Entropy on abelian groups

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Abstract

A notion of algebraic entropy for automorphisms of abelian groups was introduced by Peters. We modify appropriately this definition to extend it to all endomorphisms of abelian groups. In particular, this algebraic entropy coincides on endomorphisms of torsion abelian groups with the one proposed by Adler, Konheim e McAndrew and studied by Weiss (and more recently by Dikranjan, Goldsmith, Salce and Zanardo).

The right Bernoulli shift β_p of the group $\bigoplus_{\mathbb{N}} \mathbb{Z}(p)$ has algebraic entropy $h(\beta_p) = \log p$. Moreover, we prove the Algebraic Yuzvinski Formula showing that the value of the algebraic entropy of an endomorphism $\phi : \mathbb{Q}^n \to \mathbb{Q}^n$ equals the Mahler measure of its characteristic polynomial $p_\phi(X) = sX^n + a_1 X^{n-1} + \ldots + a_N$ over $\mathbb{Z}$, that is, $h(\phi) = \log s + \sum_{|\lambda_i|>1} \log |\lambda_i|$, where $\{\lambda_i : i = 1, \ldots, N\}$ are the eigenvalues of ϕ.

It turns out that this formula, along with the other normalization given by the value on the right Bernoulli shifts, and other three natural properties (namely, invariance under conjugation, the Addition Theorem and the continuity with respect to direct limits) determine uniquely the algebraic entropy.

As another application of the Algebraic Yuzvinski Formula we deduce an extension of both Weiss Bridge Theorem and Peters Bridge Theorem about the connection between the algebraic entropy of an endomorphism $\phi : G \to G$ and the topological entropy of its Pontryagin dual $\widehat{\phi} : G \to G$. Indeed, we see that $h(\phi) = h_{top}(\widehat{\phi})$.

A relevant tool for studying the connection of the algebraic entropy with other dynamical aspects of the endomorphisms ϕ of the abelian groups G, is the Pinsker subgroup $\mathbf{P}(G, \phi)$ of G with respect to ϕ, which is the maximum ϕ-invariant subgroup of G where the restriction of ϕ has zero algebraic entropy. It turns out that $\mathbf{P}(G, \phi)$ is also the largest ϕ-invariant subgroup of G where the restriction of ϕ has polynomial growth, as well as the smallest ϕ-invariant subgroup of G such that the induced endomorphism $\overline{\phi} : G/\mathbf{P}(G, \phi) \to G/\mathbf{P}(G, \phi)$ has no quasi-periodic points.