Silting modules

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1. RING EPIMORPHISMS

Let A be a ring.
We denote by $Mod(A)$ the category of all right A-modules,
and by $mod(A)$ the category of all finitely presented right A-modules.

Definition. A ring homomorphism $f : A \to B$ is a

- **ring epimorphism** if it is an epimorphism in the category of rings with unit (equivalently: the functor given by restriction of scalars $f_* : Mod(B) \hookrightarrow Mod(A)$ is full);
- (Geigle-Lenzing 1991)
  homological ring epimorphism if it is a ring epimorphism and $Tor_i^A(B,B) = 0$ for all $i > 0$ (equivalently: the functor given by restriction of scalars $f_* : D(Mod(B)) \hookrightarrow D(Mod(A))$ is full).

Two ring epimorphisms $f_1 : A \to B_1$ and $f_2 : A \to B_2$ are said to be **equivalent** if there is a ring isomorphism $h : B_1 \to B_2$ such that $f_2 = h \circ f_1$. We then say that they lie in the same **epiclass** of A.

Theorem. (Gabriel-de la Peña 1987) There is a bijection between:

(1) epiclasses of ring epimorphisms $A \to B$;
(2) bireflective subcategories $\mathcal{X}_B$ of $Mod(A)$,
i.e., full subcategories of $Mod(A)$ closed under products, coproducts, kernels and cokernels.

Moreover, epiclasses of A form a poset with respect to the following partial order:
given $f_1 : A \to B_1$ and $f_2 : A \to B_2$, we set

$$f_1 \geq f_2$$

if there is a ring homomorphism $g : B_1 \to B_2$ such that $g \circ f_1 = f_2$.
This corresponds to the partial order on bireflective subcategories given by inclusion.

2. SILTING MODULES

Definition. A bounded complex of projective A-modules σ is said to be **silting** if

(1) $Hom_{D(A)}(\sigma, \sigma^{(I)}[i]) = 0$, for all sets I and $i > 0$.
(2) the smallest triangulated subcategory of $D(A)$ containing $Add(\sigma)$ is $K^b(Proj(A))$.

In the compact case see work of

- Keller-Vossieck 1988
- Aihara-Iyama 2012
- Keller-Nicolàs 2013, Koenig-Yang 2014, Mendoza-Sáenz-Santiago-Souto Salorio 2013, Wei 2013

establishing a close relationship with t-structures and co-t-structures in the derived category. These connections extend to the non-compact case.

For 2-term complexes see work of

- Hoshino-Kato-Miyachi 2002

inspiring the following result

Proposition. Let σ be 2-term complex in $K^b(Proj(A))$ and $T = H^0(\sigma)$.
Then σ is a silting complex if and only if the classes

$$Gen(T) = \{M \in Mod(A) \mid \text{ there is an epimorphism } T^{(I)} \to M \text{ for some set } I\}$$

and

$$\mathcal{D}_\sigma := \{X \in Mod(A) | Hom_A(\sigma, X) \text{ is surjective}\}$$

coincide.

Definition. A module T is **silting** if there is a projective presentation σ of T such that $Gen(T) = \mathcal{D}_\sigma$.

Examples.

(1) T is silting with respect to an injective presentation $P_{-1} \overset{\sigma}{\hookrightarrow} P_0$ iff $Gen(T) = \operatorname{Ker}\operatorname{Ext}^1_A(T,-)$, i. e. T is a *tilting* module (of projective dimension at most one, possibly non finitely generated).

(2) If A is a finite dimensional algebra over a field, and $T \in mod(A)$, then
T is silting iff it is *support* τ*-tilting* in the sense of Adachi-Iyama-Reiten 2014.

(3) Let A be the path algebra of the quiver Q having two vertices, 1 and 2, and countably many arrows from 1 to 2. Let $P_i = e_iA$ be the indecomposable projective A-module for $i = 1, 2$. Then $T := S_2$ with the projective presentation

$$0 \longrightarrow P_1^{(\mathbb{N})} \xrightarrow{\sigma} P_2 \longrightarrow T \longrightarrow 0,$$

is a silting module (of projective dimension one) which is not tilting.

Indeed, it is not tilting as the class $Gen(T)$ consists precisely of the semisimple injective A-modules, so $Gen(T) = \operatorname{Ker}\operatorname{Hom}_A(P_1,-) \subsetneq \operatorname{Ker}\operatorname{Ext}^1_A(T,-)$. But T is silting with respect to the projective presentation γ of T obtained as the direct sum of σ with the trivial map $P_1 \to 0$, since $\mathcal{D}_\gamma = \operatorname{Ker}\operatorname{Ext}^1_A(T,-) \cap \operatorname{Ker}\operatorname{Hom}_A(P_1,-) = \operatorname{Ker}\operatorname{Hom}_A(P_1,-) = Gen(T)$.

3. THE HEREDITARY CASE

Let now A be a hereditary ring.

Fact 1. T is silting iff T is tilting over $\overline{A} = A/ann(T)$.
(since A is hereditary, $ann(T)$ is an idempotent ideal of A, so $Mod(\overline{A})$ is closed under extensions!)

Fact 2. If T is silting, then there is an exact sequence

$$(*) \qquad A \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0$$

such that T_0 and T_1 lie in $Add(T)$ and ϕ is a left $Gen(T)$-approximation.

Definition. A silting module T is called **minimal** if there is a sequence $(*)$ as above where ϕ is left minimal.

Fact 3. Given a homological ring epimorphism $f : A \to B$ with

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow 0$$

the module

$$T = B \oplus C$$

is a minimal silting module.

Fact 4. Given a sequence $(*)$ as above where ϕ is left minimal, and denoting

$$T_1{}^{\perp} = \{X \in Mod(A) \mid \mathrm{Hom}_A(T_1, X) = \mathrm{Ext}^1_A(T_1, X) = 0\},$$

we have that

$$\mathcal{X} = T_1{}^{\perp} \cap Mod(\overline{A})$$

is bireflective and extension closed, hence it coincides with the essential image of f_* for some homological epimorphism $f : A \to B$.

Definition. Two silting modules are **equivalent** if they generate the same torsion class (which means that they have the same additive closure).

Theorem. Let A be a hereditary ring. There is a bijection between

(1) equivalence classes of minimal silting A-modules;
(2) epiclasses of homological ring epimorphisms of A.

which restricts to a bijection between

(1) equivalence classes of minimal tilting A-modules;
(2) epiclasses of injective homological ring epimorphisms of A.

Corollary 1. Let A be a hereditary ring. Then a tilting A-module T is minimal if and only if there is an injective homological ring epimorphism $f : A \to B$ such that T is equivalent to the tilting module $B \oplus B/A$.

Corollary 2. If A is a finite dimensional hereditary algebra, there is a bijection between

(1) equivalence classes of finitely generated support tilting A-modules;
(2) epiclasses of homological ring epimorphisms $A \to B$ with B finite dimensional;

For this result, cf. Ingalls-Thomas 2009, Marks 2015, and see also Igusa-Schiffler 2010, Ringel 2015 for a combinatorial interpretation in terms of the poset of noncrossing partitions. The latter corresponds to the poset of epiclasses defined at the beginning by $\leq$, and it is not a lattice in general. However, relaxing the condition that B is finite dimensional, we obtain

Corollary 3. The poset of *all* homological ring epimorphisms $A \to B$ of a hereditary ring A is a lattice.

To prove this, one uses that by a result of Schofield and Krause-Stovicek 2010, the following statements are equivalent for a hereditary ring A:

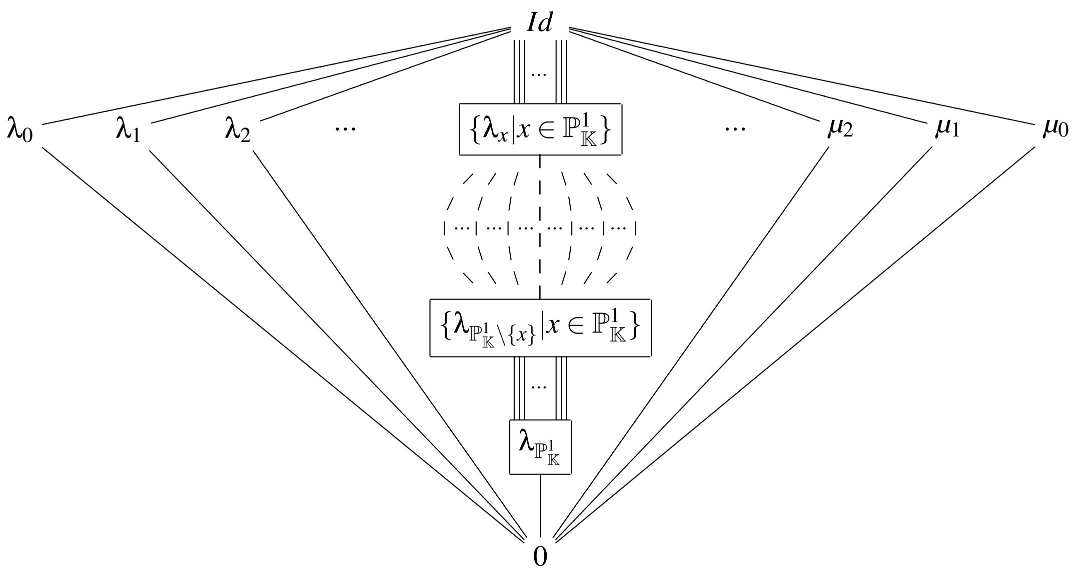
(1) $f : A \to B$ is a homological ring epimorphism

(2) there is a set $\mathcal{U} \subset mod(A)$ such that f is the **universal localization** of A at $\mathcal{U}$,
i. e. for any $U \in \mathcal{U}$ there is a projective presentation $P_{-1} \xrightarrow{\sigma_U} P_0 \longrightarrow U \to 0$ such that
(i) $\sigma_U \otimes_A B$ is an isomorphism for all $U \in \mathcal{U}$, and
(ii) f is universal with respect to (1), that is, $f \geq f'$ for all $f' : A \to B'$ satisfying (i).

Examples.

(1) Let A be a tame hereditary finite dimensional algebra with a tube $\mathcal{U}$ of rank 2, and let S_1, S_2 be the simple regular modules in $\mathcal{U}$. The universal localization $f_i : A \to A_{\{S_i\}}$ of A at S_i has a finite dimensional target $A_{\{S_i\}}$ for $i = 1, 2$. The meet of f_1 and f_2, however, is the universal localization $A \to A_{\{S_1,S_2\}}$ of A at $\mathcal{U}$ and $A_{\{S_1,S_2\}}$ is an infinite dimensional algebra (by results of Crawley-Boevey).

(2) The lattice of homological ring epimorphisms of the Kronecker algebra has the following shape



Here the λ_i are the homological ring epimorphisms corresponding to preprojective silting modules, the μ_i correspond to preinjective silting modules, and the ring epimorphisms in frames are those with infinite dimensional target, that is, those of the form $\lambda_{\mathcal{U}}$ with $\emptyset \neq \mathcal{U} \subseteq \mathbb{P}^1_{\mathbb{K}}$. The interval between Id and $\lambda_{\mathbb{P}^1_{\mathbb{K}}}$ represents the dual poset of subsets of $\mathbb{P}^1_{\mathbb{K}}$.

Up to equivalence, there is just one additional silting module L which is not minimal and thus does not appear in the lattice above. It is called Lukas tilting module and it generates the class of all modules without preprojective summands.