the company president at the root. The organizers of the party have assigned a real number to each employee measuring how 'fun' the employee is. In order to keep things social, there is one restriction on the guest list: an employee cannot attend the party if their immediate supervisor is present. On the other hand, the president of the company *must* attend the party, even though she has a negative fun rating; it's her company, after all. Give an algorithm that makes a guest list for the party that maximizes the sum of the 'fun' ratings of the guests.

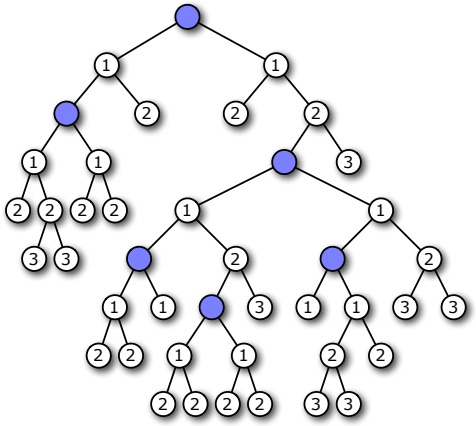
23. Let T be a rooted binary tree with n vertices, and let $k \le n$ be a positive integer. We would like to mark k vertices in T so that every vertex has a nearby marked ancestor. More formally, we define the *clustering cost* of any subset K of vertices as

$$cost(K) = \max_v cost(v, K),$$

where the maximum is taken over all vertices v in the tree, and

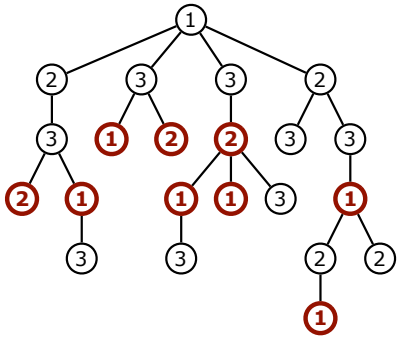
$$cost(v,K) = \begin{cases} 0 & \text{if } v \in K \\ \infty & \text{if } v \text{ is the root of } T \text{ and } v \notin K \\ 1 + cost(parent(v)) & \text{otherwise} \end{cases}$$

Describe and analyze a dynamic-programming algorithm to compute the minimum clustering cost of any subset of k vertices in T. For full credit, your algorithm should run in $O(n^2k^2)$ time.



A subset of 5 vertices with clustering cost 3

24. Oh, no! You have been appointed as the gift czar for Giggle, Inc.'s annual mandatory holiday party! The president of the company, who is certifiably insane, has declared that *every* Giggle employee must receive one of three gifts: (1) an all-expenses-paid six-week vacation anywhere in the world, (2) an all-the-pancakes-you-can-eat breakfast for two at Jumping Jack Flash's Flapjack Stack Shack, or (3) a burning paper bag full of dog poop. Corporate regulations prohibit any employee from receiving the same gift as his/her direct supervisor. Any employee who receives a better gift than his/her direct supervisor will almost certainly be fired in a fit of jealousy. How do you decide what gifts everyone gets if you want to minimize the number of people that get fired?



A tree labeling with cost 9. Bold nodes have smaller labels than their parents.
This is *not* the optimal labeling for this tree.

More formally, suppose you are given a rooted tree T, representing the company hierarchy. You want to label each node in T with an integer 1, 2, or 3, such that every node has a different label from its parent.. The *cost* of an labeling is the number of nodes that have smaller labels than

their parents. Describe and analyze an algorithm to compute the minimum cost of any labeling of the given tree T. (Your algorithm does *not* have to compute the actual best labeling—just its cost.)

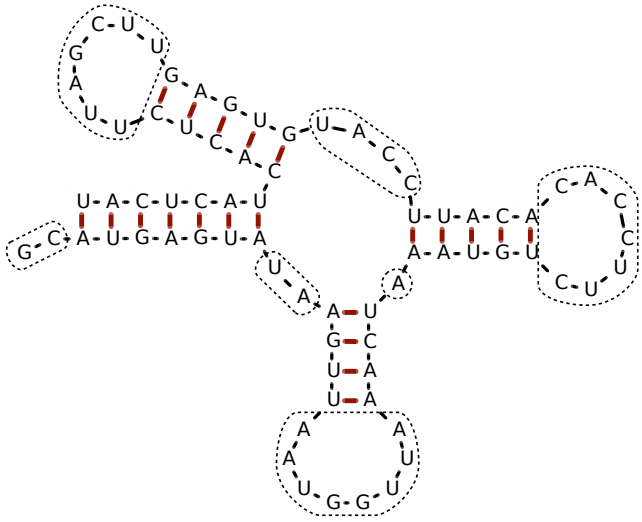
25. Ribonucleic acid (RNA) molecules are long chains of millions of nucleotides or *bases* of four different types: adenine (A), cytosine (C), guanine (G), and uracil (U). The *sequence* of an RNA molecule is a string $b[1..n]$, where each character $b[i] \in \{A, C, G, U\}$ corresponds to a base. In addition to the chemical bonds between adjacent bases in the sequence, hydrogen bonds can form between certain pairs of bases. The set of bonded base pairs is called the *secondary structure* of the RNA molecule.

    We say that two base pairs (i, j) and (i', j') with $i < j$ and $i' < j'$ ***overlap*** if $i < i' < j < j'$ or $i' < i < j' < j$. In practice, most base pairs are non-overlapping. Overlapping base pairs create so-called *pseudoknots* in the secondary structure, which are essential for some RNA functions, but are more difficult to predict.

    Suppose we want to predict the best possible secondary structure for a given RNA sequence. We will adopt a drastically simplified model of secondary structure:

    - Each base can be paired with at most one other base.
    - Only A-U pairs and C-G pairs can bond.
    - Pairs of the form $(i, i+1)$ and $(i, i+2)$ cannot bond.
    - Overlapping base pairs cannot bond.

    The last restriction allows us to visualize RNA secondary structure as a sort of fat tree.



Example RNA secondary structure with 21 base pairs, indicated by heavy red lines.
Gaps are indicated by dotted curves. This structure has score $2^2 + 2^2 + 8^2 + 1^2 + 7^2 + 4^2 + 7^2 = 187$

(a) Describe and analyze an algorithm that computes the maximum possible *number* of bonded base pairs in a secondary structure for a given RNA sequence.

(b) A *gap* in a secondary structure is a maximal substring of unpaired bases. Large gaps lead to chemical instabilities, so secondary structures with smaller gaps are more likely. To account for this preference, let's define the *score* of a secondary structure to be the sum of the *squares*

of the gap lengths.[10] Describe and analyze an algorithm that computes the minimum possible score of a secondary structure for a given RNA sequence.

26. Describe and analyze an algorithm to solve the traveling salesman problem in $O(2^n \operatorname{poly}(n))$ time. Given an undirected n-vertex graph G with weighted edges, your algorithm should return the weight of the lightest cycle in G that visits every vertex exactly once, or ∞ if G has no such cycles. *[Hint: The obvious recursive algorithm takes $O(n!)$ time.]*

*27. Let $\mathscr{A} = \{A_1, A_2, \ldots, A_n\}$ be a finite set of strings over some fixed alphabet Σ. An *edit center* for $\mathscr{A}$ is a string $C \in \Sigma^*$ such that the maximum edit distance from C to any string in $\mathscr{A}$ is as small as possible. The *edit radius* of $\mathscr{A}$ is the maximum edit distance from an edit center to a string in $\mathscr{A}$. A set of strings may have several edit centers, but its edit radius is unique.

$$\mathit{EditRadius}(\mathscr{A}) = \min_{C \in \Sigma^*} \max_{A \in \mathscr{A}} \mathit{Edit}(A, C) \qquad \mathit{EditCenter}(\mathscr{A}) = \arg\min_{C \in \Sigma^*} \max_{A \in \mathscr{A}} \mathit{Edit}(A, C)$$

(a) Describe and analyze an efficient algorithm to compute the edit radius of three given strings.

(b) Describe and analyze an efficient algorithm to approximate the edit radius of an arbitrary set of strings within a factor of 2. (Computing the edit radius exactly is NP-hard unless the number of strings is fixed.)

*28. Let $D[1..n]$ be an array of digits, each an integer between 0 and 9. An *digital subsequence* of D is an sequence of positive integers composed in the usual way from disjoint substrings of D. For example, $3, 4, 5, 6, 8, 9, 32, 38, 46, 64, 83, 279$ is an increasing digital subsequence of the first several digits of π:

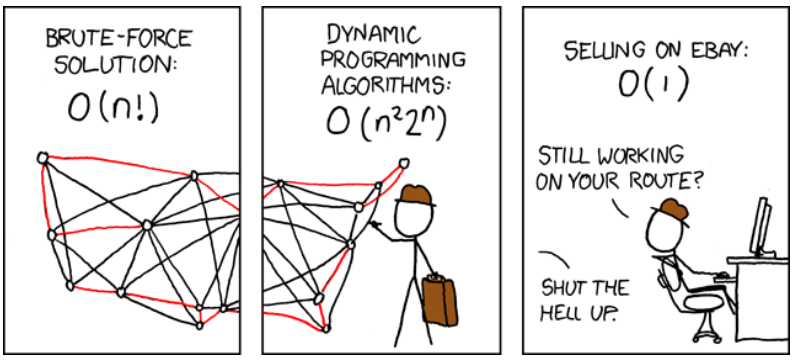
$$\underline{3}, 1, \underline{4}, 1, \underline{5}, 9, 2, \underline{6}, 5, 3, 5, \underline{8}, \underline{9}, 7, 9, \underline{3,2}, \underline{3,8}, \underline{4,6}, 2, \underline{6,4}, 3, 3, \underline{8,3}, \underline{2,7,9}$$

The *length* of a digital subsequence is the number of integers it contains, *not* the number of digits; the preceding example has length 12.

Describe and analyze an efficient algorithm to compute the longest increasing digital subsequence of D. *[Hint: Be careful about your computational assumptions. How long does it take to compare two k-digit numbers?]*

For full credit, your algorithm should run in $O(n^4)$ time; faster algorithms are worth extra credit. The fastest algorithm I know for this problem runs in $O(n^{3/2} \log n)$ time, but this requires several tricks, in both the algorithm and its analysis.

[10] This score function has absolutely no connection to reality; I just made it up. Real RNA structure prediction requires *much* more complicated scoring functions.



— Randall Munroe, *xkcd* (http://xkcd.com/399/)