Triangulations in $\mathbb{R}^2$

Marco Caliari

June 9, 2014

1 Some definitions

We have a set of sites $S = \{s_1, s_2, \ldots, s_m\}$ in $\mathbb{R}^2$, assuming that no four sites are co-circular.

The *circumcenter* of a triangle is the point where the three (perpendicular) *bisectors* meet. It is the center of triangle's circumcircle.

A *graph* is a set of sites and arcs (called edges) connecting some of the sites.

A *planar graph* is a graph with no intersecting edges.

A *planar straight line graph* is a planar graph whose edges are segments.

A *connected graph* is a graph in which any pairs of sites is connected by a finite sequence of edges.

The *convex hull* of a set of points is the smallest convex set containing all the points.

The bisectors between pairs of sites are straight lines that partition the plane into m convex regions, one corresponding to each site s_i (by induction). Each region is called *Voronoi polygon* of s_i: it is the set of points which are closer to s_i than to the remaining sites in S. The partition of the plane is called a *Voronoi diagram* of S. The vertices and the edges of the convex regions are called *Voronoi vertices* and *Voronoi edges*.

Proposition. *The number of Voronoi vertices is* $2(m-1)-h$ *and the number of Voronoi edges is* $3(m-1)-h$ *where* h *is the number of vertices on the convex hull of* S*.*

Proof. First of all, we consider a circle intersecting the unbounded edges of the Voronoi diagram and consider the resulting planar conncected graph. There are h edges in each unbounded Voronoi polygon. For this graph, $2E = 3V$, where E is the number of edges and V the number of points. In fact, there are three edges departing from any point and in this way any

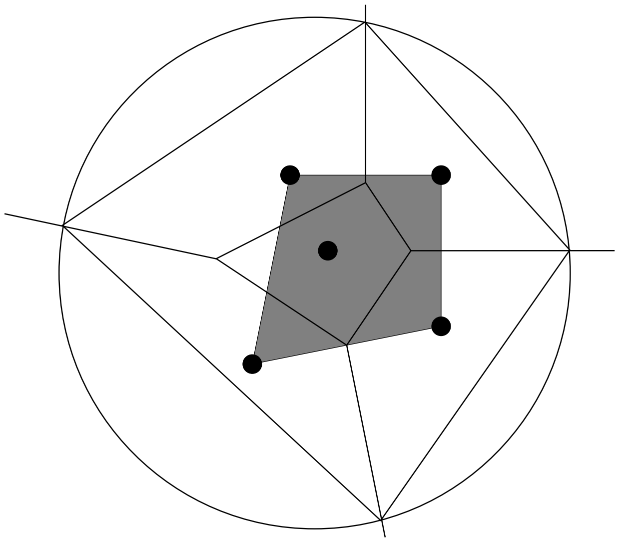

Figure 1: Voronoi's diagram.

edge is counted twice. Then, we use Euler's formula $V - E + F = 2$ getting $V = 2(F - 2)$ and $E = 3(F - 2)$. With respect to Figure 1 it is $V = 8$, $E = 12$ and $F = 6$. The number of Voronoi points is $V - h$ and the number of Voronoi edges is $E - h$. Since $F = m + 1$, we have $2(m + 1 - 2) - h$ Voronoi vertices and $3(m + 1 - 2) - h$ Voronoi edges. □

Proposition. *The circle with center a Voronoi vertex and passing through the (three) sites that defines its center has no other sites in its interior.*

Proof. By definition the center is the Voronoi vertex closest to the three sites. If another site would be inside the circle, then it would be closest to the vertex, in contradiction with its property. □

The *dual* of a Voronoi diagram is obtained by joining pairs of sites whose Voronoi polygons are adjacent.

A *triangulation* of sites is a set of straight line segments which intersect only at the sites and so that every region internal to the convex hull is a triangle.

Proposition. *The dual of a Voronoi diagram is a triangulation.*

It is called *Delaunay* triangulation. It is unique if no four sites are cocircular. Otherwise, the dual of a Voronoi diagram contains regions which are not triangles. In this case, any triangulation obtained by adding edges to the dual of a Voronoi diagram is a Delaunay triangulation.

A Delaunay triangulation maximizes the minimum angle of the triangles among all the triangulations.

2 Some other definitions

A family of triangulations $\mathcal{T}_h$ is said *regular* if there exists a constant $\delta > 0$, indipendent of h, such that

$$\frac{h_K}{\rho_K} \leq \delta, \quad \forall K \in \mathcal{T}_h$$

where h_K is the diameter and ρ_K the radius of the inscribed circle of the triangle K. Regularity excludes very deformed triangles, that is with small angles. In this sense, Delaunay triangulations are optimal.

A *constrained Delaunay triangulation* of a planar straight line graph is a triangulation in which each segment of the graph is present as a single edde in the triangulation. It is not truly a Delaunay triangularion.

A *conforming Delaunay triangulation* of a planar straight line graph is a true Delaunay triangulation in which each segment may have been subdivided into several edges by the insertion of additional vertices, called *Steiner points*.

Steiner points are also inserted to meet constraints on the minimum angle and maximum triangle area.

A *constrained conforming Delaunay triangulation* of a planar straight line graph is a constrained Delaunay triangulation that includes Steiner points. It is not truly a Delaunay triangulation, but usually takes fewer vertices.

3 Algorithms

There are several algorithms to make a Delaunay triangulation. It holds the folowing

Proposition. *The Delaunay triangulation of a set of m points can be computed in $\mathcal{O}(m \log m)$ operations, using $\mathcal{O}(m)$ storage.*

References

[1] *Voronoi Diagram and Delaunay Triangulation*, `http://asishm.myweb.cs.uwindsor.ca/cs557/F10/handouts/VDandDT.pdf`.

[2] Mark de Berg, Otfried Cheong, Marc van Kreveld, and Mark Overmars, *Computational Geometry: Algorithms and Applications*, 2008, Springer–Verlag, `http://www.cs.uu.nl/geobook/interpolation.pdf`.

[3] J. R. Shewchuk, *Triange: A Two-Dimensional Quality Mesh Generator and Delaunay Triangulator*, 2005, `http://www.cs.cmu.edu/~quake/triangle.html`.