Relational semantics for Basic Logic*

Damiano Macedonio[†] and Giovanni Sambin[‡]

[†] Dipartimento di informatica,
Università "Ca' Foscari" di Venezia (Italy).
[‡] Dipartimento di matematica pura e applicata,
Università di Padova (Italy).

e-mail: `mace@dsi.unive.it`, `sambin@math.unipd.it`

December 14, 2001

We present here a mathematical interpretation of the fragment without implications of basic logic (for an introduction to basic logic and its motivations see [13]). The models are just relational monoids $(M, \cdot, 1)$ equipped with an arbitrary relation R. The relation R induces two closure operators on subsets of M, which are obtained by combining polarities, as in [2]. The idea (due to Ferruccio Guidi, see [7] and [8]) is to interpret formulae in subsets of M which are closed in this sense. The definition of the evaluation for each connective is based on the equivalences characterizing that connective (called *definitional equations* in [13]). The proof of validity is then immediate. Contrary to what happens in other logics, here the evaluation of a sequent $\Gamma \vdash \Delta$ cannot be reduced in general to the evaluation of a sequent of the form $\varphi \vdash \psi$ or to the evaluation of a single formula. In fact, here the "comma" in the lists Γ and Δ can be replaced by a connective only when Γ (or Δ) consist of only two formulae (property of *visibility*, see [13]).

Completeness is proved as usual by way of a generic model, built up from syntax.

Basic logic has been explained (in [13]) in terms of the principle of reflection: each connective reflects at object level a link between assertions at the metalevel. This provides each connective with a clear meaning, that is with a semantics. Still, there are two good reasons to introduce a complete mathematical interpretation (what is commonly called a semantics). One is that it could be useful to some readers to grasp the meaning of basic logic. The other is that the relational monoids in which the relation is strongly symmetric (that is, satisfies $x \cdot y\, R\, z \;\rightarrow x \cdot z\, R\, y$) turn out to be exactly the phase spaces introduced by J.-Y. Girard as semantics of linear logic in [6]. This should highlight in which sense linear logic (without exponentials!) is a proper extension of basic logic.

The structural rule of exchange was introduced in the sequent calculus **B** of basic logic simply for reasons of convenience, to avoid duplications of implications. Since we here omit implications, it is very natural to consider the sequent calculus obtained dropping also the rule of exchange. In fact exchange is valid in a relational monoid if and only if the monoid operation is commutative. Thus the relational semantics introduced here applies also to a non commutative basic logic.

Finding a mathematical interpretation for the full calculus of basic logic **B**, with implications, remain an open problem. A convincing solution might involve a deeper understanding of implication in basic logic, and may be a reformulation of its rules.

*This research was partially supported by the project *"Computazione e logica: la logica di base come strumento per un nuovo approccio unitario"*, Università di Padova.

1 The basic calculus

In this section we introduce the basic sequent calculus called $\mathbf{B}^-$. It is obtained from the sequent calculus $\mathbf{B}$ of basic logic (introduced in [13]) simply by deleting the implications and exchange rules (i.e. the only structural rules of $\mathbf{B}$). In particular calculus $\mathbf{B}^-$ is non-commutative.

The language $\mathcal{L}$ of $\mathbf{B}^-$ consists of propositional constants $\top$, $\bot$, 1 and 0, propositional variables $p, q, \ldots$, additive connectives $\oplus$ and $\&$, and multiplicative connectives $\otimes$ and $\parr$. We denote by small Greek letters $\varphi, \psi, \mu \ldots$ the formulae of $\mathcal{L}$ and by capital Greek letters $\Gamma, \Delta, \Sigma, \ldots$ (possibly empty) lists of formulae. In a sequent $\Gamma \vdash \Delta$, both the antecedent Γ and the consequent Δ are called contexts.

The inference rules of $\mathbf{B}^-$ are justified, as in [13], by solving some definitional equations. We refer to [13] for a discussion on this methodology, and here we report shortly only the results.

We assume at the beginning only axioms of the form $\varphi \vdash \varphi$ and *cut* rules, as at the bottom of table 1. Each connective and logical constant is associated with a definitional equation, which completely describes its behaviour.

Definition 1 (Definitional equations). *Each connective and each logical constant is required to satisfy the corresponding equation, as follows:*

$$\begin{array}{rll}
\otimes: & \textit{For all } \Delta, & \psi \otimes \varphi \vdash \Delta \textit{ iff } \psi, \varphi \vdash \Delta; \\
\parr: & \textit{For all } \Gamma, & \Gamma \vdash \varphi \parr \psi \textit{ iff } \Gamma \vdash \varphi, \psi; \\
\oplus: & \textit{For all } \Delta, & \psi \oplus \varphi \vdash \Delta \textit{ iff } \psi \vdash \Delta \textit{ and } \varphi \vdash \Delta; \\
\&: & \textit{For all } \Gamma, & \Gamma \vdash \varphi \& \psi \textit{ iff } \Gamma \vdash \varphi \textit{ and } \Gamma \vdash \psi; \\
1: & \textit{For all } \Delta, & 1 \vdash \Delta \textit{ iff } \quad \vdash \Delta; \\
\bot: & \textit{For all } \Gamma, & \Gamma \vdash \bot \textit{ iff } \Gamma \vdash \quad ; \\
0: & \textit{For all } \Delta, & \varphi \vdash \Delta \textit{ and } 0 \vdash \Delta \textit{ iff } \varphi \vdash \Delta; \\
\top: & \textit{For all } \Gamma, & \Gamma \vdash \psi \textit{ and } \Gamma \vdash \top \textit{ iff } \Gamma \vdash \psi.
\end{array}$$

The inference rules of $\mathbf{B}^-$, listed in table 1, are obtained by solving all definitional equations. This means that the inference rules for a connective or a constant are obtained from the corresponding definitional equation by applying axioms and cut rules. And that, conversely, the definitional equations become formally derivable in $\mathbf{B}^-$. We refer to [13] for the derivation of the solution. Definitional equations are very important in this paper, since they provide with the right intuitions for the definition of evaluation of formulae, in section 3.

2 Relational monoids

The basic structures which will be used to give a relational semantics for the calculus $\mathbf{B}^-$ are just monoids equipped with a binary relation. We call them *relational monoids.* We now define them formally. Then we will introduce Birkhoff's polarities (see [2]) and we will repeat their first properties, which we will often use in the following.

Definition 2. $\mathcal{M} = (M, \cdot, 1, R)$ *is a relational monoid if*

M is a set;

$\cdot : M \times M \longrightarrow M$ is a binary operation such that

> $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *for all* $x, y, z \in M$ *(associativity),*
>
> $1 \cdot x = x \cdot 1 = x$ *for all* $x \in M$ *(neutral element);*

R is a binary relation of M on itself.

Axioms

$$\varphi \vdash \varphi$$

Operational rules

Multiplicatives

$$\frac{\psi, \varphi \vdash \Delta}{\psi \otimes \varphi \vdash \Delta}\ \otimes L \qquad \frac{\Gamma \vdash \varphi, \psi}{\Gamma \vdash \varphi \parr \psi}\ \parr R$$

$$\frac{\psi \vdash \Delta_1 \quad \varphi \vdash \Delta_2}{\psi \parr \varphi \vdash \Delta_1, \Delta_2}\ \parr L \qquad \frac{\Gamma_2 \vdash \varphi \quad \Gamma_1 \vdash \psi}{\Gamma_2, \Gamma_1 \vdash \varphi \otimes \psi}\ \otimes R$$

$$\frac{\vdash \Delta}{1 \vdash \Delta}\ 1L \qquad \frac{\Gamma \vdash}{\Gamma \vdash \bot}\ \bot R$$

$$\bot \vdash \quad \bot L \qquad \vdash 1 \quad 1R$$

Additives

$$\frac{\psi \vdash \Delta \quad \varphi \vdash \Delta}{\psi \oplus \varphi \vdash \Delta}\ \oplus L \qquad \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \,\&\, \psi}\ \& R$$

$$\frac{\psi \vdash \Delta}{\psi \,\&\, \varphi \vdash \Delta}\ \& L \quad \frac{\varphi \vdash \Delta}{\psi \,\&\, \varphi \vdash \Delta}\ \& L \qquad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \oplus \psi}\ \oplus R \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \oplus \psi}\ \oplus R$$

$$0 \vdash \Delta \quad 0L \qquad \Gamma \vdash \top \quad \top R$$

Cut rules

$$\frac{\Gamma \vdash \varphi \quad \Gamma_1, \varphi, \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta}\ cutL \qquad \frac{\Gamma \vdash \Delta_2, \varphi, \Delta_1 \quad \varphi \vdash \Delta}{\Gamma \vdash \Delta_2, \Delta, \Delta_1}\ cutR$$

Table 1: Basic sequent calculus $\mathbf{B}^-$.

Through the relation R in M, every element $z \in M$ determines two subsets of M: the subset $z^{\leftarrow}$ of elements that are in *left* relation with z and the subset $z^{\rightarrow}$ of elements that are in *right* relation with z. Formally[1]:

$$z^{\leftarrow} \equiv \{x \in M \,:\, x\,R\,z\};$$
$$z^{\rightarrow} \equiv \{y \in M \,:\, z\,R\,y\}.$$

We can extend this definition to subsets[2] of M so that we can obtain the *polarities* introduced by Birkhoff in [2]. For any subset $A \subseteq M$ we define its *left polar* $A^{\leftarrow}$ and its *right polar* $A^{\rightarrow}$ as follows.

Definition 3. *For any $A \subseteq M$, we put:*

$$A^{\leftarrow} \equiv \{x \in M \,:\, x\,R\,z \text{ for all } z \,\epsilon\, A\} = \bigcap_{z \,\epsilon\, A} z^{\leftarrow}; \tag{1}$$
$$A^{\rightarrow} \equiv \{y \in M \,:\, z\,R\,y \text{ for all } z \,\epsilon\, A\} = \bigcap_{z \,\epsilon\, A} z^{\rightarrow}. \tag{2}$$

Note that there is no ambiguity of notation, since $\{z\}^{\leftarrow} = z^{\leftarrow}$ and $\{z\}^{\rightarrow} = z^{\rightarrow}$ for every $z \in M$. Moreover, for any $A \subseteq M$ the following equivalences hold by definition:

$$x \,\epsilon\, A^{\leftarrow} \text{ iff } A \subseteq x^{\rightarrow} \quad \text{and} \quad y \,\epsilon\, A^{\rightarrow} \text{ iff } A \subseteq y^{\leftarrow}.$$

By using them the reader can easily verify that the following lemma holds.

Lemma 4. *For any A_1, A_2, B_1, B_2 subsets[3] of M:*

$$A_1 \subseteq A_2 \quad \textit{implies} \quad A_2^{\rightarrow} \subseteq A_1^{\rightarrow}, \tag{3}$$
$$B_1 \subseteq B_2 \quad \textit{implies} \quad B_2^{\leftarrow} \subseteq B_1^{\leftarrow}, \tag{4}$$
$$A_1 \subseteq A_1^{\rightarrow\leftarrow} \quad \textit{and} \quad B_1 \subseteq B_1^{\leftarrow\rightarrow}. \tag{5}$$

Conditions (3)–(5) are just the definition to saying that the correspondences $A \mapsto A^{\rightarrow}$ and $B \mapsto B^{\leftarrow}$ define a Galois connection (see [2]) between the complete lattice $(\mathcal{P}(M), \subseteq)$ and itself, where $\mathcal{P}(M)$ is the class of subsets of M and $\subseteq$ determines the order. Conditions (3)–(5) are equivalent to a very important condition which we will use for soundness.

Lemma 5. *Conditions* (3), (4) *and* (5) *are equivalent to condition:*

$$A \subseteq B^{\leftarrow} \textit{ iff } B \subseteq A^{\rightarrow} \qquad \textit{for all } A, B \subseteq M. \tag{6}$$

Proof. We obtain the direction left-right of (6) by applying (3) and (5), and the direction right-left of (6) by applying (4) and (5). Vice versa, we obtain (5) by applying (6) to $A_1^{\rightarrow} \subseteq A_1^{\rightarrow}$ (if we put $A = A_1$ and $B = A_1^{\rightarrow}$). We obtain (3) in this way: if we assume $A_1 \subseteq A_2$, then $A_1 \subseteq A_2^{\rightarrow\leftarrow}$ by (5), and so $A_2^{\rightarrow} \subseteq A_1^{\rightarrow}$ by applying (6) to $A = A_1$ and $B = A_2^{\rightarrow}$. We can obtain (4) symmetrically. $\square$

The condition (6) says that the two (contravariant) operators $(\cdot)^{\rightarrow}$ and $(\cdot)^{\leftarrow}$ are adjoint on the right (see [3] p.81).

The following are immediate consequences of lemma 4, or equivalently of (6).

Corollary 6. *For any $A, B \subseteq M$:*

$$A^{\rightarrow\leftarrow\rightarrow} = A^{\rightarrow} \qquad \textit{and} \qquad B^{\leftarrow\rightarrow\leftarrow} = B^{\leftarrow}. \tag{7}$$

[1] Here and in whole paper $\equiv$ is the sign for definitional equality, when a definition is first given, the definiendum will always be at the left and the definiens at the right.

[2] We adopt the definitions and notations for subsets introduced and justified in [14]. So, for any set M, $A \subseteq M$ means that A is a propositional function over M. To denote that a is an element of the subset A, we write $a \,\epsilon\, A$.

[3] With $A^{\rightarrow\leftarrow}$ and $B^{\leftarrow\rightarrow}$ we mean $(A^{\rightarrow})^{\leftarrow}$ and $(B^{\leftarrow})^{\rightarrow}$ respectively. In general terms, an exponential with more than one arrow is to be intended as the application of arrows from left to right. As an example: $B^{\leftarrow\rightarrow\leftarrow\rightarrow}$ means $(((B^{\leftarrow})^{\rightarrow})^{\leftarrow})^{\rightarrow}$, and so on.

Proof. By (5), we have $A^{\rightarrow} \subseteq A^{\rightarrow\leftarrow\rightarrow}$ and $A \subseteq A^{\rightarrow\leftarrow}$, hence $A^{\rightarrow\leftarrow\rightarrow} \subseteq A^{\rightarrow}$ by (3). The proof that $B^{\leftarrow\rightarrow\leftarrow} = B^{\leftarrow}$ is similar. □

Corollary 7. *The operators* $(\cdot)^{\rightarrow\leftarrow} : \mathcal{P}(M) \longrightarrow \mathcal{P}(M)$ *and* $(\cdot)^{\leftarrow\rightarrow} : \mathcal{P}(M) \longrightarrow \mathcal{P}(M)$ *are closure operators.*

Proof. It easy to see that the conditions for closure operators holds, in fact for any $A, A_1 \subseteq M$: $A \subseteq A^{\rightarrow\leftarrow}$ by (5); $A^{\rightarrow\leftarrow} = A^{\rightarrow\leftarrow\rightarrow\leftarrow}$ by (7); $A \subseteq A_1$ implies $A^{\rightarrow\leftarrow} \subseteq A_1^{\rightarrow\leftarrow}$ by (3) and (4). The case of $(\cdot)^{\leftarrow\rightarrow}$ is similar. □

Now we can define two classes of subsets of M which we will use to define the evaluation of formulae.

Definition 8 (Saturated subsets). *For any $A, B \subseteq M$, we say that A is left saturated if $A = A^{\rightarrow\leftarrow}$ and that B is right saturated if $B = B^{\leftarrow\rightarrow}$. Moreover we define $Sat^{\leftarrow}(M)$ and $Sat^{\rightarrow}(M)$ as the collection of left saturated and right saturated subsets of M respectively.*

The justification of the adjectives "left" and "right" to saturated subsets derives from corollary 6. In fact, by (7), left and right saturated subsets of M are just those of the form $B^{\leftarrow}$ and $A^{\rightarrow}$ respectively.

The collections $Sat^{\leftarrow}(M)$ and $Sat^{\rightarrow}(M)$ are complete lattices, where meet (gub) is the intersection $\cap$ and join (lub) is the saturation of the union $\cup$. M is the maximum among both left and right saturated subsets. The saturations of the empty subset, $\emptyset^{\rightarrow\leftarrow}$ and $\emptyset^{\leftarrow\rightarrow}$, are the minimum among left and right saturated subsets, respectively.

The next theorem shows a very important correspondence between left and right saturated subsets. Such correspondence is useful to our interpretation of formulae of language $\mathcal{L}$.

Theorem 9. *The correspondences $A \mapsto A^{\rightarrow}$ and $B \mapsto B^{\leftarrow}$ define a dual isomorphism between the complete lattices of left and right saturated subsets of M. In particular, if A_1, A_2 are left saturated subsets and B_1, B_2 are right saturated subsets, then:*

$$(A_1 \cap A_2)^{\rightarrow} = (A_1^{\rightarrow} \cup A_2^{\rightarrow})^{\leftarrow\rightarrow} \qquad (B_1 \cap B_2)^{\leftarrow} = (B_1^{\leftarrow} \cup B_2^{\leftarrow})^{\rightarrow\leftarrow} \tag{8}$$

$$(A_1 \cup A_2)^{\rightarrow} = A_1^{\rightarrow} \cap A_2^{\rightarrow} \qquad (B_1 \cup B_2)^{\leftarrow} = B_1^{\leftarrow} \cap B_2^{\leftarrow} \tag{9}$$

$$\emptyset^{\rightarrow} = M \qquad \emptyset^{\leftarrow} = M \tag{10}$$

$$M^{\rightarrow} = \emptyset^{\leftarrow\rightarrow} \qquad M^{\leftarrow} = \emptyset^{\rightarrow\leftarrow} \tag{11}$$

Proof. By (7), the correspondences $A \mapsto A^{\rightarrow}$ and $B \mapsto B^{\leftarrow}$ are inverse of each other; hence they are one-one and onto. Finally, by (3) and (4), they invert inclusion and so they interchange join with meet. □

3 Soundness

In any relational monoid $\mathcal{M} = (M, \cdot, 1, R)$ we can interpret formulae of the language $\mathcal{L}$ as saturated subsets, as we define in this section. Here we also prove a soundness theorem for such interpretation, while a completeness theorem is given in next section.

The idea we follow is to think of M as the set of resources in a production cycle. For any resources $x, y \in M$ we read xRy as: *the resource x can produce the resource y*. We call x the (possible) *ingredient* and y the (possible) *product*.

In this way the polarities we defined in section 2 assume a particular meaning. In fact for any resource x the subset $x^{\rightarrow}$ determines the subset of all the resources (products) which can be produced with x as ingredient. On the other hand, the subset $x^{\leftarrow}$ determines the subset of all the resources (ingredients) that can give x as product.

In section 2 we pointed out that any element in $Sat^{\leftarrow}(M)$ is of the form $B^{\leftarrow}$, i.e. it is the subset of the *ingredients* which can produce every resource in B. Equivalently any element in $Sat^{\rightarrow}(M)$ is of the form $A^{\rightarrow}$, i.e. it is the subset of the *products* which can be obtained by using any resource of A. So intuitively we can think of an element in the collection $Sat^{\leftarrow}(M)$ as a subset of (possible) *ingredients*, and we can think to an element in $Sat^{\rightarrow}(M)$ as a subset of (possible) *products*.

We intend the operation $\cdot$ in M as the composition of resources. If we combine the resource x with y (in this order), then we obtain the resource $x \cdot y$. In $x \cdot y$ the resources x and y are connected to each other, we cannot isolate x or y. In particular 1 is the resource that does not modify the resource which it is combined with.

The combination between two subsets A, B of resource is just the subset $A \cdot B$ formed by all the possible combinations between a resource of A a resource of B, namely the algebraic product between subsets $A \cdot B \equiv \{x \cdot y : x \epsilon A, y \epsilon B\}$.

We associate any formula φ with a pair of saturated subset of M: a subset of ingredients (left saturated) and a subset of products (right saturated).

Theorem 9 says that every left saturated subset (ingredients) determine one and only one right saturated subset (products), so we do not have to choose two saturated subsets to evaluate a formula: for example, we can chose a left saturated subset and automatically we have also the right saturated one by applying the operator $(\cdot)^{\rightarrow}$. This is our choice.

Let Frm be the set of formulae in the language $\mathcal{L}$. We want to define

$$V(\cdot) : Frm \longrightarrow \; Sat^{\leftarrow}(M)$$

that is the *evaluation of formulae*. It will associate every formula φ with a subset $V(\varphi)$ of ingredients, and, clearly, with the subset $V(\varphi)^{\rightarrow}$ of products.

For any propositional variable p the value $V(p)$ in $Sat^{\leftarrow}(M)$ is assumed to be given. Then we apply induction on connectives. We first look at the interpretation of a sequent $\Gamma \vdash \Delta$ to explain the definition of V.

So suppose that V is already defined on all formulae, and let us define the evaluation of the contexts that form a sequent. If we read the sequent $\Gamma \vdash \Delta$ as Γ *can produce* Δ *in the calculus* $\boldsymbol{B}^{-}$, then it becomes natural to associate Γ with ingredients and Δ with products. It is simple to associate $\Gamma = \varphi_1, \ldots, \varphi_m$ with the combination of ingredients $Ingr(\Gamma) \equiv V(\varphi_1) \cdot \ldots \cdot V(\varphi_m)$, and $\Delta = \psi_1, \ldots, \psi_n$ with the combination of products $Prod(\Delta) \equiv V(\psi_1)^{\rightarrow} \cdot \ldots \cdot V(\psi_n)^{\rightarrow}$.

A particular case is that of the empty context. The behaviour of the empty context in the set of formulae and the one of the neutral element in the monoid are very much alike. In fact the empty list [ ] is neutral respect to the composition with formulae, as we will see for the syntactic model. So we define $Ingr([\,]) \equiv \{1\}$ and $Prod([\,]) \equiv \{1\}$.

Formally, for any context $\Sigma = \sigma_1, \ldots, \sigma_m$, where $m \geq 0$, we define:

$$Ingr(\Sigma) \equiv \{1\} \cdot V(\sigma_1) \cdot \ldots \cdot V(\sigma_m) \tag{12}$$

$$Prod(\Sigma) \equiv \{1\} \cdot V(\sigma_1)^{\rightarrow} \cdot \ldots \cdot V(\sigma_m)^{\rightarrow} \tag{13}$$

Note that there is no ambiguity because the operation of monoid is associative. Moreover both products contain the subset $\{1\}$; this fact allows us to evaluate the empty context as we have just said. If the context is formed by one or more formula, then the subset $\{1\}$ does not influence the product. In fact $\{1\}$ is neutral in the product between subsets, that is $\{1\} \cdot A = A \cdot \{1\} = A$ for all $A \subseteq M$. If the contexts are formed by exactly one formula φ, then $Ingr(\varphi) = V(\varphi)$, namely the ingredients associated with φ, and $Prod(\varphi) = V(\varphi)^{\rightarrow}$, namely the products associated with φ.

Intuitively we say that a sequent $\Gamma \vdash \Delta$ is valid if *every resource associated with* Γ *can produce every resource associated with* Δ. Formally we say that the sequent $\Gamma \vdash \Delta$ is valid in the monoid $\mathcal{M}$ iff $Ingr(\Gamma) \subseteq Prod(\Delta)^{\leftarrow}$ (*the resources associated with* Γ *are ingredients for the resources associated with* Δ) or equivalently, by (6), iff $Prod(\Delta) \subseteq Ingr(\Gamma)^{\rightarrow}$ (*the resources associated with* Δ *are products of the resources associated with* Γ).

Now we can make a step back and define the evaluation V on formulae. We use the intuition we have just given and the definitional equations for $\mathbf{B}^{-}$. In fact we revise the definition 1 using the idea of the production cycle.

Connective $\&$. The definitional equation says that: $Ingr(\Gamma) \subseteq Prod(\varphi \& \psi)^{\leftarrow}$ *iff* $Ingr(\Gamma) \subseteq Prod(\varphi)^{\leftarrow}$ *and* $Ingr(\Gamma) \subseteq Prod(\psi)^{\leftarrow}$. Note that for any single formula σ: $Prod(\sigma)^{\leftarrow} = V(\sigma)$. So the equation is equivalent to: $Ingr(\Gamma) \subseteq V(p \& q)$ *iff* $Ingr(\Gamma) \subseteq V(p)$ *and* $Ingr(\Gamma) \subseteq V(q)$. This means that we have to associate the connective $\&$ whit meet (intersection) for left saturated subsets, and so we have to define:

$$V(\varphi \& \psi) \equiv V(\varphi) \cap V(\psi). \tag{14}$$

Connective $\oplus$. The definitional equation says that: $Prod(\Delta) \subseteq Ingr(\psi \oplus \varphi)^{\rightarrow}$ *iff* $Prod(\Delta) \subseteq Ingr(\psi)^{\rightarrow}$ *and* $Prod(\Delta) \subseteq Ingr(\varphi)^{\rightarrow}$. Here for any single formula σ: $Ingr(\sigma)^{\rightarrow} = V(\sigma)^{\rightarrow}$. So the equation is equivalent to: $Prod(\Delta) \subseteq V(\psi \oplus \varphi)^{\rightarrow}$ *iff* $Prod(\Delta) \subseteq V(\psi)^{\rightarrow}$ *and* $Prod(\Delta) \subseteq V(\varphi)^{\rightarrow}$. This means that we have to associate the connective $\oplus$ with meet (intersection) for right saturated subsets, and so:

$$V(\psi \oplus \varphi)^{\rightarrow} \equiv V(\psi)^{\rightarrow} \cap V(\varphi)^{\rightarrow}. \tag{15}$$

Finally we obtain by (8):

$$V(\psi \oplus \varphi) = V(\psi \oplus \varphi)^{\rightarrow\leftarrow} = (\, V(\psi)^{\rightarrow} \cap V(\varphi)^{\rightarrow}\,)^{\leftarrow} = (\, V(\psi) \cup V(\varphi)\,)^{\rightarrow\leftarrow}. \tag{16}$$

that is the join for left saturated subsets.
Connective $⅋$. Definition 1 says that $Ingr(\Gamma) \subseteq Prod(\varphi ⅋ \psi)^{\leftarrow}$ *iff* $Ingr(\Gamma) \subseteq Prod(\varphi, \psi)^{\leftarrow}$. This means that we have to define:

$$V(\varphi ⅋ \psi) \equiv Prod(\varphi, \psi)^{\leftarrow} = (\, V(\varphi)^{\rightarrow} \cdot V(\psi)^{\rightarrow}\,)^{\leftarrow}. \tag{17}$$

Connective $\otimes$. Definition 1 says that $Prod(\Delta) \subseteq Ingr(\psi \otimes \varphi)^{\rightarrow}$ *iff* $Prod(\Delta) \subseteq Ingr(\psi, \varphi)^{\rightarrow}$. This means we have to define $Ingr(\psi \otimes \varphi)^{\rightarrow} = Ingr(\psi, \varphi)^{\rightarrow}$ and so:

$$V(\psi \otimes \varphi) \equiv Ingr(\psi, \varphi)^{\rightarrow\leftarrow} = (\, V(\psi) \cdot V(\varphi)\,)^{\rightarrow\leftarrow}. \tag{18}$$

Constant 1. By definition 1: $Prod(\Delta) \subseteq Ingr(1)^{\rightarrow}$ *iff* $Prod(\Delta) \subseteq Ingr([\;])^{\rightarrow}$. So the only possibility we have is to define:

$$V(1) \equiv Ingr([\;])^{\rightarrow\leftarrow} = \{1\}^{\rightarrow\leftarrow}. \tag{19}$$

Constant $\bot$. By definition 1: $Ingr(\Gamma) \subseteq Prod(\bot)^{\leftarrow}$ *iff* $Ingr(\Gamma) \subseteq Prod([\;])^{\leftarrow}$. So we have to define:

$$V(\bot) \equiv Prod([\;])^{\leftarrow} = \{1\}^{\leftarrow}. \tag{20}$$

Constant 0. Definition 1 says that the subset of products associated with 0 must be as big as possible. The biggest right saturated subset is M. Therefore we have to define $V(0)^{\rightarrow} \equiv M$ and so:

$$V(0) \equiv M^{\leftarrow} = \emptyset^{\rightarrow\leftarrow}. \tag{21}$$

Constant $\top$. Definition 1 says that the subset of ingredients associated with $\top$ must be as big as possible. The biggest left saturated subset is M again. So we have to define:

$$V(\top) \equiv M. \tag{22}$$

The previous intuitive explanations justify the following formal definition.

Definition 10 (Inductive definition of validity). *Let $\mathcal{M}$ be a relational monoid. A given assignment V of subsets $V(p), V(q), \ldots$ of $Sat^{\leftarrow}(M)$ to propositional variables $p, q, \ldots$ is extended to an evaluation V of all formulae by the inductive clauses:*

$$\begin{array}{cc}
V(\top) \equiv M & V(0) \equiv \emptyset^{\rightarrow\leftarrow} \\
V(1) \equiv \{\,1\}^{\rightarrow\leftarrow} & V(\bot) \equiv \{\,1\}^{\leftarrow} \\
V(\varphi \& \psi) \equiv V(\varphi) \cap V(\psi) & V(\psi \oplus \varphi) \equiv (\, V(\psi) \cup V(\varphi)\,)^{\rightarrow\leftarrow}
\end{array}$$

$$V(\psi \otimes \varphi) \equiv (\, V(\psi) \cdot V(\varphi)\,)^{\rightarrow\leftarrow} \qquad\qquad V(\varphi \,⅋\, \psi) \equiv (\, V(\varphi)^{\rightarrow} \cdot V(\psi)^{\rightarrow}\,)^{\leftarrow}$$

For any list $\Sigma = \varphi_1, \ldots, \varphi_m$ (whit $m \geq 0$) we put:

$$Ingr(\Sigma) \equiv \{\,1\,\} \cdot V(\varphi_1) \cdot \ldots \cdot V(\varphi_m); \tag{23}$$
$$Prod(\Sigma) \equiv \{\,1\,\} \cdot V(\varphi_1)^{\rightarrow} \cdot \ldots \cdot V(\varphi_m)^{\rightarrow}. \tag{24}$$

A sequent $\Gamma \vdash \Delta$ is said to be valid under the evaluation V if $Ingr(\Gamma) \subseteq Prod(\Delta)^{\leftarrow}$ (or equivalently if $Prod(\Delta) \subseteq Ingr(\Gamma)^{\rightarrow}$), and valid in $\mathcal{M}$ if it is valid under any V in $\mathcal{M}$.

Theorem 11 (Soundness). *Let $\mathcal{M}$ be any relational monoid. If the sequent $\Gamma \vdash \Delta$ is deducible in $\boldsymbol{B}^-$, then $\Gamma \vdash \Delta$ is valid in $\mathcal{M}$.*

Proof. Rather than a boring proof showing that axioms are valid, and that each rule preserves validity,as is usually done, we obtain a full proof by showing the validity of definitional equations. In fact, this is equivalent to the validity of rules. We have already done it when we have introduced the evaluation of formulae! So it is needed only to prove the validity for *cut* rules. This holds since the combination of subsets preserves inclusion. □

4 Completeness

The proof of completeness theorem is based on the construction of a particular relational monoid: the *syntactic model*. We will prove that a sequent is valid in the syntactic model if and only if it is derivable in $\mathbf{B}^-$. In the sequent $\Gamma \vdash \Delta$ we consider the antecedent Γ as the ingredient and the consequent Δ as the product.

Definition 12 (Syntactic model). *The syntactic model is the structure*

$$\mathcal{F} \equiv (Frm^*, \circ, [\,], \vdash_{\boldsymbol{B}^-})$$

where:

a. Frm^* is the set of all finite lists we can create with formulae of $\mathcal{L}$ (including the empty list)

b. $\circ$ is the concatenation between lists; i.e. if Γ_1 and Γ_2 are lists, then $\Gamma_1 \circ \Gamma_2 \equiv \Gamma_1, \Gamma_2$.

c. $[\,]$ is the empty list.

d. relation $\vdash_{\boldsymbol{B}^-}$ is defined in this way: $\Gamma \vdash_{\boldsymbol{B}^-} \Delta$ if and only if $\Gamma \vdash \Delta$ is derivable in $\boldsymbol{B}^-$.

We can easily verify that $\mathcal{F}$ is indeed a relational monoid. In fact, the concatenation between lists is associative and $[\,]$ is the neutral element, since for every finite list of formulae Γ, $\Gamma, [\,] = [\,], \Gamma = \Gamma$.

We can define the operators $(\cdot)^{\leftarrow}$ and $(\cdot)^{\rightarrow}$ in $\mathcal{F}$; let us look at how they behave on $\mathcal{P}(Frm^*)$. If Σ is a list of formulae, definition 3 says that:

$$\Sigma^{\rightarrow} = \{\Delta \in Frm^* : \Sigma \vdash_{\mathbf{B}^-} \Delta\} \quad \text{consequents of } \Sigma; \tag{25}$$
$$\Sigma^{\leftarrow} = \{\Gamma \in Frm^* : \Gamma \vdash_{\mathbf{B}^-} \Sigma\} \quad \text{antecedents of } \Sigma. \tag{26}$$

Generally for any $A, B \subseteq Frm^*$:

$$A^{\rightarrow} = \{\Delta \in Frm^* : \Gamma \vdash_{\mathbf{B}^-} \Delta \text{ for all } \Gamma \,\epsilon\, A\}\,; \tag{27}$$
$$B^{\leftarrow} = \{\Gamma \in Frm^* : \Gamma \vdash_{\mathbf{B}^-} \Delta \text{ for all } \Delta \,\epsilon\, B\}\,. \tag{28}$$

Now we can prove an important lemma. If we consider a formula φ in $\mathcal{L}$, then its left saturation is formed by all its antecedents and its right saturation is formed by all its consequents.

Lemma 13. *For any formula φ of $\mathcal{L}$: $\{\varphi\}^{\rightarrow\leftarrow} = \{\varphi\}^{\leftarrow}$ and $\{\varphi\}^{\leftarrow\rightarrow} = \{\varphi\}^{\rightarrow}$.*

Proof. For any Γ and φ,

$$\begin{array}{lll} \Gamma \,\epsilon\, \{\varphi\}^{\rightarrow\leftarrow} & \text{iff} & \Gamma \vdash_{\mathbf{B}^-} \Delta \;\text{ for all } \Delta \,\epsilon\, \{\varphi\}^{\rightarrow} \\ & \text{iff} & \dfrac{\varphi \vdash_{\mathbf{B}^-} \Delta}{\Gamma \vdash_{\mathbf{B}^-} \Delta} \;\text{ for all } \Delta \\ & \text{iff} & \Gamma \vdash_{\mathbf{B}^-} \varphi \;\text{(one direction by choosing } \varphi \vdash \varphi, \text{ vice versa by } \mathit{cut}) \\ & \text{iff} & \Gamma \,\epsilon\, \{\varphi\}^{\leftarrow}. \end{array}$$

The case of $\{\varphi\}^{\leftarrow\rightarrow}$ is perfectly symmetric. □

We have used an important equivalence for calculus $\mathbf{B}^-$ in the above proof, at the next to last step. It is an instance a more general lemma:

Lemma 14. *In the calculus* $\boldsymbol{B}^-$ *the following hold:*

a. for any list Δ *and* $m \geq 1$*, the sequent* $\varphi_1, \ldots, \varphi_m \vdash \Delta$ *is derivable iff*

$$\frac{\Gamma_1 \vdash \varphi_1 \;\ldots\; \Gamma_m \vdash \varphi_m}{\Gamma_1, \ldots, \Gamma_m \vdash \Delta} \;\textit{for all } \Gamma_1, \ldots, \Gamma_m\,; \tag{29}$$

b. for any list Γ *and* $n \geq 1$*, the sequent* $\Gamma \vdash \psi_1, \ldots, \psi_n$ *is derivable iff*

$$\frac{\psi_1 \vdash \Delta_1 \;\ldots\; \psi_n \vdash \Delta_n}{\Gamma \vdash \Delta_1, \ldots, \Delta_n} \;\textit{for all } \Delta_1, \ldots, \Delta_n\,. \tag{30}$$

Proof. a. By $\varphi_1, \ldots, \varphi_m \vdash \Delta$ we can derive (29) using *cutL* rule m times:

$$\dfrac{\Gamma_m \vdash \varphi_m \qquad \dfrac{\dfrac{\Gamma_1 \vdash \varphi_1 \quad \varphi_1, \ldots, \varphi_m \vdash \Delta}{\Gamma_1, \varphi_2, \ldots, \varphi_m \vdash \Delta}\,\mathit{cutL}}{\vdots \atop \Gamma_1, \ldots, \Gamma_{m-1}, \varphi_m \vdash \Delta}}{\Gamma_1, \ldots, \Gamma_m \vdash \Delta}\,\mathit{cutL}$$

Vice versa, by (29) if we consider axioms $\varphi_i \vdash \varphi_i$ $(i = 1, \ldots, m)$, then we obtain the sequent $\varphi_1, \ldots, \varphi_m \vdash \Delta$. Case *b.* is symmetric: by using n *cutR* rules and by considering the axioms $\psi_i \vdash \psi_i$ $(i = 1, \ldots, n)$. □

If we consider the contexts with exactly two formulae, then this equivalence involves also the multiplicative connectives. We have just to consider the definitional equations.

Corollary 15 (Multiplicatives). *In the calculus* $\boldsymbol{B}^-$*:*

a. for every list Δ*, the following are equivalent:*

$$\psi \otimes \varphi \vdash \Delta\,; \qquad \psi, \varphi \vdash \Delta\,; \qquad \frac{\Gamma_1 \vdash \varphi \quad \Gamma_2 \vdash \psi}{\Gamma_1, \Gamma_2 \vdash \Delta} \;\textit{for all } \Gamma_1, \Gamma_2\,;$$

b. for every list Γ*, the followings are equivalent:*

$$\Gamma \vdash \varphi \invamp \psi\,; \qquad \Gamma \vdash \varphi, \psi\,; \qquad \frac{\varphi \vdash \Delta_1 \quad \psi \vdash \Delta_2}{\Gamma \vdash \Delta_1, \Delta_2} \;\textit{for all } \Delta_1, \Delta_2\,.$$

Proof. *a.* The first equivalence is just definitional equation for $\otimes$. The second one is a particular case of previous lemma 14 where $m = 2$. Case *b.* is symmetric. □

Now we introduce the *canonical evaluation* V of formulae in $\mathcal{F}$. We evaluate every propositional variable p with the subset of Frm^* that is made by all the antecedents of p. This kind of subset is left saturated by lemma 13. Our choice is respected by every kind of formula in $\mathcal{L}$, that is for every formula $\varphi \in \mathcal{L}$ the evaluation $V(\varphi)$ is the left saturated subset of Frm^* formed by all the antecedents (*ingredients*) of φ. Obviously the right saturated subset of Frm^* associated to φ is just the subset formed by all the consequents (*products*) of φ. We prove it formally:

Lemma 16 (Canonical evaluation). *Let us define $V(p) \equiv \{p\}^{\leftarrow}$ for every propositional variable p. Then for every formula φ of $\mathcal{L}$, $V(\varphi) = \{\varphi\}^{\leftarrow}$. And for every context Σ, $Ingr(\Sigma)^{\rightarrow} = \{\Sigma\}^{\rightarrow}$ and $Prod(\Sigma)^{\leftarrow} = \{\Sigma\}^{\leftarrow}$.*

Proof. First we consider the formulas of $\mathcal{L}$. By lemma 13 and (6), for every formula φ:

$$V(\varphi) = \{\varphi\}^{\leftarrow} \quad \text{iff} \quad V(\varphi)^{\rightarrow} = \{\varphi\}^{\rightarrow}. \tag{31}$$

so we can prove the first equality or the second one equivalently.

We proceed by induction on the structure of formulae. The thesis is verified on propositional variables by hypothesis. We prove the thesis on constants using the definitional equations.

$$V(\top) \equiv Frm^* = \{\Gamma \in Frm^* : \Gamma \vdash_{\mathbf{B}^-} \top\} = \{\top\}^{\leftarrow}.$$

$$V(0)^{\rightarrow} \equiv \emptyset^{\rightarrow} = (Frm^*)^{\leftarrow\rightarrow} = Frm^* = \{\Delta \in Frm^* : 0 \vdash_{\mathbf{B}^-} \Delta\} = \{0\}^{\rightarrow}.$$

$$V(\bot) \equiv \{[\,]\}^{\leftarrow} = \{\Gamma \in Frm^* : \Gamma \vdash_{\mathbf{B}^-} \quad\} = \{\Gamma \in Frm^* : \Gamma \vdash_{\mathbf{B}^-} \bot\} = \{\bot\}^{\leftarrow}.$$

$$V(1)^{\rightarrow} \equiv \{[\,]\}^{\rightarrow\leftarrow\rightarrow} = \{\Delta \in Frm^* : \quad \vdash_{\mathbf{B}^-} \Delta\} = \{\Delta \in Frm^* : 1 \vdash_{\mathbf{B}^-} \Delta\} = \{1\}^{\rightarrow}.$$

The inductive steps are verified as follows

$$\begin{aligned}
V(\varphi \parr \psi) &\equiv (\,V(\varphi)^{\rightarrow} \circ V(\psi)^{\rightarrow}\,)^{\leftarrow} \\
&= \{\Delta_1, \Delta_2 : \Delta_1 \,\epsilon V(\varphi)^{\rightarrow} \text{ and } \Delta_2 \,\epsilon V(\psi)^{\rightarrow}\}^{\leftarrow} \\
&= \{\Gamma : \Gamma \vdash_{\mathbf{B}^-} \Delta_1, \Delta_2 \text{ for all } \Delta_1 \,\epsilon V(\varphi)^{\rightarrow} \text{ and } \Delta_2 \,\epsilon V(\psi)^{\rightarrow}\} \quad \text{by (1)} \\
&= \{\Gamma : \Gamma \vdash_{\mathbf{B}^-} \Delta_1, \Delta_2 \text{ for all } \Delta_1, \Delta_2 \text{ s.t. } \varphi \vdash_{\mathbf{B}^-} \Delta_1 \text{ and } \psi \vdash_{\mathbf{B}^-} \Delta_2\} \quad \text{Induction} \\
&= \{\Gamma : \Gamma \vdash_{\mathbf{B}^-} \varphi \parr \psi\} \quad \text{by corollary 15.}b \\
&= \{\varphi \parr \psi\}^{\leftarrow} \quad \text{by (26).}
\end{aligned}$$

$$\begin{aligned}
V(\psi \otimes \varphi)^{\rightarrow} &\equiv (\,V(\psi) \circ V(\varphi)\,)^{\rightarrow\leftarrow\rightarrow} \\
&= \{\Gamma_1, \Gamma_2 : \Gamma_1 \,\epsilon\, V(\psi) \text{ and } \Gamma_2 \,\epsilon\, V(\varphi)\}^{\rightarrow} \\
&= \{\Delta : \Gamma_1, \Gamma_2 \vdash_{\mathbf{B}^-} \Delta \text{ for all } \Gamma_1 \,\epsilon\, V(\psi) \text{ e } \Gamma_2 \,\epsilon\, V(\varphi)\} \quad \text{by (2)} \\
&= \{\Delta : \Gamma_1, \Gamma_2 \vdash_{\mathbf{B}^-} \Delta \text{ for all } \Gamma_1, \Gamma_2 \text{ s.t. } \Gamma_1 \vdash_{\mathbf{B}^-} \psi \text{ e } \Gamma_2 \vdash_{\mathbf{B}^-} \varphi\} \quad \text{Induction} \\
&= \{\Delta : \psi \otimes \varphi \vdash_{\mathbf{B}^-} \Delta\} \quad \text{by corollary 15.}a \\
&= \{\psi \otimes \varphi\}^{\rightarrow} \quad \text{by (25).}
\end{aligned}$$

$$\begin{aligned}
V(\varphi \,\&\, \psi) &\equiv V(\varphi) \cap V(\psi) \\
&= \{\Gamma : \Gamma \,\epsilon\, V(\varphi) \text{ and } \Gamma \,\epsilon\, V(\psi)\} \\
&= \{\Gamma : \Gamma \vdash_{\mathbf{B}^-} \varphi \text{ and } \Gamma \vdash_{\mathbf{B}^-} \psi\} \quad \text{Induction} \\
&= \{\Gamma : \Gamma \vdash_{\mathbf{B}^-} \varphi \,\&\, \psi\} \quad \text{by definition 1} \\
&= \{\varphi \,\&\, \psi\}^{\leftarrow} \quad \text{by (26).}
\end{aligned}$$

$$
\begin{aligned}
V(\psi \oplus \varphi)^{\rightarrow} &\equiv (\, V(\varphi) \cup V(\psi)\,)^{\rightarrow\leftarrow\rightarrow} \\
&= V(\varphi)^{\rightarrow} \cap V(\psi)^{\rightarrow} \\
&= \{\Delta : \Delta \,\epsilon\, V(\psi)^{\rightarrow} \text{ and } \Delta \,\epsilon\, V(\varphi)^{\rightarrow}\} \\
&= \{\Delta : \psi \vdash_{\mathbf{B}^-} \Delta \text{ and } \varphi \vdash_{\mathbf{B}^-} \Delta\} \quad \text{Induction} \\
&= \{\Delta : \psi \oplus \varphi \vdash_{\mathbf{B}^-} \Delta\} \quad \text{by definition 1} \\
&= \{\psi \oplus \varphi\}^{\rightarrow} \quad \text{by (25).}
\end{aligned}
$$

Finally we consider the lists of formulae. If $\Sigma = [\,]$, then the lemma is verified by definition 10; in fact neutral element of syntactic model is just $[\,]$. If $\Sigma = \sigma_1, \ldots, \sigma_m$ with $m \geq 1$, then:

$$
\begin{aligned}
Ingr(\Sigma)^{\rightarrow} &\equiv (\, \{[\,]\} \circ V(\sigma_1) \circ \ldots \circ V(\sigma_m)\,)^{\rightarrow} \\
&= \{\Gamma_1, \ldots, \Gamma_m : \Gamma_1 \,\epsilon\, V(\sigma_1), \ldots, \Gamma_2 \,\epsilon\, V(\sigma_m)\}^{\rightarrow} \\
&= \{\Delta : \Gamma_1, \ldots, \Gamma_m \vdash_{\mathbf{B}^-} \Delta \text{ for all } \Gamma_1 \,\epsilon\, V(\sigma_1), \ldots, \Gamma_m \,\epsilon\, V(\sigma_m)\} \quad \text{by (2)} \\
&= \{\Delta : \Gamma_1, \ldots, \Gamma_m \vdash_{\mathbf{B}^-} \Delta \text{ for all } \Gamma_1 \vdash_{\mathbf{B}^-} \sigma_1, \ldots, \Gamma_m \vdash_{\mathbf{B}^-} \sigma_m\} \\
&= \{\Delta : \Sigma \vdash_{\mathbf{B}^-} \Delta\} \quad \text{by lemma } 14.a \\
&= \{\Sigma\}^{\rightarrow} \quad \text{by (2)}
\end{aligned}
$$

The case $Prod(\Sigma)^{\leftarrow}$ is symmetric. □

The reader can observe that $V(\varphi)^{\rightarrow}$ (i.e. right saturated subset associated to a formula φ) is just the subset of all consequents of φ as we have anticipated. Moreover $Ingr(\Sigma)^{\rightarrow}$ is the subset of all the consequents (i.e products in production cycle $\mathbf{B}^-$) of Σ and $Prod(\Sigma)^{\leftarrow}$ is the subset of all the antecedents (i.e. ingredients in production cycle) of Σ, as we have said at the beginning of this section.

Finally we prove the completeness theorem.

Theorem 17 (Completeness). *The sequent $\Gamma \vdash \Delta$ is derivable in the calculus $\boldsymbol{B}^-$ if and only if $\Gamma \vdash \Delta$ is valid in every relational monoid.*

Proof. The direction from left to right is theorem 11 of soundness. From right to left, if $\Gamma \vdash \Delta$ is valid in every relational monoid then $Ingr(\Gamma) \subseteq Prod(\Delta)^{\leftarrow}$ in the syntactic for the canonical evaluation. This means that $Ingr(\Gamma) \subseteq \{\Delta\}^{\leftarrow}$ by lemma 16. The function $(\cdot)^{\rightarrow\leftarrow}$ is a closure operator and $\{\Delta\}^{\leftarrow}$ is left saturated, so $Ingr(\Gamma)^{\rightarrow\leftarrow} \subseteq \{\Delta\}^{\leftarrow}$ and then, by lemma 16, $\{\Gamma\}^{\rightarrow\leftarrow} \subseteq \{\Delta\}^{\leftarrow}$. By (5) we obtain $\Gamma \,\epsilon\, \{\Delta\}^{\leftarrow}$, and this means that $\Gamma \vdash_{\mathbf{B}^-} \Delta$. □

5 Extensions

The relational semantics can be extended to the commutative calculus $\mathbf{B}^-_{exch}$ that we obtain by adding the rules of *exchange* to $\mathbf{B}^-$:

$$
\frac{\Gamma_1, \Sigma, \Pi, \Gamma_2 \vdash \Delta}{\Gamma_1, \Pi, \Sigma, \Gamma_2 \vdash \Delta}\; exchL \qquad\qquad \frac{\Gamma \vdash \Delta_1, \Pi, \Sigma, \Delta_2}{\Gamma \vdash \Delta_1, \Sigma, \Pi, \Delta_2}\; exchR \tag{32}
$$

To obtain a new completeness theorem for calculus $\mathbf{B}^-_{exch}$ we have just to restrict the relational semantics to *commutative relational monoids*, namely the relational monoids $\mathcal{M} = (M, \cdot, 1, R)$ where the operation $\cdot$ is commutative ($x \cdot y = y \cdot x$ for all $x, y \in M$).

The evaluation for formulae and contexts, and the definition of validity of sequents remain the same of section 3. For completeness we have to consider the syntactic model

$$
\mathcal{F}' \equiv (Frm^{\circledast}, \circ, [\,], \vdash_{exch})
$$

where $Frm^{\circledast}$ is the set of all non ordered lists of formulae of $\mathcal{L}$, and $\vdash_{exch}$ is the derivability in $\mathbf{B}^-_{exch}$, i.e. for $\Gamma, \Delta \in Frm^{\circledast}$:

$$
\Gamma \vdash_{exch} \Delta \quad \text{iff} \quad \Gamma \vdash \Delta \text{ is derivable in } \mathbf{B}^-_{exch} \tag{33}
$$

It is easy to see that $\mathcal{F}'$ is a commutative relational monoid[4]. Moreover all lemmas and corollaries that we have proved for $\mathbf{B}^-$ and $\mathcal{F}$ in section 4 are still verified for $\mathbf{B}^-_{exch}$ and $\mathcal{F}'$. So we easily obtain a completeness theorem by following the proof of theorem 17. We have just to verify the validity of the new *exch* rules by using the commutativity of operation in $\mathcal{M}$.

Theorem 18 (Completeness for $\mathbf{B}^-_{exch}$). *The sequent $\Gamma \vdash \Delta$ is deducible by calculus $\mathbf{B}^-_{exch}$ if and only if $\Gamma \vdash \Delta$ is valid into every commutative relational monoid.*

We can extend $\mathbf{B}^-_{exch}$ to obtain Classical Linear Logic without exponentials (see [13]). Girard's phase spaces provide with a set-theoretic semantics for linear logic (see [6]). Phase spaces can be seen as a particular case of relational monoids: we can show that a phase space is a commutative relational monoid whit a strongly symmetric relation[5]. We define that the relation R is strongly symmetric if

$$\text{for all } x, y, z \in M: \text{ if } x \cdot y\, R\, z \text{ then } x \cdot z\, R\, y\,. \tag{34}$$

Note that a strongly symmetric relation is symmetric; in fact we can choose $x = 1$ in (34). If R is symmetric, the operators $(\cdot)^{\leftarrow}$ e $(\cdot)^{\rightarrow}$ coincide, and we indicate them with $(\cdot)^{-}$.

Now we prove that the operator $(\cdot)^{-}$ for strongly symmetric monoids is the operator $(\cdot)^{\perp}$ of space phases. Moreover the evaluation of formulae in phase spaces coincides with the evaluation we have just defined if we consider only strongly symmetric monoids.

Theorem 19 (Phase spaces and strongly symmetric monoids). *Any phase space is a strongly symmetric monoid, and vice versa any strongly symmetric monoid is a phase space.*

Proof. Let $(M, \perp)$ be a phase space. Then $(M, \cdot)$ is a commutative monoid and it become a strongly symmetric monoid $(M, \cdot, 1, R)$ if we define:

$$\text{for all } x, y \in M: \quad x\, R\, y \;\equiv\; x \cdot y \,\epsilon\, \perp.$$

Obviously the relation R is strongly symmetric. Moreover the operators $(\cdot)^{-}$ and $(\cdot)^{\perp}$ coincide on subsets of M. In fact for any $A \subseteq M$:

$$\begin{aligned} A^{-} &\equiv \{y \in M : x\, R\, y \text{ for all } x \,\epsilon\, A\} \\ &= \{y \in M : x \cdot y \,\epsilon\, \perp \text{ for all } x \,\epsilon\, A\} \equiv A^{\perp}. \end{aligned}$$

In particular

$$\{1\}^{-} = \{y \in M : 1 \cdot y \,\epsilon\, \perp\} = \{y \in M : y \,\epsilon\, \perp\} = \perp \tag{35}$$

Vice versa, let $(M, \cdot, 1, R)$ be a strongly symmetric monoid; then we reduce it to a phase space by defining $\perp \equiv \{1\}^{-}$. In such way, for any $A \subseteq M$:

$$\begin{aligned} A^{\perp} &\equiv \{y \in M : x \cdot y \,\epsilon\, \perp \text{ for all } x \,\epsilon\, A\} \\ &= \{y \in M : x \cdot y\, R\, 1 \text{ for all } x \,\epsilon\, A\} \\ &= \{y \in M : x \cdot 1\, R\, y \text{ for all } x \,\epsilon\, A\} \quad \text{by (34)} \\ &= \{y \in M : x\, R\, y \text{ for all } x \,\epsilon\, A\} \equiv A^{-}. \end{aligned}$$

□

[4] Note that the relation $\vdash_{exch}$ between not ordered lists is well defined. In fact the calculus $\mathbf{B}^-_{exch}$, with rules of exchange, does not consider the position of formulae into the contexts; namely if $\Gamma \vdash \Delta$ is derivable in $\mathbf{B}^-_{exch}$, then also $\Gamma' \vdash \Delta'$ is derivable in $\mathbf{B}^-_{exch}$, where Γ' and Δ' are permutations of Γ and Δ respectively.

[5] In connection with linear logic, the use of Birkhoff's polarities appears also in [1], [4], [15].

References

[1] A. Avron, *The semantics and proof Theory of linear logic*. Theoretical Computer Sience 57 (1988), pp. 161-184.

[2] G. Birkhoff, *Lattice Theory*. AMS Colloquium publications 25 (1967), pp. 122-131.

[3] P. Freyd, *Abelian categories: an introduction to the theory of functors*. Harper and Row, New York (1964).

[4] J. Gallier, *Constructive logics: linear logic and proofnets*. Technical Report, CIS Departement University of Pennsylvania (1991).

[5] G. Gentzen, *Untersuchungen über das logische Schliessen*. Mathematische Zeitschrift, 39 (1935), pp. 176-210. English translation in "*The collected papers of Gerhard Gentzen*", M.E. Szabo ed., North-Holland (1969), pp. 68-131.

[6] J. Girard, *Linear Logic*. Theoretical Computer Science, 50 (1987), pp. 1-102.

[7] F. Guidi, *Galois connections on the basic pair*. Manuscript (25 June 1999).

[8] ——, *Basic pairs as semantics for the conjunctive fragment of the calculus BS*. Manuscript (20 September 2000).

[9] D. Macedonio, *Semantica relazionale per la logica di base*. Tesi di laurea in Matematica, Università di Padova, March 2001.

[10] H. Ono, *Semantics for Substructural Logics*, in Substructural Logics, K. Dosěn and P. Schroeder-Heister eds., Oxford (1993), Clarendon, pp. 259-291.

[11] H. Ono, Y. Komori, *Logics without the contraction rule*. Journal of Symbolic Logic, 50 (1985), pp. 169-201.

[12] G. Sambin, *Pretopologies and completeness proofs*. Journal of Symbolic Logic, 60 (1995), pp. 861-878.

[13] G. Sambin, G. Battilotti, C. Faggian, *Basic logic: reflection, symmetry, visibility*. Journal of Symbolic Logic, 65 (2000), pp. 979-1013.

[14] G. Sambin, S. Valentini, *Building up a toolbox for Martin-Löf's type theory: subset theory*, in Twenty-five years of constructive type theory. Proceedings of the Congress held in Venice, October, 1995, G. Sambin and J. Smith eds., Oxford U. P. (1998), pp. 221-244.

[15] A. Ursini, *Semantical investigations of linear logic*. Rapp. Matematico 291 (1995), Università di Siena.